

Locally Refined B-splines for isogeometric representation and Isogeometric Analysis

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Introductory remarks

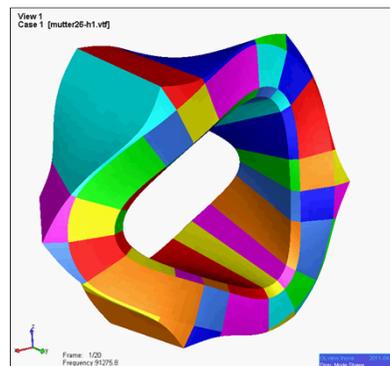
The idea of LR B-splines is inspired by T-splines

- The work on LR B-splines started in June 2009 when I wanted to understand T-splines better. However, I did this without looking at the details of the T-spline approach, contemplating the concept "Local Refinement of B-spline represented functions".
- Comparing my thoughts on "Local Refinement of B-spline represented functions" with T-splines, we realized we had an alternative approach. The LR B-spline refinement is specified in the parameter domain, while for T-splines the B-splines are inferred from T-splines vertex grid,
- The LR B-splines can be given a T-spline type interface for specifying refinement. Having such an interface the data provided will be used directly to refine in the parameter domain, rather than reason in the T-spline type vertex grid.



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Free vibration of a Nut – 3-variate NURBS elements



Example by: Knut
Morten Okstad,
SINTEF IKT



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Introductory remarks - continued

- We prefer to refine directly in the mesh in the parameter domain, as we then deal with only one mesh that reflects the piecewise polynomial structure, rather than the multiple meshes of T-splines.
- It was also obvious that the idea of local refinement posed many new questions related to what we now denote B-splines over Box-partitions. So rather than first proposing new algorithms we: Prof Tom Lyche University of Oslo, Tor Dokken and Kjell Fredrik Pettersen from SINTEF wanted to build a theoretical foundation.
- The theoretical work took much longer time than originally anticipated. The preprint (March 2012) of the paper can be downloaded from:
 - <http://www.sintef.no/upload/IKT/9011/geometri/LR-splines%20SINTEF%20Preprint%20-%20signatures.pdf>



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Introductory remarks – continued 2

We have presented our progress at conferences, workshops and project meetings, and got feedback, and difficult questions:

- Bernard Mourrain, INRIA opened his work on the dimension of spline spaces over T-subdivisions to us, giving us a first dimension formula for the 2-variate case. This triggered the development by Bernard Mourrain and Kjell Fredrik Pettersen of d -variate dimension formula supporting local variation of continuities.
- Tomas Sauer, Univ. Giesen, proposed that we should use Box-partition as the starting point, not the mesh, an idea that significantly simplified the theory (and reduced the number of definitions, and took some months to put in place).
- For ensuring linear independency we first only devised a strategy of dimension increase by one/increase the number of B-splines by one:
 - Carla Manni point out that we had to handle internal knotlines with multiplicity higher than 1
 - Kjetil Johannesen point out that we needed to handle "gap filling".
 These requirements forced us to better understand these more complex situations.
- Many others have commented and made us aware of prior work relevant for LR B-splines.



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Introduction - The larger picture

Some of you will have seen some/many of these slides before, but I feel it important to paint a larger picture.



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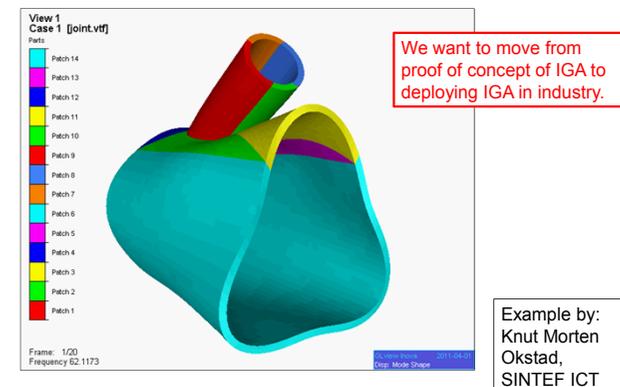
Structure of the 4 lectures

- Hour 1 gives an introduction and sets the scene.
- Hour 2 will be focused on spline spaces over box-partitions and their relevance for FEA. The lectures are thus relaxant for both:
 - T-splines
 - PHT-splines
 - LR B-splines
 - Hierarchical splines
- Hour 3 and 4 will address LR B-splines



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Free vibration of a Tubular Joint – 3-variate NURBS elements



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Independent evolution of Computer Aided Design (CAD) and Finite Element Analysis (FEA)

- CAD and FEA evolved in different communities before electronic data exchange
 - FEA developed to improve analysis in engineering
 - CAD developed to improve the design process
 - Information exchange was drawing based, consequently the mathematical representation used posed no problems
 - Manual modelling of the element grid
 - Implementations used approaches that best exploited the limited computational resources and memory available.
- FEA was developed before the NURBS theory
 - FEA evolution started in the 1940s and was given a rigorous mathematical foundation around 1970 (1973: [Strang](#) and [Fix's An Analysis of The Finite Element Method](#))
 - B-splines: 1972: DeBoor-Cox Algorithm, 1980: Oslo Algorithm

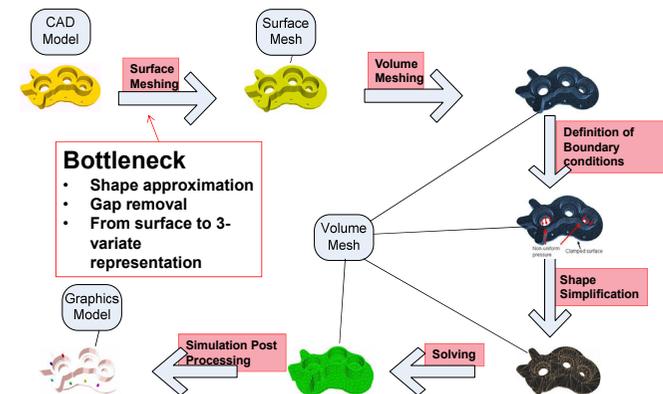
Why are splines important to isogeometric analysis?

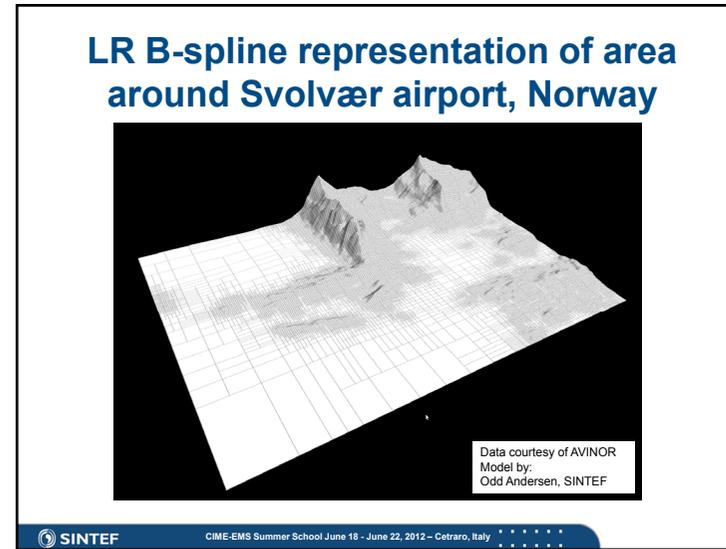
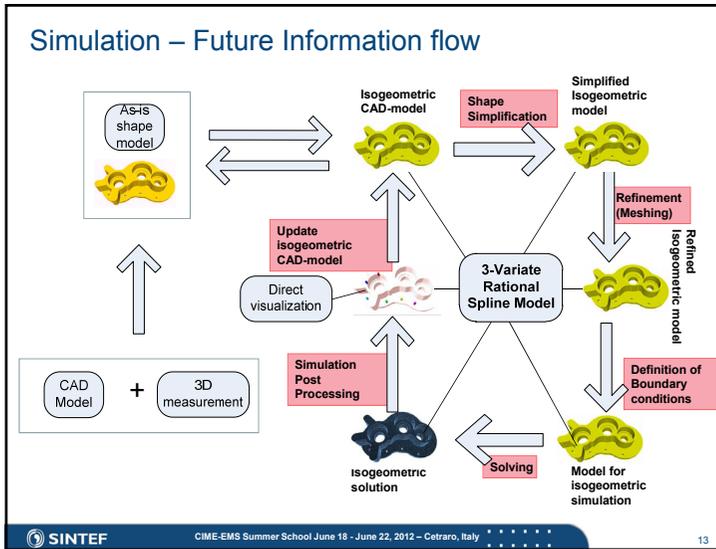
- Splines are polynomial, same as Finite Elements
- B-Splines are very stable numerically
- B-splines represent regular piecewise polynomial structure in a more compact way than Finite Elements
- NonUniform rational B-splines can represent elementary curves and surfaces exactly. (circle, ellipse, cylinder, cone...)
- Efficient and stable methods exist for refining the piecewise polynomials represented by splines
 - Knot insertion (Boehms Algorithm, and Oslo Algorithm, 1980)
 - B-spline has a rich set of refinement methods

Why have NURBS not been used in FEA?

- Shape representation in CAD and FEA
 - Finite Elements are 3-variate polynomials most often of degree 1 or 2 and cannot represent many shapes accurately.
 - NURBS (NonUniform Rational B-Spline) surfaces used in CAD can represent elementary curves and surfaces exactly. (circle, ellipse, cylinder, cone...) in addition to free form shapes.
 - Mathematical representation in CAD and FEA chosen based on what was computationally feasible in the early days of the technology development.
- We needed someone with high standing in FEA to promote the idea of splines in analysis
 - Prof. Tom Hughes, University of Texas at Austin, did this in 2005
 - This has triggered a new drive in spline research after a quiet period.

Traditional simulation pipeline





Challenge 1: Create “as-is” model

- CAD-models describes the object as planned
 - Combines elementary surfaces (plane, cylinder, cone, sphere, torus and NURBS)
- Models aimed at visual purpose most often represent shape by (texture mapped) triangulations
- Laser scanning efficiently produce millions of points on the geometry
 - Extracting information from 3D datasets is complex
 - A industry is established related to model building from laser scans
 - Using the datasets for validation and updating of 3D models (CAD) is challenging
- The projects following projects address these issue partly using LR B-splines:
 - “3D Airports for Remotely Operated Towers” in SESAR JU (EU)
 - The STREP TERRIFIC – EU ICT-program Factory of the Future
 - New IP- IQmulus on large GIS data sets (currently under contract negotiations) (October 2012- September 2016)

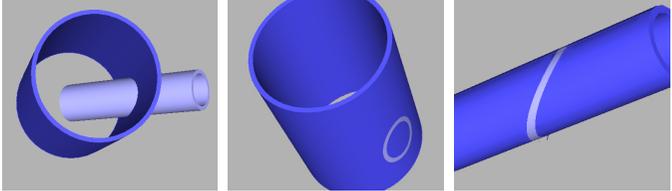
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Challenge 2: Create 3-variant isogeometric model

- The “As-is” shape model describes mathematically only the inner and outer hulls (surfaces) of the object using triangulations, elementary surfaces or NURBS surfaces.
- The isogeometric model is analysis/simulation suitable and describes the volumes mathematically by watertight structures of blocks of 3-variant rational splines
- Building an isogeometric model is a challenge:
 - There is a mismatch between the surface patch structure of the “As-is” model, and a suited block structure of an Isogeometric 3-variant rational spline model.
 - Augmented spline technology is needed such as the novel Locally Refined Splines.
- Projects addressing this
 - Isogeometry (2008-2012)- KMB project funded by the Norwegian Research council
 - TERRIFIC (2011-2014) – STREP funded by the EU ICT

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Example: Isogeometric tube joint - Intersection

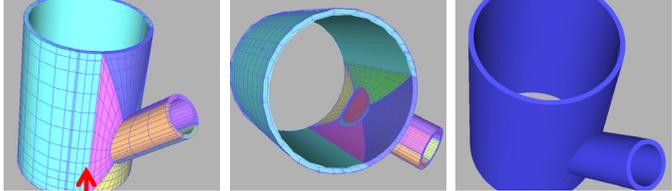


- Two independent pipes coming from CAD and described as 3-variate volumes
- The intersection of the pipes calculated.
- The original large pipe is split in 3 volumes
- The intersection of the pipes calculated.
- The original small pipe is split in 3 volumes

Example by: Vibeke Skytt, SINTEF ICT

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Example: Isogeometric tube joint – match spline spaces(B-splines)



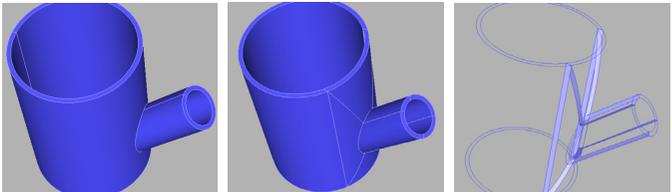
- Spline space refined to have matching lines in each hexahedral NURBS-block to produce a watertight representation
- Same as to the left, different view
- The final isogeometric tube joint.

Matching spline spaces of adjacent tensor product B-splines volumes drastically increases the number of vertices. (Local refinement needed)

Example by: Vibeke Skytt, SINTEF ICT

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Example: Isogeometric tube joint – Composing volumes



- The relations between the sub volumes produced by the intersection are established
- These volumes do not satisfy the hexahedral (box structure) of the need isogeometric sub volumes
- The volumes split to produce hexahedral volumes
- The internal faces produced by the splitting process

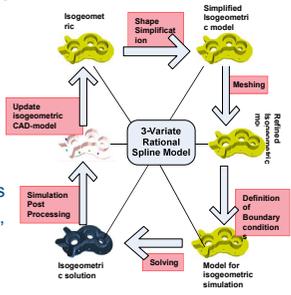
Example by: Vibeke Skytt, SINTEF ICT

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Challenge 3: Isogeometric analysis

First introduced in 2005 by T.J.R. Hughes

- Replace traditional Finite Elements by NURBS - NonUniform Rational B-splines
- Accurate representation of shape
- Allows higher order methods
- Perform much better than traditional Finite Elements on benchmarks
- Refinement of analysis models without remeshing
- Exact coupling of stationary and rotating grids
- Augmented spline technology is needed, e.g., Locally Refined Splines



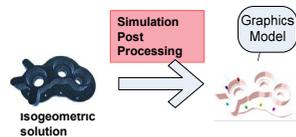
Projects:

- ICADA (2009-2014)– KMB Project funded by Norwegian Research Council and Statoil
- Exciting (2008-2012)
- TERRIFIC (2011-2014) – STREP EU ICT-program

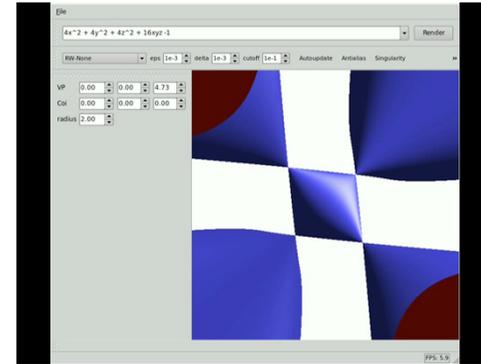
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Challenge 4: Isogeometric visualization

- Traditional visualization technology is triangle based (tessellation)
 - The isogeometric model has to be approximated with triangles for visualization
 - Results are degraded and information lost
- Need for visualization solutions exploiting the higher order representations
 - Higher order representations are more advanced and can better represent singularities in the solution
 - Create view dependent tessellation of splines on the GPU
 - Direct ray tracing on the GPU:
- Project: Cloudviz (2011-2014)



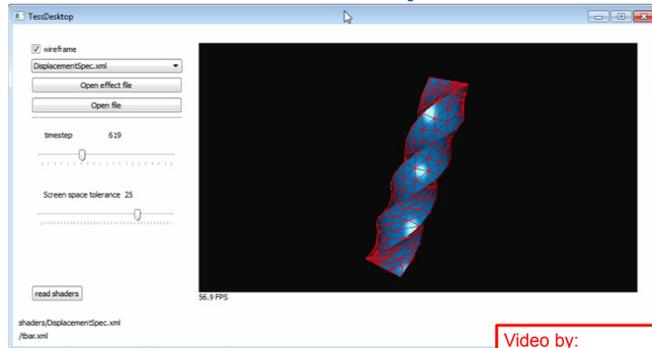
Direct ray-casting on the GPU



Video by:
Johan S. Seland,
SINTEF ICT

Click her for video: <http://www.youtube.com/watch?v=ORFhU3diakA>

Isogeometric view dependent tessellation of the spline model

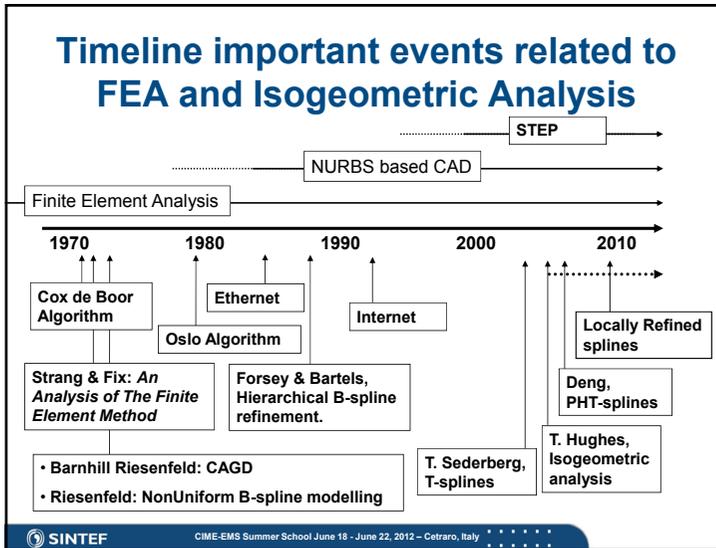


Video by:
Jon Hjelmervik,
SINTEF ICT

Click her for video:
http://www.youtube.com/watch?v=K0sDBx8yEt0&list=UU_GWvrs307zpjUWvQxWwHA

From stand alone computers and systems to integrated information flows

- As long as communication between computers was hard, information exchange remained paper based
 - The Ethernet invented by Xerox Parc in 1973-1975,
 - ISO/IEEE 802/3 standard in 1984
 - Deployment in industry started, simple communication between computers
- CAD Data Exchange introduced
 - IGES - *Initial Graphics Exchange Specification* Version 1.0 in 1980
 - STEP - *Standard for the Exchange of Product model data* started in 1984 as a successor of IGES, SET and VDA-FS, Initial Release in 1994/1995, deployment in industry started
- The Internet opened to all 1991
 - Start of deployment of data exchange between processes over the Internet



CAD patch structure not an obvious guide to isogeometric block structure

- We would like considerably fewer NURBS blocks than the number of surfaces patches in the CAD-model
- The object has three main parts
 - The “torus” like part
 - The “cylindrical” handle
 - The transition between these
- Not obvious how this can be represented as a composition of NURBS blocks
 - Acute angles
 - Extraordinary points
 - Singular points

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CAD has to change to support isogeometric analysis

- Example: Patch structure of a fairly simple CAD-object
 - Object designed patch by patch to match the desired shape
 - Shape designed for production

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Current CAD technology is here to stay

- The major part of revenue of CAD vendors comes from industries that don't suffer from the CAD to analysis bottleneck. However, advanced industry suffers.
- Current CAD is standardized in ISO 10303 STEP.
- The driving force for isogeometric CAD has to be industries that has the most to gain from the novel approach, e.g.,
 - aeronautics, defense, space and automotive industries
- Isogeometric CAD: A next natural step in CAD evolution?
- ISO 10303 STEP should also include isogeometric CAD

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Two approaches to isogeometric CAD

1. Build the block structure one block at the time
 - User responsible for block interfaces and interfaces to outer and inner hulls.
 - Similar to surface modeling without trimming
 - Can be template based
2. Design the tri-variate block structure in an already existing ISO 10303 STEP type CAD model
 - The user controls the block structure. The blocks snap together and to outer and inner hulls.
 - Similar to designing surfaces into a point cloud in reverse engineering
 - Last week (June 2012) my colleague Vibeke Skytt participated in an ISO meeting in Stockholm to find out how ISO 10303 STEP can better support the idea of isogeometric analysis.



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Some spline history

Based on Tom Lyches talk at the SIAM conference on Applied Algebraic Geometry in Raleigh, South Carolina, October 2011



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NURBS lack local refinement

- The regular structure of tensor product NURBS does not allow local refinement
- 1988: Forsey & Bartels: Hierarchical B-spline refinement.
 - $f(s,t) = f_1(s,t) + f_2(s,t) + \dots + f_n(s,t)$
 - The spline space of f_{i+1} is a refinement of the spline space of f_i
- 1998: Rainer Kraft, Adaptive und linear unabhängige multilevel B-splines und ihre Anwendungen. PhD Thesis
- 2003: T. Sederberg, T-splines
 - Compact one level of hierarchical B-splines in the surface control grid. Generalization based on the control grid of B-spline surfaces
- 2006: Deng, PHT-splines
 - C^1 Patchwork of bi-cubic Hermite surface patches allowing T-joints between patches
- 2009: Locally Refined B-splines (LR B-splines), addressing local refinement from the viewpoint of CAGD and Analysis
- 2010 Hierarchical B-splines and IGA – Jüttler, Giannelli, et. al.



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Applications of splines today

- signal and image processing,
- animation,
- computational geometry,
- computer-aided design,
- computer-aided manufacturing,
- computer graphics,
- control theory,
- geometric design
- image analysis,
- medical visualization,
- optimization,
- partial differential equations,
- robotics,
- statistics.
- isogeometric analysis.
-



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Signal and image processing

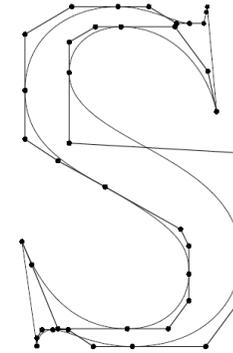
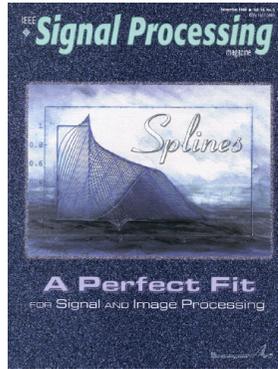


Figure: The letter S in the Postscript font Times Roman.

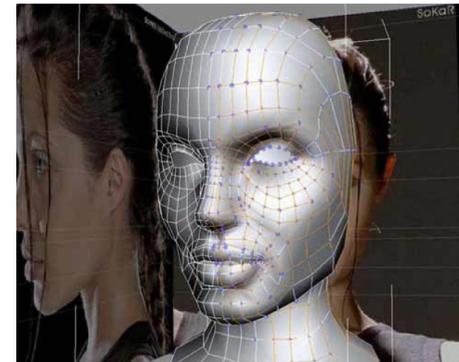
Parameter estimation

Parameter estimation in systems biology models using spline approximation

Choujun Zhan  and Lam F Yeung 
 Department of Electronic Engineering, City University of Hong Kong, Hong Kong, PR China

BMC Systems Biology 2011, **5**:14

Many faces of splines



Isaac Jacob Schoenberg, 1903-1990, the father of splines



Schoenberg, I. J., Contributions to the problem of approximation of equidistant data by analytic functions, Quart. Appl. Math 4(1946), 45-99 and 112-141.



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The word spline



Figure: A physical spline with ducks.

- Physical spline: a flexible beam with minimal potential energy.
- Kept in place using "ducks".
- Nonlinear spline: minimal integrated square curvature



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Samuel Karlin wrote

"Schoenberg is noted worldwide for his realization of the importance of spline functions for general mathematical analysis and in approximation theory, their key relevance in numerical procedures for solving differential equations with initial and/or boundary conditions, and their role in the solution of a whole host of variational problems. The fundamental papers by Schoenberg [two papers in 1946] form a monument in the history of the subject, as well as its inauguration."



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What is a spline

- It is not a spline unless it minimizes something (for a long time the French view)
- Variational approach.
- A spline is a linear combination of B-splines, Carl de Boor.
- Any piecewise polynomial is a spline, Larry Schumaker.
- Constructive approach.



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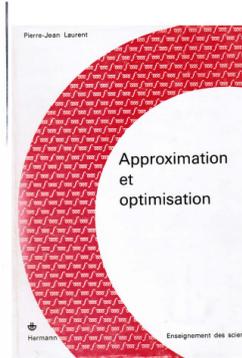
The variational approach

The Theory of Splines
and Their Applications

J. H. AHLBERG
UNIVERSITY OF CALIFORNIA, BERKELEY
E. N. NILSON
PURE & APPLIED MATHEMATICS
J. L. WALSH
UNIVERSITY OF MICHIGAN



Ahlberg Nilson, Walsh, 1967



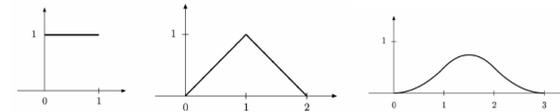
P-J Laurent, 1972



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Cardinal B-Splines - Uniform knots

$$B[0, 1, \dots, p+1] = B[0, 1, \dots, p] * B[0, 1], \quad p = 1, 2, 3, \dots$$



$p = 0,$

$p = 1,$

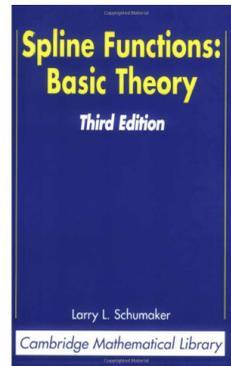
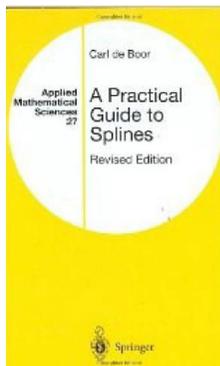
$p = 2$

- ▶ Studied in a monograph by Schoenberg
- ▶ Computed by repeated averaging (Lane Riesenfeld)



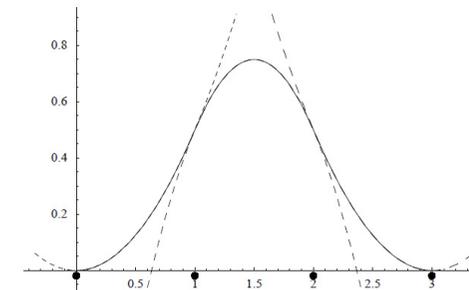
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The constructive approach



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A B-spline is a Piecewise Polynomial



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The first CAGD (Computer Aided Geometric Design) meeting Salt Lake City 1974



- R.F. Riesenfeld, Applications of B-spline Approximation to Geometric Problems of Computer-Aided Design, Ph.D. Thesis, Syracuse University, 1973.
- K.J. Versprille, Computer-Aided Design Applications of the Rational B-splines, Ph.D. Thesis, Syracuse University 1975.

de Boor algorithm

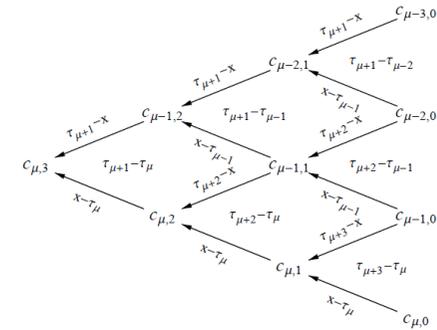
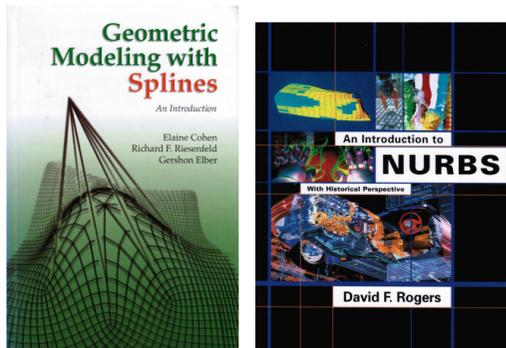


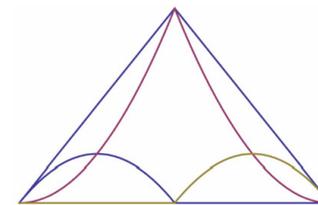
Figure: A triangular algorithm for computing the value of a cubic spline with B-spline coefficients c at $x \in [\tau_{\mu}, \tau_{\mu+1}]$.

Introduction to NURBS



Knot insertion algorithms (h-refinement)

- Boehm's algorithm, 1980. One knot at a time.
- The Oslo algorithm, 1980. All knots simultaneously.
- Related to discrete B-splines and polar forms (blossoming).
- Can also do degree raising (p-refinement), Cohen, L, Schumaker 1985.



Knot insertion

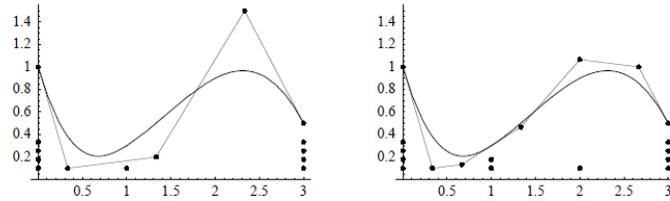


Figure: A cubic spline with one interior knot (a). In (b) the same spline is represented with two extra knots (the knot at $x = 1$ is now double).

Local refinement essential for efficient
Isogeometric Analysis,

Second lecture

Splines spaces over box partitions

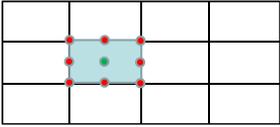
Block structured grids, local refinement and spline bases

Tor Dokken



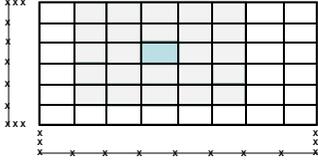
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Bi-quadratic example



Bi-quadratic finite element

- Shape functions local to each element
- C^0 - continuity by matching edge nodes
- C^1 - continuity imposes many relations between nodes



Bi-quadratic B-spline

- Shape functions (B-splines) not local to a single element
- Continuity embedded in the B-spline definition through the knot vectors
- C^1 - continuity straight forward



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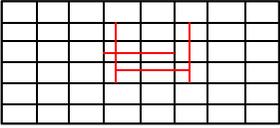
Isogeometric Analysis challenge traditional FEA approaches to block structured grids.

- Block structured FEA
 - Composed of coupled blocks of meshes with a regular structure
- Spline function represented by tensor product B-splines
 - The B-splines space defined by univariate knot vectors t_1, t_2 one in each parameter direction and the polynomial degrees p_1, p_2 . A regular structure of not lines
 - Continuity defined by multiplicity of values
 - A tensor product B-spline not local to an element
- Higher order continuity a challenge when blocks meet in an extraordinary point
 - Simple when four surfaces meet
 - Challenge independent of T-splines, LR B-splines....



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Local refinement of a block



- We want to insert new local line segment in the grid that result in new elements (and degrees of freedom) located where they are needed.
- Questions:
 - What is the dimension of the resulting spline space?
 - How to find a suitable basis?
 - Which restrictions has to be imposed on the refinement?
- The questions are independent of the approach is T-splines Locally Refined B-splines, PHT-splines or Hierarchical B-splines.

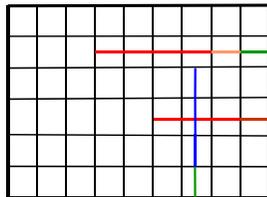


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Dimension increase for local refinement in the 2-variate case well understood.

Example dimension increase of spline space when

- Bi-cubic $p = 3$.
- C^2 -continuity across interior mesh lines



Mesh line segment length 1 starting at the boundary, T-joint at other end. Dimension increase 1.

Mesh line segment length 1 extending existing mesh line segment of length 4 or more, T-joint at other end. Dimension increase 1.

Interior mesh line segment length 4, T-joints at both ends. Dimension increase 1.

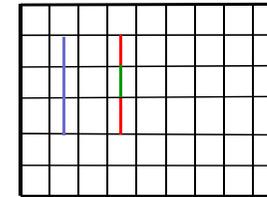
Mesh line segment length 1 gap filling. Dimension increase 4.

Mesh line segment length 1 extension of existing mesh line segment to the boundary. Dimension increase 4



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Short lines do not give dimension increase bi-cubic example



Interior mesh line segment length 3 (or shorter), T-joints at both ends. No dimension increase.

Interior mesh line segment length 3 (or shorter), T-joints at both ends. No dimension increase.

Gap filling mesh line segment length 1, resulting mesh line segment less than 4 (or shorter), T-joints at both ends. No dimension increase.

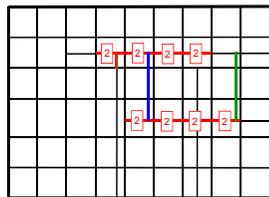
- Mesh line segment ending in T-joints has to have length $(p + 1)$ to give dimension increase by 1.



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Increasing interior multiplicity in the bi-cubic case

- When multiplicity is 2 we have C^1 -continuity across interior mesh lines in the bi-cubic case.
- The edges on the boundary have multiplicity 4.



Interior mesh line segment length 4, increase multiplicity to 2, lower multiplicity at both ends, dimension increase 1.

Interior mesh line segment length 3, ending in T-joints with orthogonal mesh line segment, one with multiplicity 1, and one with multiplicity 2, dimension increase 1.

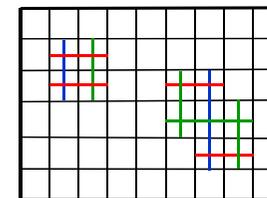
Interior mesh line segment length 3, ending in T-joints with orthogonal mesh line segment of multiplicity 2, dimension increase 2.

Extend existing mesh line segment by length 1, ending in T-joint with orthogonal mesh line segment with multiplicity 2, dimension increase 2,



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Using short lines to localize refinement bi-cubic example



Interior mesh line segment length 2, T-joints at both ends. No dimension increase.

Interior mesh line segment length 3, T-joints at both ends. No dimension increase.

Interior mesh line segment length 4, T-joints at both ends. Dimension increase 1.

- Note: we are only addressing refinement of the spline space. Currently no relation to T-spline, LR B-splines, PHT-splines, Hierarchical B-splines etc.



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Bernard Mourrain: *On the dimension of spline spaces on planar T-subdivisions*

- A T-subdivision is a 2-variate specialization of what we will denote an μ -extended box-mesh.
- Dimension of space of bivariate functions, piecewise polynomial of bidegree (p_1, p_2) and class C^{r_1, r_2} over a planar T-subdivision:

$$\dim \mathcal{S}_{p_1, p_2}^{r_1, r_2} = (p_1 + 1)(p_2 + 1)f_2 - (p_1 + 1)(r_2 + 1)f_1^v - (p_2 + 1)(r_1 + 1)f_1^h + (r_1 + 1)(r_2 + 1)f_0 + h_{p_1, p_2}^{r_1, r_2}$$
- f_2, f_1^v, f_1^h, f_0 are respectively the number of 2-faces(elements), horizontal interior edges, vertical interior edges and interior vertices
- $h_{p_1, p_2}^{r_1, r_2}$ is a homology term that is zero provided all sequences of interior edges that are not attached to the boundary respectively consists of at least p_1 or p_2 consecutive edges.

Find a piecewise polynomial basis for the locally refined block mesh (1)

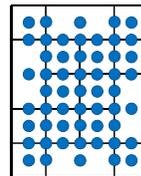
- Using Mourrain's dimension formula when it applicable or the more general formula to be presented soon enables in the cases where the homology terms are zero to calculate the spline space over the extended box-mesh.
- We now want to find a basis that span this box splines.
- Some candidate approaches for bases:
 - Hierarchical B-splines
 - PHT-splines
 - T-splines
 - LR B-splines

Dimension formula, example

- Dimension of T-mesh space, $(p_1 = p_2 = 3, r_1 = r_2 = 2)$:

$$\dim \mathcal{S}_{p_1, p_2}^{r_1, r_2} = (p_1 + 1)(p_2 + 1)f_2 - (p_1 + 1)(r_2 + 1)f_1^v - (p_2 + 1)(r_1 + 1)f_1^h + (r_1 + 1)(r_2 + 1)f_0$$

- f_2 = number of 2-faces
- f_1^h = number of interior horizontal edges
- f_1^v = number of interior vertical edges
- f_0 = number of interior vertices



$f_0 = 12$

$\dim \mathcal{S}_{3,3}^{2,2} = 4 \times 4 \times 16 - 4 \times 3 \times 13 - 4 \times 3 \times 14 + 3 \times 3 \times 12 = 40$

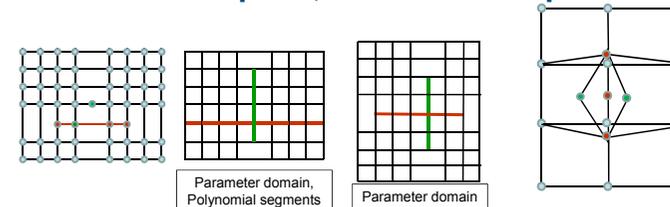
Find a piecewise polynomial basis for the locally refined block mesh (2)

- **Hierarchical B-splines** have nested levels of refinement, with a nested selection of regions to use at each level.
 - The spline spaced by the extended box-mesh produced by merging the trimmed knotlines of all levels (the element structure) is not spanned by the Hierarchical B-spline basis.
- **PHT-splines** are special basis produced for specific degrees and continuities for very general box-meshes.
 - There exists, e.g., a $C^{1,1}$ bi-cubic PHT-spline basis.
 - PHT-splines do not, as far as I know, offer a general solution.
- Hierarchical B-splines do not offer a basis than spans the spline space of the extended box-mesh.

Find a piecewise polynomial basis for the locally refined block mesh (3)

- **T-splines**
 - Refinement defined in the T-spline vertex T-grid following the T-spline rules, recently new extension.
 - Box-mesh (Bezier surfaces) inferred from T-grid
 - Rational scaling of B-splines to form partition of unity
 - **LR B-splines**
 - Refinement directly on B-splines in box-mesh, mesh-lines projected on to the surface provides a T-grid structure
 - More choices of refinements than T-splines
 - Scaled B-splines to form partition of unity
- Same challenges of both methods to ensure:
 - Spanning of spline space defined by the Box-mesh
 - Linear independence of the B-splines

The refinement approaches of T-spline and LR-spline, bi-cubic example

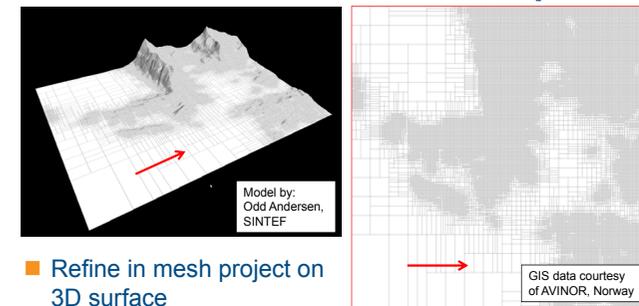


- Adding two close vertices by T-spline refinement creates 11 new polynomial segments and 5 vertices (original T-spline rules)
- Adding a minimal "+" structure by LR B-splines creates 2 vertices and 8 new polynomial segments
- Position of vertices in parameter domain average of internal knots

Direct modeling by vertices

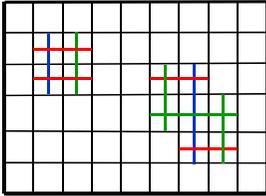
- Both T-splines and LR B-splines have bases that are a partition of unity.
 - Consequently the coefficient/vertices have a geometric interpretation, and the surfaces can be directly manipulated by these.
- The refinement allowed for T-splines is restricted by the requirement that the vertex grid of T-splines, T-joint, recently extensions by L-joints and I-joints.
 - No such structure is imposed on the coefficients of the general LR B-splines. The only restriction, at least one B-spline has to be refined.
 - Direct modeling by vertices is difficult, so in CAD the location of vertices is most often calculated by algorithms.

Local refinement for LR B-splines



- Refine in mesh project on 3D surface
- Refine in 2D mesh in parameter domain
- Automatic checking:
 - Spline space filled?
 - LR B-spline basis exists?
 - Automatic corrections possible

Support of short lines to localize refinement



- In the $C^{1,1}$ bi-cubic case possibly supported by PHT-splines.
- Currently not supported by T-splines or LR B-splines
 - Challenge: How to extend LR B-spline theory to allow minimal support B-splines in the mesh that are not a result of B-spline refinement.

Summary

Method:	Use B-splines	Fill the spline space of the mesh	Guaranteed basis	Geometric approach to refinement
PHT-splines	No	For specific continuities	For specific continuities	In parameter domain (mapped on surface)
Hierarchical B-splines	Yes	No	Yes	Nested regions
T-splines	Yes	Often	If analysis suitable	In vertex T-grid
LR B-splines	Yes	Often	Tools exist ensure that a basis is made	In parameter domain (mapped on surface)

The theory of LR B-splines applicable to T-splines

Lecture 3

Box-partitions and dimension of spline spaces over Box-partition

Definition of LR B-splines
some geometric properties

Tor Dokken

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A box in \mathbb{R}^d

Given an integer $d \geq 0$. A box in \mathbb{R}^d is a Cartesian product

$$\beta = J_1 \times \dots \times J_d \subseteq \mathbb{R}^d,$$

where each $J_k = [a_k, b_k]$ with $a_k \leq b_k$ is a closed finite interval in \mathbb{R}^d . We also write $\beta = [\mathbf{a}, \mathbf{b}]$, where $\mathbf{a} = [a_1, \dots, a_d]$, and $\mathbf{b} = [b_1, \dots, b_d]$.

- The interval J_k is said to be **trivial** if $a_k = b_k$ and **non-trivial** otherwise.
- The **dimension** of β , denoted $\dim \beta$, is the number of non-trivial intervals J_k in β .
 - We call β an l -box or and (l, d) -box if $\dim \beta = l$.

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Box-partitions

- Box-partitions - Rectangular subdivision of regular domain d -box \mathbb{R}^d

$$\Omega \subseteq \mathbb{R}^d$$

$$\Omega = [a_1, b_1] \times \dots \times [a_d, b_d]$$

$$a_i < b_i, 1 \leq i \leq d$$

- Subdivision of Ω into smaller d -boxes

$$\mathcal{E} = \{\beta_1, \dots, \beta_n\}$$

$$\beta_1 \cup \beta_2 \cup \dots \cup \beta_n = \Omega$$

$$\beta_i \cap \beta_j = \emptyset, i \neq j$$

$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$

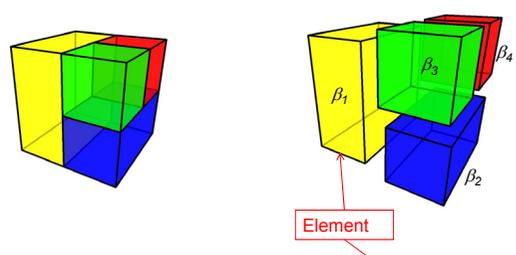
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Important boxes

- If $\dim \beta = d$ then β is called an **element**.
- If $\dim \beta = d - 1$ there exists exactly one k such that $J_k = [a]$ is trivial. Then β is called a **mesh-rectangle**, a k -**mesh-rectangle** or a (k, a) -**mesh-rectangle**

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Example: 3D-mesh



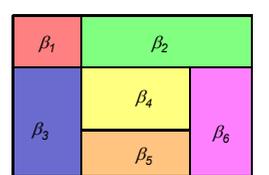
$\Omega \subseteq \mathbb{R}^3$
 $\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$

Illustration by: Kjell Fredrik Pettersen, SINTEF


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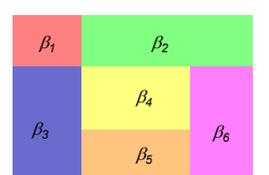

Example: 2D-mesh

dim = 2 (Elements)



$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$

dim = 2 (Elements)



$\mathcal{F}_2(\mathcal{E}) = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\} = \mathcal{E}$

Illustration by: Kjell Fredrik Pettersen, SINTEF

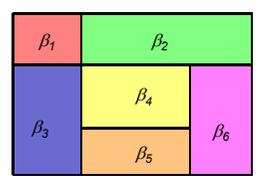

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Lower-dimensional boxes in the mesh

- $\mathcal{F}_l(\mathcal{E})$ is the set of l -boxes describing the mesh topology, $0 \leq l \leq d$
- $\mathcal{F}_d(\mathcal{E})$ is the same as \mathcal{E}
- For $l < d$: $\mathcal{F}_l(\mathcal{E})$ is the set of l -boxes where higher-dimensional boxes in \mathcal{E} intersect, or at boundary of Ω

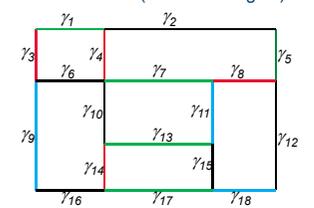

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Example: 2D-mesh



$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$

dim = 1 (Mesh-rectangles)

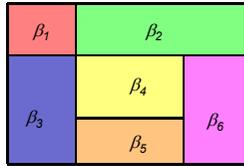


$\mathcal{F}_1(\mathcal{E}) = \{\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_{18}\}$

Illustration by: Kjell Fredrik Pettersen, SINTEF


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Example: 2D-mesh



$$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$$

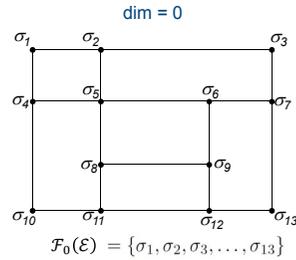


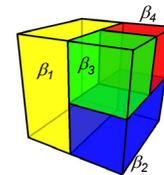
Illustration by: Kjell Fredrik Pettersen, SINTEF



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Example: 3D-mesh

dim = 2 (Mesh-rectangles)



$$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$$

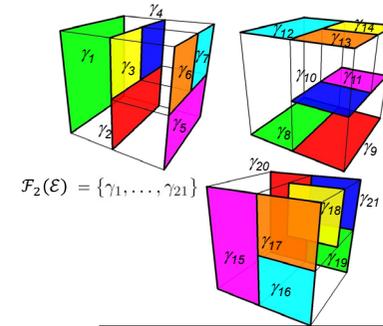


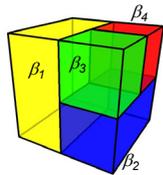
Illustration by: Kjell Fredrik Pettersen, SINTEF

11

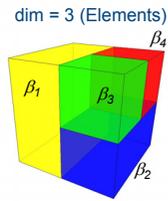


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Example: 3D-mesh



$$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$$



$$\mathcal{F}_3(\mathcal{E}) = \{\beta_1, \beta_2, \beta_3, \beta_4\} = \mathcal{E}$$

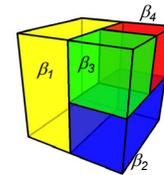
Illustration by: Kjell Fredrik Pettersen, SINTEF



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Example: 3D-mesh

dim = 1



$$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$$

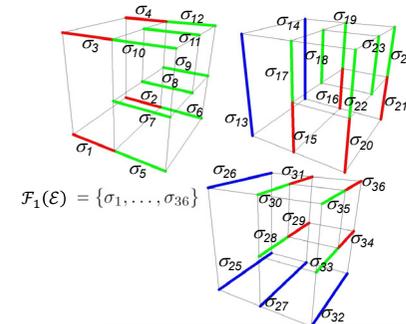
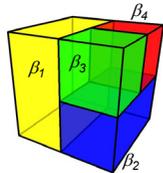


Illustration by: Kjell Fredrik Pettersen, SINTEF

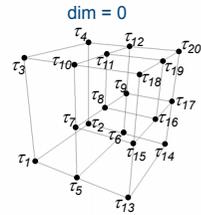


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Example: 3D-mesh



$$\mathcal{E} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$$

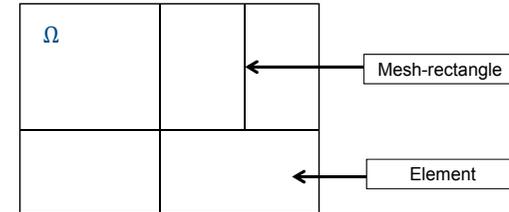


$$\mathcal{F}_0(\mathcal{E}) = \{\tau_1, \dots, \tau_{20}\}$$

Illustration by: Kjell Fredrik Pettersen, SINTEF

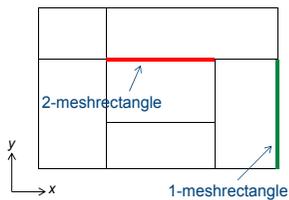
Box-partition

- $\Omega \subseteq \mathbb{R}^d$ a d -box in \mathbb{R}^d .
- A finite collection \mathcal{E} of d -boxes in \mathbb{R}^d is said to be a **box partition** of Ω if
 1. $\beta_1^o \cap \beta_2^o = \emptyset$ for any $\beta_1^o, \beta_2^o \in \mathcal{E}$, where $\beta_1^o \neq \beta_2^o$.
 2. $\cup_{\beta \in \mathcal{E}} \beta = \Omega$.



Mesh-rectangles

Example. 2D



Example. 3D

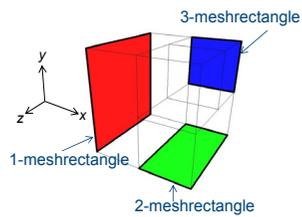
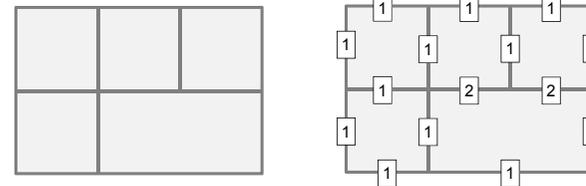


Illustration by: Kjell Fredrik Pettersen, SINTEF

μ -extended box-mesh (adding multiplicities)



- A multiplicity μ is assigned to each mesh-rectangle
- Supports variable knot multiplicity for Locally Refined B-splines, and local lower order continuity across mesh-rectangles.

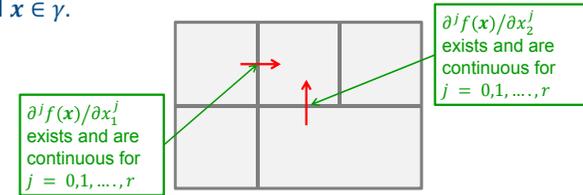
Comment

- When our work started out we used the index grid for knots.
 - However, this posed challenges with respect to mesh-rectangles of multiplicity higher than one and the uniqueness of refinement.
 - The generalized dimension formula uses multiplicity, thus to ensure
 - Not straight forward to understand what happens with spline space dimensionality when two knotline segments converge and obtain the same knot value.
- To make a consistent theory we discarded the index grid.

Continuity across mesh-rectangles

Given a function $f: [a, b] \rightarrow \mathbb{R}$, and let $\gamma \in \mathcal{F}_{d-1,k}(\mathcal{E})$ be any k -mesh-rectangle in $[a, b]$ for some $1 \leq k \leq d$.

We say that $f \in C^r(\gamma)$ if the partial derivatives $\partial^j f(x)/\partial x_k^j$ exists and are continuous for $j = 0, 1, \dots, r$ and all $x \in \gamma$.

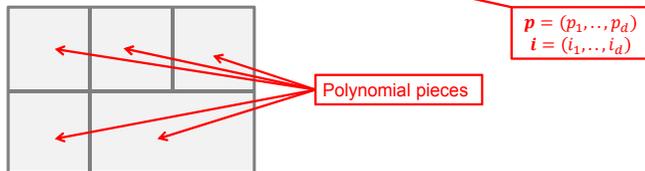


Polynomials of component degree

On each element the spline is a polynomial.

We define polynomials of component degree at most $p_k, k = 1, \dots, d$ by:

$$\Pi_p^d = \left\{ f: \mathbb{R}^d \rightarrow \mathbb{R}: f(x) = \sum_{0 \leq i \leq p} c_i x^i, c_i \text{ in } \mathbb{R} \text{ for all } i \right\}$$

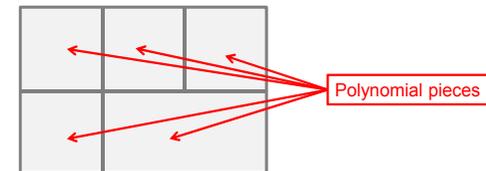


Piecewise polynomial space

We define the piecewise polynomial space

$$\mathbb{P}_p(\mathcal{E}) = \{f: [a, b] \rightarrow \mathbb{R}: f|_\beta \in \Pi_p^d, \beta \in \tilde{\mathcal{E}}\},$$

where $\tilde{\mathcal{E}}$ is obtained from \mathcal{E} using half-open intervals as for univariate B-splines.



Spline space

We define the spline space

$$S_p(\mathcal{M}, \mu) = \{f \in \Pi_p^d(\mathcal{E}(\mathcal{M})) : f \in C^{p_k - \mu(\gamma)}(\gamma), \forall \gamma \in \mathcal{F}_{d-1,k}(\mathcal{E}(\mathcal{M})), k = 1, \dots, d\}$$

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LR B-splines over μ -extended box-mesh

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How to measure dimensional of spline space of degree p over a μ -extended box partition (\mathcal{M}, μ) .

- Dimension formula developed (Mourrain, Pettersen)

$$\dim S_p(\mathcal{M}, \mu) = \sum_{\ell=0}^d (-1)^{d-\ell} \left(\sum_{\beta \in \mathcal{F}_\ell(\mathcal{M})} \prod_{k=1}^d (p_k - \mu_k(\beta) + 1) \right) - \sum_{q=0}^{d-1} (-1)^{d-q} \dim H_q(\tilde{\mathcal{S}}(\mathcal{N}))$$

Combinatorial values calculated from topological structure

Summing over all ℓ -boxes of all dimensions

Dimension influenced by mesh-rectangle multiplicity

Homology terms

- In the case of 2-variate LR-splines always zero

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Refinement by inserting mesh-rectangles giving a constant split

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Interactively specifying mesh-rectangles – 2-variate case

Idea developed together with Peter Nørtoft Nielsen and Odd Andersen, SINTEF

- A mesh-rectangle is defined by the two the extreme corners
 - $(a, b) = (a, a + v)$
 - Where $v = (v_1, v_2)$, with just one of v_1, v_2 zero
- Given two points p and q as input. Snap p to the nearest mesh-rectangle to create \tilde{p} , remember the class of k -mesh-rectangles snapped to, $k \in \{1,2\}$. Snap q to the nearest parallel k -meshrectangle to create \tilde{q} .
 - Make c where $c_j = \min(\tilde{p}_j, \tilde{q}_j)$
 - Make w where $w_j = |\tilde{p}_j - \tilde{q}_j|$
 - Define $a = (a_1, a_2)$, $a_j = \begin{cases} p_k, j \neq k \\ c_j, j = k \end{cases}$
 - Define $v = (v_1, v_2)$, $v_j = \begin{cases} 0, j \neq k \\ w_j, j = k \end{cases}$
- Provided $v_j > 0$, for $j = k$, we have a well defined mesh-rectangle define by $(a, b) = (a, a + v)$
- When creating mesh-rectangles automatic checks can be run with respect to the increase of dimensionality, and if the resulting B-splines form a basis.

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μ -extended LR-mesh

A μ -extended LR-mesh is a μ -extended box-mesh (\mathcal{M}, μ) where either

- (\mathcal{M}, μ) is a tensor-mesh with knot multiplicities or
- $(\mathcal{M}, \mu) = (\tilde{\mathcal{M}} + \gamma, \tilde{\mu}_\gamma)$ where $(\tilde{\mathcal{M}}, \tilde{\mu})$ is a μ -extended LR-mesh and γ is a constant split of $(\tilde{\mathcal{M}}, \tilde{\mu})$.

All multiplicities not shown are 1.

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Interactively specifying mesh-rectangles

- A mesh-rectangle is defined by two the extreme corners
 - $(a, b) = (a, a + v)$
 - Where $v = (v_1, \dots, v_d)$, with just one $v_k = 0$.
- Given two points p and q as input. Snap p to the nearest mesh-rectangle to create \tilde{p} . Snap q to the nearest parallel meshrectangle to create \tilde{q} . Remember the index r of these parallel r -mesh-rectangles.
 - Make c where $c_j = \min(\tilde{p}_j, \tilde{q}_j)$
 - Make w where $w_j = |\tilde{p}_j - \tilde{q}_j|$
 - Find $k \neq r$ such that $w_k \leq \min_{\substack{0 \leq j \leq d \\ j \neq r}} w_j$
 - Define $a = (a_1, \dots, a_d)$, $a_j = \begin{cases} c_j, j \neq k \\ p_k, j = k \end{cases}$
 - Define $v = (v_1, \dots, v_d)$, $v_j = \begin{cases} w_j, j \neq k \\ 0, j = k \end{cases}$
- Provided $v_j > 0$, for $j \neq k$, we have a well defined mesh-rectangle define by $(a, b) = (a, a + v)$

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LR B-spline

Let (M, μ) be an μ -extended LR-mesh in \mathbb{R}^d . A function $B: \mathbb{R}^d \rightarrow \mathbb{R}$ is called an LR B-spline of degree p on (M, μ) if B is a tensor-product B-spline with minimal support in (M, μ) .

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Splines on a μ -extended LR-mesh

We define as sequence of μ -extended LR-meshes $(\mathcal{M}_1, \mu_1), \dots, (\mathcal{M}_q, \mu_q)$ with corresponding collections of minimal support B-splines $\mathcal{B}_1, \dots, \mathcal{B}_q$.

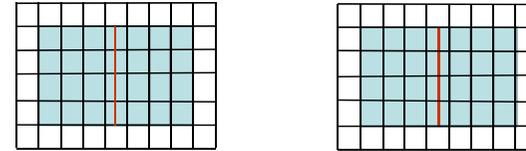
For $j = 1, \dots, q - 1$ creating $(\mathcal{M}_{j+1}, \mu_{j+1}) = (\mathcal{M}_j + \gamma_j, \mu_j, \gamma_j)$ from (\mathcal{M}_j, μ_j) involves inserting a mesh-rectangle γ_j that **increases the number of B-splines**. More specifically:

- γ_j splits (\mathcal{M}_j, μ_j) in a constant split.
- at least on B-spline in \mathcal{B}_j does not have minimal support in $(\mathcal{M}_{j+1}, \mu_{j+1})$.

After inserting γ_j we start a process to generate a collection of minimal support B-splines \mathcal{B}_{j+1} over $(\mathcal{M}_{j+1}, \mu_{j+1})$ from \mathcal{B}_j .

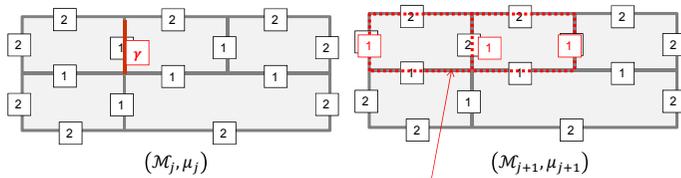
LR B-Spline Refinement step Cubic example: One line

- Insert knot line segments that at least span the width of one basis function



- Four B-splines functions that do not have minimal Support in the refined mesh
- 4 B-splines to be removed
- 5 B-splines to be added
- Dimension increase 1

Going from (\mathcal{M}_j, μ_j) to $(\mathcal{M}_{j+1}, \mu_{j+1})$



B-spline from \mathcal{B}_j that has to be split to generate \mathcal{B}_{j+1}

Refinement of LR B-splines is focused on the tensor product B-splines

Let d be a positive integer, suppose $\mathbf{p} = (p_1, \dots, p_d)$ has nonnegative components (the degrees), and let $\mathbf{y}_k = (y_{k,1}, \dots, y_{k,p_k+2})$ be a nondecreasing (knot) sequence $k = 1, \dots, d$. We define a tensor product B-spline $B[\mathbf{y}_1, \dots, \mathbf{y}_d]: \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$B[\mathbf{y}_1, \dots, \mathbf{y}_d](x_1, \dots, x_d) = \prod_{k=1}^d B[\mathbf{y}_k](x_k).$$

Tensor product B-spline.

Product of Univariate B-splines.

Refinement of a tensor product B-spline

- The support of B is given by the cartesian product $\text{supp}(B) := [y_{1,1}, \dots, y_{1,p_k+2}] \times \dots \times [y_{d,1}, \dots, y_{d,p_k+2}]$.
- Suppose we insert a knot z in $(y_{k,1}, \dots, y_{k,p_k+2})$ for some $1 \leq k \leq d$. Then

$$B[Y] = \alpha_1 B[Y_1] + \alpha_2 B[Y_2]$$

- Where Y_1 and Y_2 are the knot vectors of the resulting tensor product B-splines, and

$$\alpha_1 := \begin{cases} 1 & y_{k,p_k+1} \leq z < y_{k,p_k+2} \\ \frac{z - y_{k,1}}{y_{k,p_k+1} - y_{k,1}} & y_{k,1} < z < y_{k,p_k+1} \\ 0 & \text{otherwise} \end{cases}$$

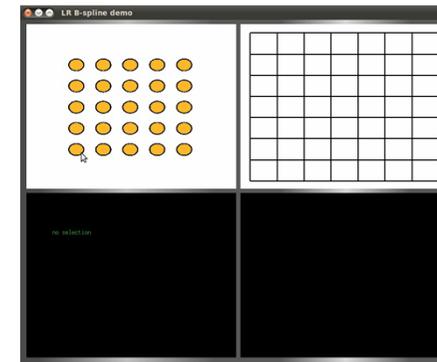
$$\alpha_2 := \begin{cases} 1 & y_{k,1} \leq z \leq y_{k,2} \\ \frac{y_{k,p_k+2} - z}{y_{k,p_k+2} - y_{k,2}} & y_{k,2} < z < y_{k,p_k+2} \\ 0 & \text{otherwise} \end{cases}$$

- α_1 and α_2 calculated by Oslo Algorithm/Boehms algorithm



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Example LR B-spline refinement



Video by PhD fellow Kjetil A. Johannessen, NTNU, Trondheim, Norway.



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LR B-splines and partition of unity

- The LR B-spline refinement starts from a partition of unity tensor product B-spline basis.
- By accumulating the weights α_1 and α_2 as scaling factors for the LR B-splines, partition of unity is maintained throughout the refinement for the scaled collection of tensor product B-splines
- The partition of unity properties gives the coefficients of LR B-splines the same geometric interpretation as B-splines and T-splines.
 - However, the spatial interrelation of the coefficients is more intricate than for T-splines as the refinement strategies are more generic than for T-splines.
 - This is, however, no problem as in general algorithms calculate the coefficients both in FEA and CAD.



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Lecture 4

LR B-splines and linear independence

+ examples

Tor Dokken


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Ensuring linear independence

Spline space spanned by B-splines before refinement
Spline space before refinement

We say that $(\mathcal{M}_{j+1}, \mu_{j+1}, \mathbf{p})$ goes hand-in-hand with $(\mathcal{M}_j, \mu_j, \mathbf{p})$ if

- $\text{span}(B)_{B \in \mathcal{B}_j} = \mathbb{S}_p(\mathcal{M}_j, \mu_j)$ and
- $\text{span}(B)_{B \in \mathcal{B}_{j+1}} = \mathbb{S}_p(\mathcal{M}_{j+1}, \mu_{j+1})$.

Spline space spanned by B-splines after refinement
Spline space after refinement

If $(\mathcal{M}_{j+1}, \mu_{j+1}, \mathbf{p})$ and $(\mathcal{M}_j, \mu_j, \mathbf{p})$ goes hand-in-hand and

$\#\mathcal{B}_{j+1} = \dim \mathbb{S}_p(\mathcal{M}_{j+1}, \mu_{j+1})$

then the B-splines of \mathcal{B}_{j+1} form a basis for $\mathbb{S}_p(\mathcal{M}_{j+1}, \mu_{j+1})$.

A basis if hand-in-hand and the number of B-splines matches the spline space dimension


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Spline space and μ -extended LR-mesh

- We introduced in Lecture 3:
 - The μ -extended box-mesh
 - The dimension formula
 - The μ -extended LR-mesh
 - The LR B-splines
- In Lecture 2 we focused on the importance of:
 - Spanning the spline space over the μ -extended box-mesh
 - Finding a basis for the spline space
- In this Lecture we focus approaches for ensuring that the LR B-splines is a basis for the spline space defined by μ -extended LR-mesh by:
 - Defining a hand in-hand-property between the LR B-splines and the spline space over the μ -extended LR-mesh
 - When the LR B-splines is a basis for the spline space over the μ -extended LR-mesh


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To ensure linear independence we have to

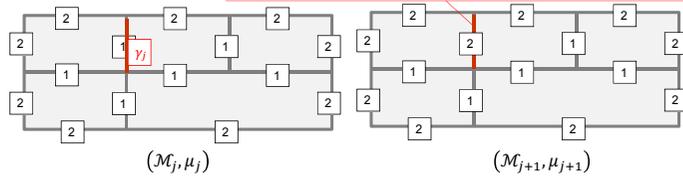
1. Determine $\dim \mathbb{S}_p(\mathcal{M}_{j+1}, \mu_{j+1})$
2. Determine if \mathcal{B}_{j+1} spans $\mathbb{S}_p(\mathcal{M}_{j+1}, \mu_{j+1})$
3. Check that $\#\mathcal{B}_{j+1} = \dim \mathbb{S}_p(\mathcal{M}_{j+1}, \mu_{j+1})$


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Difference in spanning properties between \mathcal{B}_j and \mathcal{B}_{j+1}

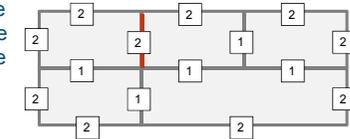
- The only B-splines in \mathcal{B}_{j+1} that model the discontinuity introduced by inserting the mesh-rectangle γ_j are those that have γ_j with multiplicity $\mu(\gamma_j)$ as part of the knot structure.
- By restricting these B-splines to γ_j we get a set of B-splines \mathcal{B}_γ restricted to γ_j with dimension one lower than the dimension of the B-splines of \mathcal{B}_{j+1} .

Pick out the B-splines with multiplicity 2 over γ_j . Intersect with γ_j to select trimmed univariate B-splines \mathcal{B}_γ .



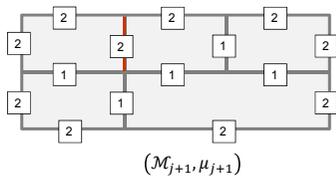
Observations

- To find the dimension of a spline space with many B-splines is more complex than finding the dimension of a spline space with few B-splines
- When assessing the B-splines \mathcal{B}_γ over γ_j we see if the refinement can be broken into a sequence of LR B-spline refinements with as low dimension increase as possible.
 - As a legal LR-spline refinement always introduces at least one B-spline linearly independent from the pre-existing, a dimension increase by just one will ensure that we go hand-in-hand.
 - If the dimension increase is greater than 1 we have to resort to assessing the B-splines \mathcal{B}_γ over γ_j .



The use of \mathcal{B}_γ

- A theorem for general dimensions and degrees states $\dim \text{span} (\mathcal{B}_\gamma)_{B \in \mathcal{B}_\gamma} \leq \dim \mathcal{S}_p(\mathcal{M}_{j+1}, \mu_{j+1}) - \dim \mathcal{S}_p(\mathcal{M}_j, \mu_j)$
- Further it is stated that \mathcal{B}_{j+1} spans $\mathcal{S}_p(\mathcal{M}_{j+1}, \mu_{j+1})$ if $\dim \text{span} (\mathcal{B}_\gamma)_{B \in \mathcal{B}_\gamma} = \dim \mathcal{S}_p(\mathcal{M}_{j+1}, \mu_{j+1}) - \dim \mathcal{S}_p(\mathcal{M}_j, \mu_j)$



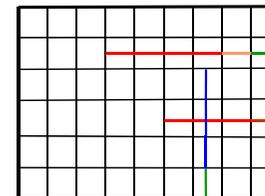
We can find $\mathcal{S}_p(\mathcal{M}_j, \mu_j)$ and $\mathcal{S}_p(\mathcal{M}_{j+1}, \mu_{j+1})$ using the dimension formula provided the homology terms are zero.

Checking the dimension of the space spanned by \mathcal{B}_γ is a constructive tool to check if \mathcal{B}_{j+1} spans the spline space required.

Example: C^2 bi-cubic refinement configurations

Dimension increase of spline space over the box-partition

- All boundary knots mesh-rectangles have multiplicity 4
- All interior mesh-rectangles have multiplicity 1



Mesh-rectangle length 1 starting at the boundary, T-joint at other end. Dimension increase 1. Trivial

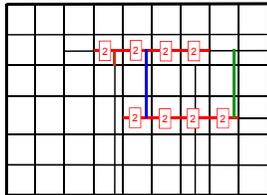
Mesh-rectangle length 1 extending existing mesh-rectangle, T-joint at other end. Dimension increase 1. Trivial.

Interior mesh-rectangle length 4, T-joints at both ends. Dimension increase 1. Trivial.

Mesh-rectangle length 1 gap filling. Dimension increase 4, \mathcal{B}_γ spans a polynomial space, Trivial to check

Mesh-rectangle length 1 extension of existing mesh-rectangle to the boundary. Dimension increase 4, \mathcal{B}_γ spans a polynomial space, Trivial to check.

Increasing interior multiplicity in the bi-cubic case



Interior mesh-rectangle length 4, increase multiplicity to 2, lower multiplicity at both ends, dimension increase 1. Trivial.

Interior mesh-rectangle length 3, ending in T-joints with orthogonal mesh rectangles, one with multiplicity 1, and one with multiplicity 2, dimension increase 1. Trivial.

Extend existing mesh by length 1, ending in T-joint with orthogonal mesh rectangles with multiplicity 2, dimension increase 2, \mathcal{B}_γ spans a polynomial space. Trivial to check.

Interior mesh-rectangle length 3, ending in T-joints with orthogonal mesh rectangles of multiplicity 2, dimension increase 2, two new B-splines. To decide if \mathcal{B}_{j+1} is a basis check if $\dim \text{span} (\mathcal{B}_\gamma)_{\mathcal{B}_\gamma} = 2$.

C^2 bi-cubic refinement configurations Cases: Dimension increase 4

Questions:

1. Do the B-splines in \mathcal{B}_{j+1} span $\mathcal{S}_{(3,3)}(\mathcal{M}_j, \mu_j)$?
2. Is \mathcal{B}_{j+1} a basis for $\mathcal{S}_{(3,3)}(\mathcal{M}_j, \mu_j)$?

If we can answer yes to question 1, and the number of B-splines in \mathcal{B}_{j+1} corresponds to the dimension of $\mathcal{S}_{(3,3)}(\mathcal{M}_j, \mu_j)$ then \mathcal{B}_{j+1} is a basis for $\mathcal{S}_{(3,3)}(\mathcal{M}_j, \mu_j)$.

C^2 bi-cubic refinement configurations Cases: Dimension increase 1

The start point is a bi-cubic tensor product B-spline basis spanning the spline space over a tensor-mesh.

Assume that before the refinement that the B-splines in \mathcal{B}_j are linear independent and span $\mathcal{S}_{(3,3)}(\mathcal{M}_j, \mu_j)$

- Assume that a B-spline in \mathcal{B}_{j+1} has a knotline containing the new mesh rectangle γ_j . This B-spline will be linearly independent from the B-splines in \mathcal{B}_j . Consequently the whole spline space defined over $\mathcal{S}_{(3,3)}(\mathcal{M}_{j+1}, \mu_{j+1})$ is spanned by \mathcal{B}_{j+1} .
- If the number of B-splines in \mathcal{B}_{j+1} corresponds to the dimension of the bi-cubic spline space over \mathcal{M}_{j+1} then the B-splines in \mathcal{B}_{j+1} are linearly independent.

C^2 bi-cubic refinement Cases: Dimension increase 4

- We have to determine if $\dim \text{span } \mathcal{B}_\gamma = 4$ or less than 4.
 - If the B-spline have a structure known for univariate B-splines, trivial to check.
 - If a more complex b-spline configuration, perform knot insertion such that the knot multiplicity at both ends of γ is 4, e.g., convert to a Bernstein basis. Check if the rank of the knot insertion matrix is 4.

Possible to increase dimension without refining B-splines

Dimension increase 1, one new B-splines (+5, -4)
 Dimension increase 1, one new B-splines (+5, -4)
 Dimension increase 1, no new B-splines
 Dimension increase 3, three new B-splines (+ 9, -6)
 • To decide if \mathcal{B}_{j+1} is a basis check if $\dim \text{span} (\mathcal{B}_{\gamma})_{\mathcal{B} \in \mathcal{B}_\gamma} = 3$.

Alternative refinement sequence
 Dimension increase 1, one new B-spline (+5, -4)
 Dimension increase 1, one new B-spline (+2, -1)
 Dimension increase 1, one new B-spline (+2, -1)
 Dimension increase 1, one new B-spline (+5, -4)
 Dimension increase 1, one new B-spline (+5, -4)

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How to determine if the collection of B-splines goes hand in hand with the spline space?

- The study of when two μ -extended meshes go hand-in-hand is simplified by considering the restriction \mathcal{B}_γ of a B-spline \mathcal{B} to a mesh-rectangle γ .
 - In the 2-variate case we can look at the B-splines of \mathcal{B}_{j+1} that have γ with multiplicity $\mu(\gamma)$ as a knotline, and determine the dimension of the univariate spline space spanned by \mathcal{B}_γ
 $\dim \text{span} (\mathcal{B}_\gamma)_{\mathcal{B} \in \mathcal{B}_\gamma}$
 - If $\dim \text{span} (\mathcal{B}_\gamma)_{\mathcal{B} \in \mathcal{B}_\gamma} = \dim \mathbb{S}_p(\mathcal{M}_{j+1}, \mu_{j+1}) - \dim \mathbb{S}_p(\mathcal{M}_j, \mu_j)$ then the spline space go hand in hand

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How to guarantee that \mathcal{B}_{j+1} is a basis for $(\mathcal{M}_{j+1}, \mu_{j+1})$ in the general case?

- Assume that \mathcal{B}_j is a basis for $\mathbb{S}_p(\mathcal{M}_j, \mu_j)$, .
- Make $(\mathcal{M}_{j+1}, \mu_{j+1}) = (\mathcal{M}_j + \gamma_j, \mu_j, \gamma_j)$
- \mathcal{B}_{j+1} is a basis for $\mathbb{S}_p(\mathcal{M}_{j+1}, \mu_{j+1})$ if
 - The B-splines of \mathcal{B}_{j+1} spans $\mathbb{S}_p(\mathcal{M}_j, \mu_j)$ (Goes hand in hand)
 - $\#\mathcal{B}_{j+1} = \dim \mathbb{S}_p(\mathcal{M}_{j+1}, \mu_{j+1})$
 - The number of B-splines corresponds to the dimension of $\mathbb{S}_p(\mathcal{M}_{j+1}, \mu_{j+1})$

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Linear dependency example

Level 1 Level 2 Level 3
Linear independency

■ = lost RMSBF ■ = new RMSBF lin. indep. $\Leftrightarrow (\# \blacksquare) = (\# \blacksquare) + 1$

$p_1 = p_2 = 2$

1 2 3 4 5 6 7 8 9

■ = 3 ■ = 4 → Linear independency

Example by: Kjell Fredrik Pettersen, SINTEF

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

■ = lost RMSBF ■ = new RMSBF lin. indep. \Leftrightarrow (# ■) = (# ■) + 1

$p_1 = p_2 = 2$

■ = 1 ■ = 2 ➔ Linear independency

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

■ = lost RMSBF ■ = new RMSBF lin. indep. \Leftrightarrow (# ■) = (# ■) + 1

$p_1 = p_2 = 2$

■ = 1 ■ = 2 ➔ Linear independency

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

■ = lost RMSBF ■ = new RMSBF lin. indep. \Leftrightarrow (# ■) = (# ■) + 1

$p_1 = p_2 = 2$

■ = 1 ■ = 2 ➔ Linear independency

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

■ = lost RMSBF ■ = new RMSBF lin. indep. \Leftrightarrow (# ■) = (# ■) + 1

$p_1 = p_2 = 2$

■ = 2 ■ = 3 ➔ Linear independency

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

■ = lost RMSBF ■ = new RMSBF lin. indep. \Leftrightarrow (# ■) = (# ■) + 1

$p_1 = p_2 = 2$

■ = 1 ■ = 2 ➔ Linear independency

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

■ = lost RMSBF ■ = new RMSBF lin. indep. \Leftrightarrow (# ■) = (# ■) + 1

$p_1 = p_2 = 2$

■ = 2 ■ = 3 ➔ Linear independency

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

■ = lost RMSBF ■ = new RMSBF lin. indep. \Leftrightarrow (# ■) = (# ■) + 1

$p_1 = p_2 = 2$

■ = 1 ■ = 2 ➔ Linear independency

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

■ = lost RMSBF ■ = new RMSBF lin. indep. \Leftrightarrow (# ■) = (# ■) + 1

$p_1 = p_2 = 2$

■ = 2 ■ = 4 ➔ Linear dependency!!!

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

$p_1 = p_2 = 2$

1 2 3 4 5 6 7 8 9

1 2 3 4 5 6 7

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

$p_1 = p_2 = 2$

1 2 3 4 5 6 7 8 9

1 2 3 4 5 6 7

3567x3456

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

$p_1 = p_2 = 2$

1 2 3 4 5 6 7 8 9

1 2 3 4 5 6 7

2356x2456

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

$p_1 = p_2 = 2$

1 2 3 4 5 6 7 8 9

1 2 3 4 5 6 7

5678x2346

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

$p_1 = p_2 = 2$

2346x1245

7
6
5
4
3
2
1

1 2 3 4 5 6 7 8 9

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

$p_1 = p_2 = 2$

4567x2345

7
6
5
4
3
2
1

1 2 3 4 5 6 7 8 9

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

$p_1 = p_2 = 2$

3456x2345

7
6
5
4
3
2
1

1 2 3 4 5 6 7 8 9

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example

$p_1 = p_2 = 2$

3468x1234

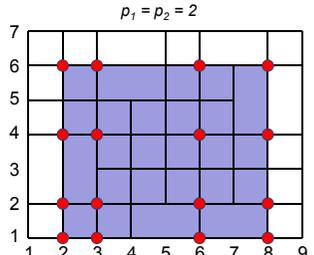
7
6
5
4
3
2
1

1 2 3 4 5 6 7 8 9

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example



2368x1246

$p_1 = p_2 = 2$

1 2 3 4 5 6 7 8 9

1 2 3 4 5 6 7

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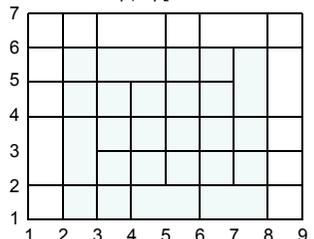
What to do to handle the situation when we produce too many B-splines to have a basis?

- We can eliminate one of the B-splines
 - We may end up with a collection of scaled B-splines that are only a partition of unity, but not a not a positive partition of unit.
 - Discard elimination strategy if the result is not a positive partiton of unity.
- Discard the problematic refinement and perform an alternative refinement close by.
- We perform additional refinements to solve the problem.

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Level 1 Level 2 Level 3
Linear independency

Linear dependency example



Linear relation

(knot value = knot position)

$$\begin{aligned}
 & 108 \cdot (5678)x(2346) \\
 & + 135 \cdot (2356)x(2456) \\
 & + 108 \cdot (3567)x(3456) \\
 & + 268 \cdot (3456)x(2345) \\
 & + 324 \cdot (4567)x(2345) \\
 & + 360 \cdot (2346)x(1245) \\
 & + 384 \cdot (3468)x(1234) \\
 & = 720 \cdot (2368)x(1246)
 \end{aligned}$$

$p_1 = p_2 = 2$

1 2 3 4 5 6 7 8 9

1 2 3 4 5 6 7

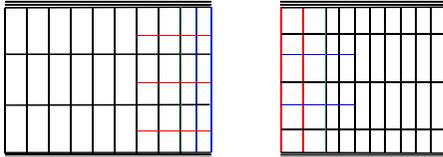
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Some examples of use of LR B-splines

- Stitching of B-spline patches
- Approximation of large data sets

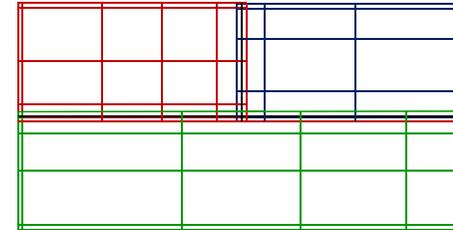
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C¹ Sticking of 2-variate B-splines Bi-quadratic case using LR B-splines

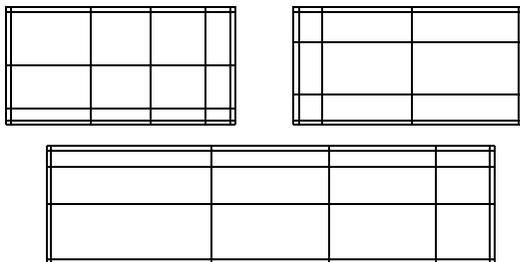


1. Adapt the edge knotlines of A to B
2. Adapt the edge knotlines of B to A
3. Insert horizontal knotline segment from B in A
4. Insert horizontal knotline segment from A in B
5. Merge the parameter domains

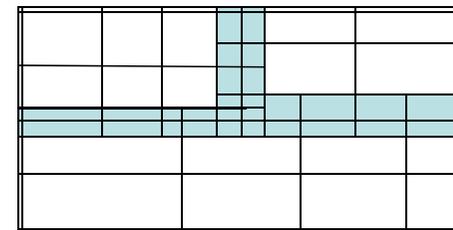
Multi-block T-joints(2) adjust boundary knotlines



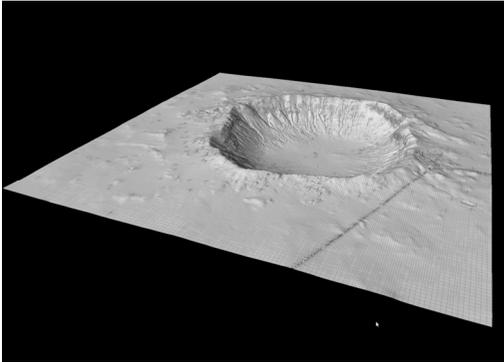
Multi-block T-joints (1) match parametrization



Multi-block T-joints(3) identify + split transition B-splines

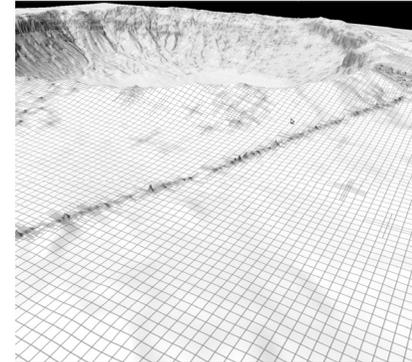


Approximation of large data set Barringer crater Arizona



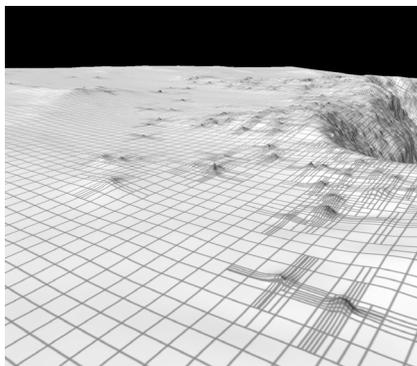
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Data along powerline? reproduced



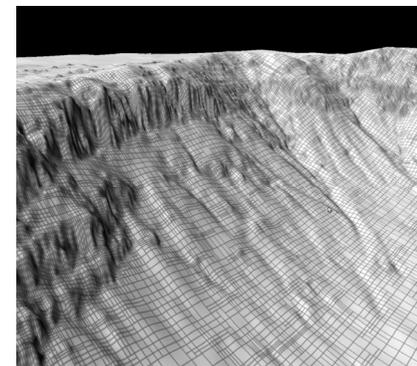
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Local refinement to adapt to fine details



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Details along inside slope



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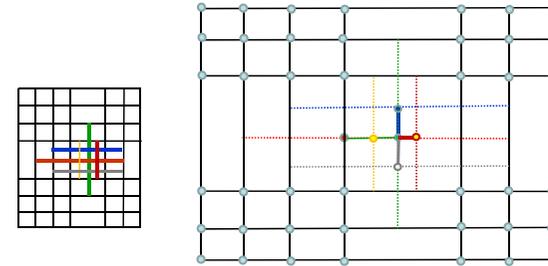
T-spline vertex grid as interface to LR B-splines

- The insertion of a vertex in a T-spline vertex grid (single vertex, T-, L-, L-joint) can be regarded as:
 - A specification of the parameter direction in which to refine
 - The parameter value to be used for the refinement
 - The location of the center of the new B-spline:
 - For odd degrees the location of the middle knotlines of the new B-spline
 - For even degrees the location of the mid-knot interval of the new B-splines
- This information is sufficient for performing refinement directly in the μ -extended box mesh
 - The hand-in-hand principle can be used for check linear independence



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Draft of concept: T-spline type vertex mesh driving LR B-spline refinement in parameter domain



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T-spline vertex grid as interface to LR B-splines - properties

- Andrea Bressan, University of Pavia, has compared T-spline and LR B-spline refinement in the case where exactly $(p + 1) \times (p + 1)$ B-splines overlap the elements of the box partition and found that in most cases the B-splines are the same.
 - Difference observed related to lines with multiplicities
- T-spline compatible LR B-spline can be defined
 - Restriction imposed on which refinement are allowed for LR B-splines
- The vertices and lines in the T-spline T-mesh have all a well defined location in the parameter domain.
 - Projecting the T-mesh/Dual mesh on to the LR-spline surface a T-spline type refinement can be specified directly in the parameter domain of the LR B-spline by specifying the location of the center point of a new B-spline in the mesh.



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Concluding remarks

- T-splines and LR B-splines are two related approaches to Local Refinement of B-splines
- Refinement of LR B-splines can be performed by
 - Insertion of mesh-rectangles
 - T-spline like refinement approaches somewhat restricting the allowed refinements.
- The possibility of using a T-spline like interface also opens up the possibility to replace the T-spline rules for creating B-splines by the LR B-spline approach thus opening up for the use of the LR B-spline results on dimensionality and linear independence.



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Current work on LR B-splines at SINTEF

- LR Splines extensions to the SINTEF GoTools C++ library is under way. EU-project: TERRIFIC.
- We work on efficient computation of stiffness matrices for LR Spline represented IGA on multi-core and many core CPUs
- We work on IGA based on LR B-splines
- We work on efficient LR B-spline visualization on GPUs
- We address representation of geographic information using LR B-splines (New EU-project starting October 1.
- We look at LR B-splines in design optimization. ITN Network SAGA.

The end

- [Click here for video of the isogeometric dancing queen.](#)
- <http://www.youtube.com/watch?v=7LGpiptQ1u4>

Simulation – Future Information flow

