Multiscale Mixed Finite Elements for the Stokes–Brinkman Equations

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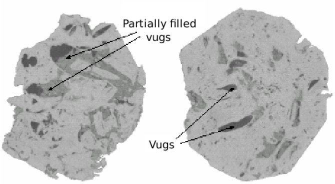
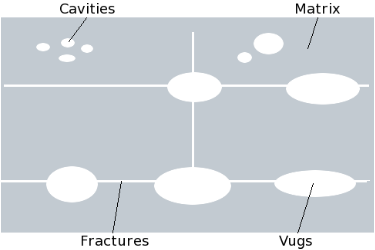
Motivation: Carbonate Reservoirs

Carbonate reservoirs contain:

- 60% of the world's oil reserves
- 40% of the world's gas reserves
- consist of free-flow and porous regions



(Liying Zhang, 2005)



(courtesy of NTNU)

Conventional Approach:

Porous region:
Darcy's law, mass conservation:

$$\begin{aligned} \mu \mathbf{K}^{-1}\vec{u}_D + \nabla p_D &= \vec{f} \quad \text{in } \Omega_D \\ \nabla \cdot \vec{u}_D &= q \quad \text{in } \Omega_D \end{aligned}$$

⇔ Interface conditions ⇔

Free-flow region:
Stokes equations:

$$\begin{aligned} -\mu \Delta \vec{u}_S + \nabla p_S &= \vec{f} \quad \text{in } \Omega_S \\ \nabla \cdot \vec{u}_S &= q \quad \text{in } \Omega_S \end{aligned}$$

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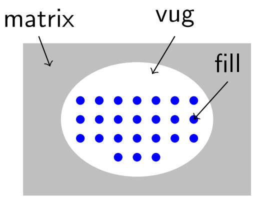

Problems:

- domains not well separated
- difficulties obtaining precise information about location and geometry
- hard to model loose fill-in material

The Stokes–Brinkman model

Introduce a single-parameter family

$$\begin{aligned} \mu \mathbf{K}^{-1}\vec{u} + \nabla p - \tilde{\mu}\Delta\vec{u} &= \vec{f} && \text{in } \Omega \\ \nabla \cdot \vec{u}_S &= q && \text{in } \Omega. \end{aligned}$$

$\tilde{\mu}$ – effective viscosity, μ – fluid viscosity

Special cases:

- $\mathbf{K} \to \infty$, $\tilde{\mu} = \mu \quad \Longrightarrow \quad$ Stokes–Brinkman $\longrightarrow$ Stokes
- $\tilde{\mu} = 0 \quad \Longrightarrow \quad$ Stokes–Brinkman $\longrightarrow$ Darcy

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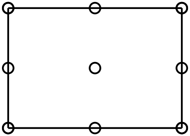
Here: $\tilde{\mu} = \mu$. For typical parameters seen in carbonate reservoirs

$$\nabla p = -\mu \mathbf{K}^{-1}\vec{u} + \tilde{\mu}\Delta\vec{u} \approx -\mu \mathbf{K}^{-1}\vec{u}$$

2-dimensional Taylor-Hood elements:

Pressure ($\mathbb{Q}_1$):

Velocity ($\mathbb{Q}_2$):



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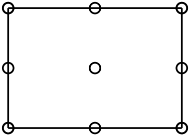

Variational formulation:

Find $\vec{u} \in V$ and $p \in Q$ such that

$$
\begin{aligned}
b(\vec{u},\, \vec{v}) - c(p,\, \vec{v}) &= 0 \qquad && \forall \vec{v} \in V \\
c(\vec{u},\, \pi) &= (q,\, \pi) \qquad && \forall \pi \in Q
\end{aligned}
$$

where $V \subset (H^1(\Omega))^2$ and $Q \subset L^2(\Omega)$

Mixed finite-element system (Stokes-Brinkman)

$$\begin{bmatrix} \boldsymbol{B}_1 & \mathbf{0} & \boldsymbol{C}_1 \\ \mathbf{0} & \boldsymbol{B}_2 & \boldsymbol{C}_2 \\ \boldsymbol{C}_1^\mathsf{T} & \boldsymbol{C}_2^\mathsf{T} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ -\boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{q} \end{bmatrix}$$

The entries in the matrices are:

$$B_{ij,k} = \int_\Omega \mu v_i K_k^{-1} v_j \; d\Omega + \int_\Omega \tilde{\mu} \left(\frac{\partial v_i}{\partial x_1} \frac{\partial v_j}{\partial x_1} + \frac{\partial v_i}{\partial x_2} \frac{\partial v_j}{\partial x_2} \right) \; d\Omega,$$

$$C_{ij,k} = \int_\Omega \frac{\partial v_i}{\partial x_k} \pi_j \; d\Omega.$$

where $k = 1, 2$ denotes the spatial dimension and $\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{K}_2 \end{bmatrix}$.

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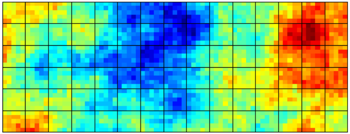
100×100 cells $\Rightarrow$ 91.003 dofs $\Rightarrow$ multiscale multiphysics method

Darcy's law on coarse scale, Stokes–Brinkman on fine scale

Flow-based upscaling:



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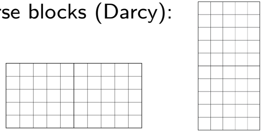
Coarse blocks (Darcy):



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Darcy–Stokes or
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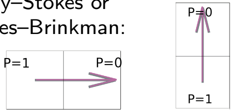

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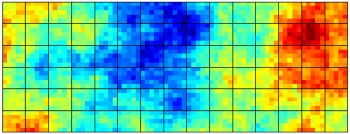

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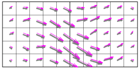

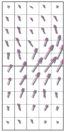

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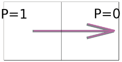

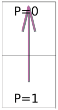

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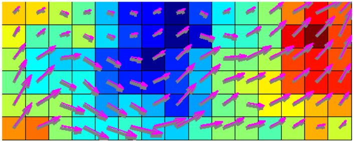

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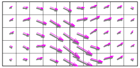

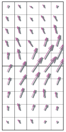

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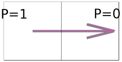

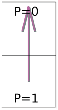

Feasible Approaches: Upscaling or Multiscale Methods

Darcy's law on coarse scale, Stokes–Brinkman on fine scale

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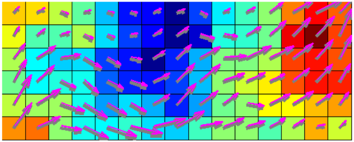

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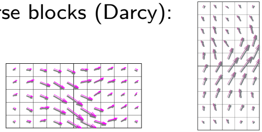

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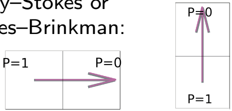
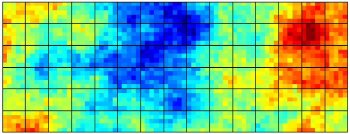


Multiscale method:



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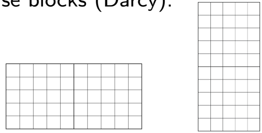

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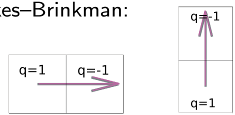
Stokes–Brinkman:



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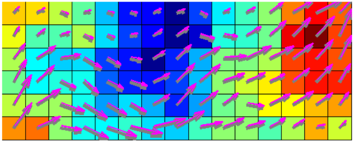

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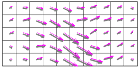

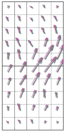

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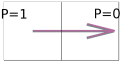

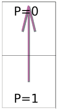

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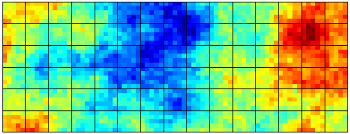

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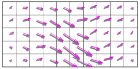

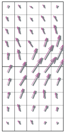

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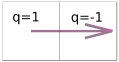

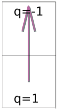

Darcy's law on coarse scale, Stokes–Brinkman on fine scale

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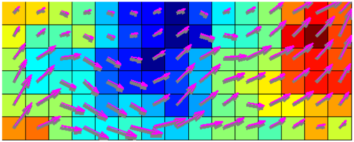

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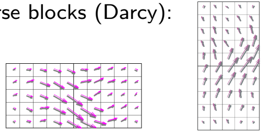

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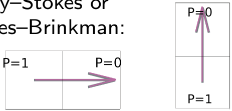

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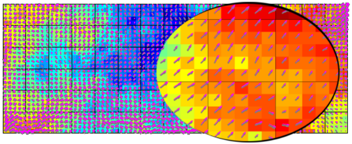

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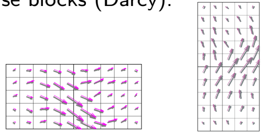

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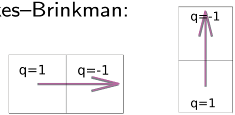
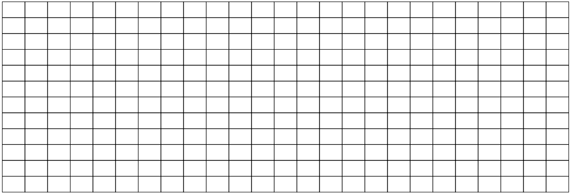


Grids and basis functions:

Fine grid with permeability attached to each cell:



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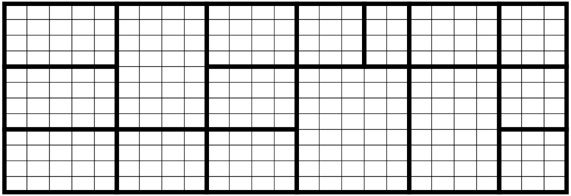

Construct a *coarse* grid, and choose the discretisation spaces U and V^{ms} such that:

Multiscale Mixed Finite Elements

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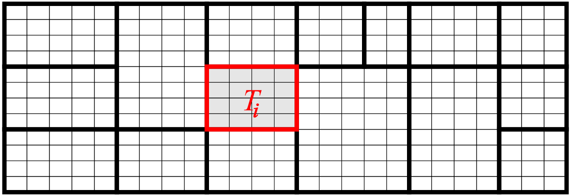

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- For each coarse block T_i, there is a basis function $\phi_i \in U$.

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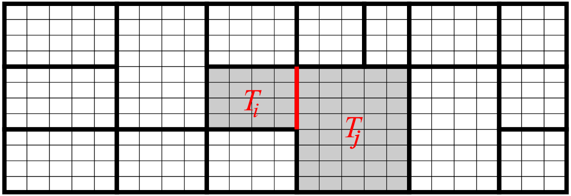

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- For each coarse block T_i, there is a basis function $\phi_i \in U$.
- For each coarse edge Γ_{ij}, there is a basis function $\psi_{ij} \in V^{\mathrm{ms}}$.

Multiscale Mixed Finite Elements

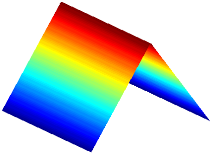
Decomposition:

- $p(x,y) = \sum_i p_i \phi_i(x,y)$ – sum over all coarse blocks
- $v(x,y) = \sum_{ij} v_{ij} \psi_{ij}(x,y)$ – sum over all block faces

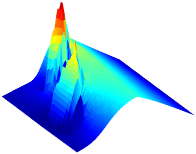
Basis ϕ_i for pressure:

$$\phi_i = \begin{cases} 1 & \text{in } T_i, \\ 0 & \text{otherwise.} \end{cases}$$

Basis ψ_{ij} for velocity:



homogeneous (RT0)



heterogeneous

Construction of Multiscale Basis Functions

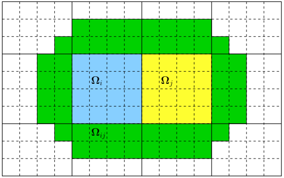
Velocity basis function ψ_{ij} solves a local system of equations:

$$\mu \mathbf{K}^{-1}\vec{\psi}_{ij} + \nabla\varphi_{ij} - \tilde{\mu}\Delta\vec{\psi}_{ij} = 0,$$

$$\nabla\cdot\vec{\psi}_{ij} = \begin{cases} w_i(\vec{x}), & \text{if } \vec{x}\in T_i, \\ -w_j(\vec{x}), & \text{if } \vec{x}\in T_j, \\ 0, & \text{otherwise.} \end{cases}$$

$w_i \propto \mathrm{trace}\,(K_i)$ drives a unit flow through Γ_{ij}.

If there is a sink/source in T_i, then $w_i \propto q_i$.



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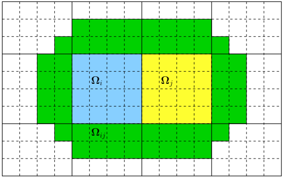
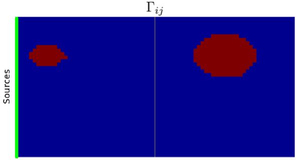
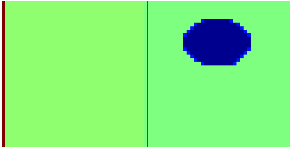
$w_i \propto \text{trace}\,(K_i)$ drives a unit flow through Γ_{ij}.

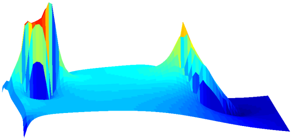
If there is a sink/source in T_i, then $w_i \propto q_i$.





Permeability

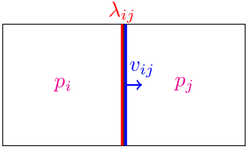

Weight function


Basis function

Hybrid Formulation (for the Darcy Equations)

Mixed method gives a saddle-point problem $\longrightarrow$ hybrid formulation:

$$\begin{bmatrix} B & C & D \\ C^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ D^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} v \\ -p \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ f \\ \mathbf{0} \end{bmatrix}$$

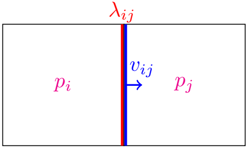


where:

$$B_{ij} = \int_\Omega \mu\, \vec{v}_i\, \mathbf{K}^{-1} \vec{v}_j\, d\Omega, \qquad C_{ij} = \int_\Omega \chi_{T_j}\, \nabla \cdot \vec{v}_i\, d\Omega, \qquad D_{ij} = \int_{\partial\Omega} |\vec{v}_i \cdot \vec{n}_j|\, ds,$$

Mixed method gives a saddle-point problem $\longrightarrow$ hybrid formulation:

$$\begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{C}^{\mathsf{T}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{D}^{\mathsf{T}} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ -\boldsymbol{p} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}$$



where:

$$B_{ij} = \int_\Omega \mu\, \vec{v}_i\, \mathsf{K}^{-1} \vec{v}_j \, d\Omega, \qquad C_{ij} = \int_\Omega \chi_{T_j} \nabla \cdot \vec{v}_i \, d\Omega, \qquad D_{ij} = \int_{\partial\Omega} |\vec{v}_i \cdot \vec{n}_j| \, ds,$$

Split the basis functions, $\boldsymbol{\psi}_{ij} = \boldsymbol{\psi}^H_{ij} - \boldsymbol{\psi}^H_{ji}$

$$\boldsymbol{\psi}^H_{ij}(E) = \begin{cases} \boldsymbol{\psi}_{ij}(E), & \text{if } E \in T_{ij} \setminus T_j \\ 0, & \text{otherwise} \end{cases} \qquad \boldsymbol{\psi}^H_{ji}(E) = \begin{cases} -\boldsymbol{\psi}_{ij}(E), & \text{if } E \in T_j \\ 0, & \text{otherwise} \end{cases}$$

Hybrid basis functions ψ^H_{ij} as columns in a matrix $\boldsymbol{\Psi}$

Coarse-scale hybrid mixed system (Darcy):

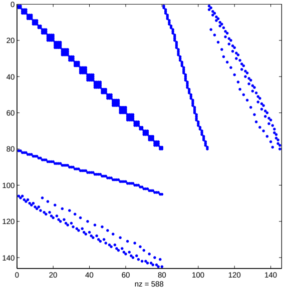
$$\begin{bmatrix} \hat{\boldsymbol{\Psi}}^{\mathsf{T}} B^{TH} \hat{\boldsymbol{\Psi}} & \hat{\boldsymbol{\Psi}}^{\mathsf{T}} C \mathcal{I} & \hat{\boldsymbol{\Psi}}^{\mathsf{T}} D \mathcal{J} \\ \mathcal{I}^{\mathsf{T}} C^{\mathsf{T}} \hat{\boldsymbol{\Psi}} & \mathbf{0} & \mathbf{0} \\ \mathcal{J}^{\mathsf{T}} D^{\mathsf{T}} \hat{\boldsymbol{\Psi}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^c \\ -\boldsymbol{p}^c \\ \boldsymbol{\lambda}^c \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{f}^c \\ \mathbf{0} \end{bmatrix}$$

$\hat{\boldsymbol{\Psi}} = \boldsymbol{\Psi} \boldsymbol{A}^{-1}$
$\boldsymbol{\Psi}$ – matrix with basis functions
$\boldsymbol{A}$ – matrix with face areas
$\boldsymbol{B}^{TH}$ – fine-scale *Darcy* TH-discretization
$\mathcal{I}$ – prolongation from blocks to cells
$\mathcal{J}$ – prolongation from block faces to cell faces

Reconstruction of fine-scale velocity

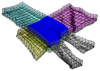
$$\boldsymbol{v}^f = \hat{\boldsymbol{\Psi}} \boldsymbol{v}^c$$

(Pressure bases may also have fine-scale structure if necessary)

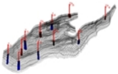


Test of the multiscale Darcy/Stokes-Brinkman method:

1. 2D sandstone reservoirs (no free-flow regions)
2. 2D vuggy reservoir (short correlation)
3. 2D fractured reservoir (long correlation)
4. 2D vuggy and fractured reservoir (short and long correlation)
5. 3D core sample

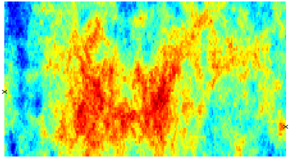


All simulations in the MATLAB Reservoir Simulation Toolbox
http://www.sintef.no/MRST (GNU Public License)

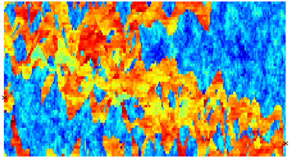


Example 1: Sandstone reservoir

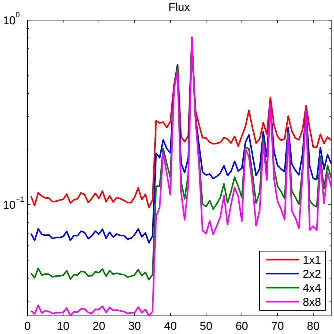
Model 2 of the 10th SPE Comparative Solution Project

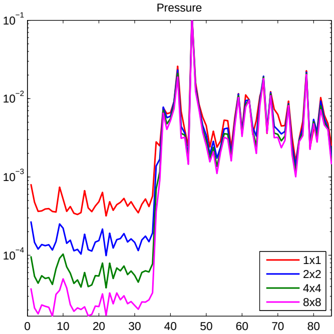


Tarbert (1–35)



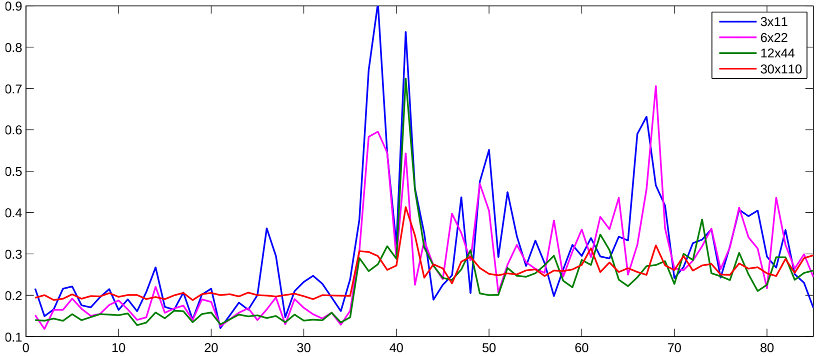
Upper Ness (36–85)



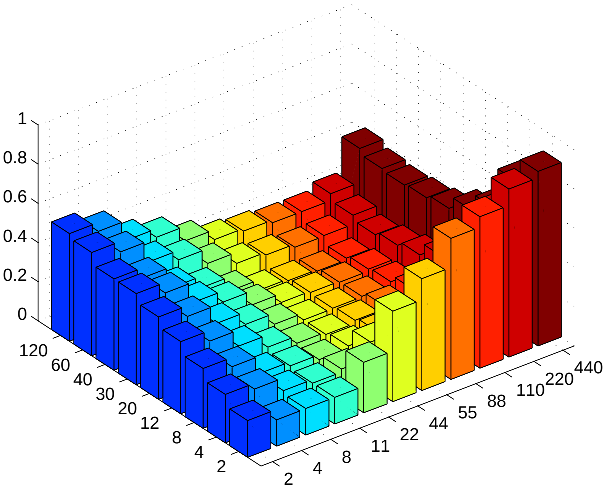


Relative discrepancy between RT0 and $\mathbb{Q}_2/\mathbb{Q}_1$

Example 1: Sandstone reservoir

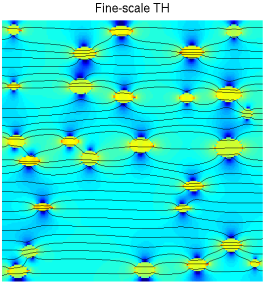


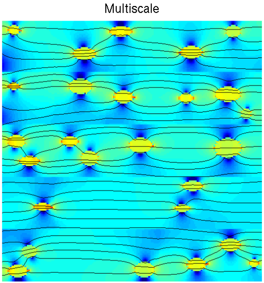
Relative discrepancy in flux (multiscale vs fine-scale)

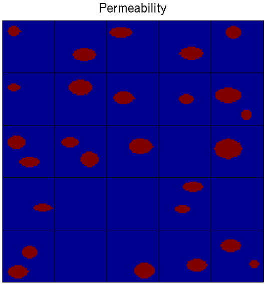


Relative discrepancy in flux (multiscale vs fine-scale)
2×2 refinement of Layer 21

Example 2: Vuggy reservoir (short correlation)







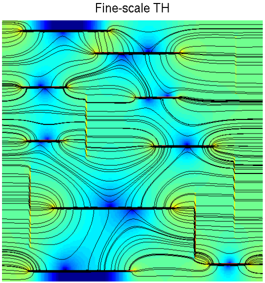
Fine-scale model: 200×200 cells

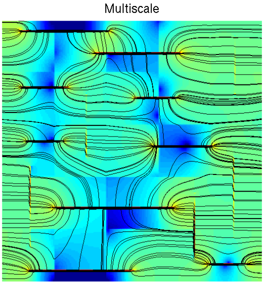
Multiscale model: 5×5 blocks

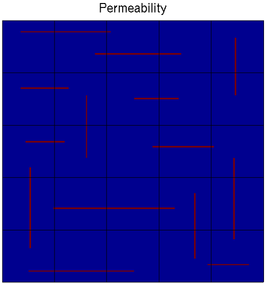
$K_{vugs} = 10^7 \times K_{matrix}$

26 random vugs (areas= 1.8–10.4 m^2), $\frac{\|v^{ms} - v\|_2}{\|v\|_2} = 0.07$

Example 3: Fractured reservoir (long correlation)







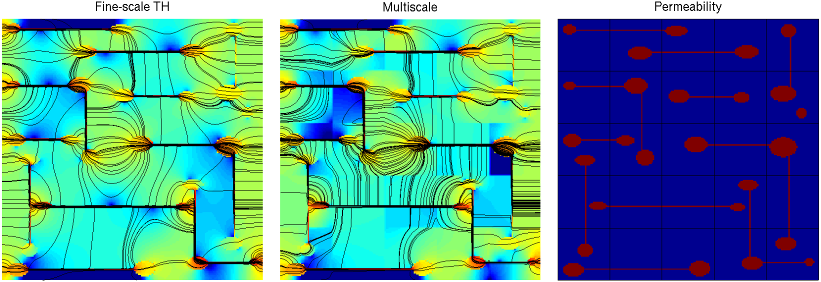
Fine-scale model: 200×200 cells

Multiscale model: 5×5 blocks

$K_{vugs} = 10^7 \times K_{matrix}$

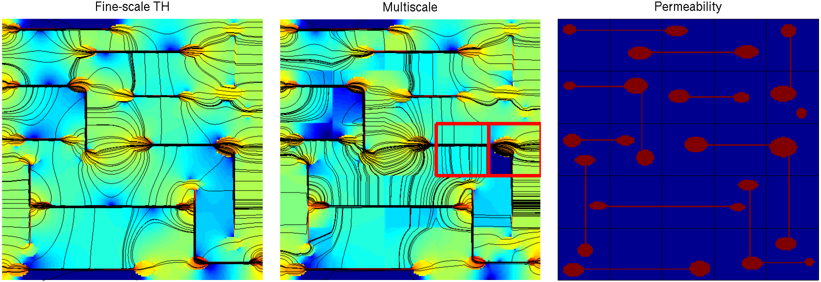
14 random fractures of varying length, $\frac{\|v^{ms} - v\|_2}{\|v\|_2} = 0.07$

Example 4: Vuggy and fractured reservoir



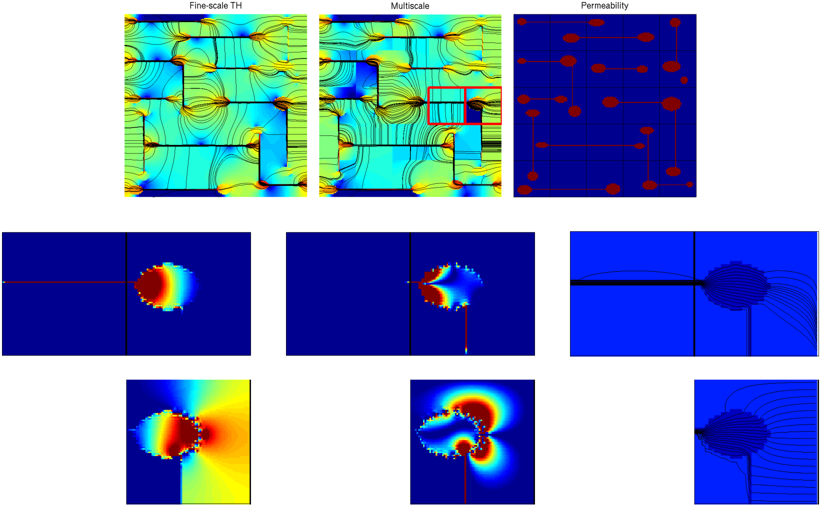
$$\frac{\|v^{ms} - v\|_2}{\|v\|_2} = 0.09$$

Example 4: Vuggy and fractured reservoir



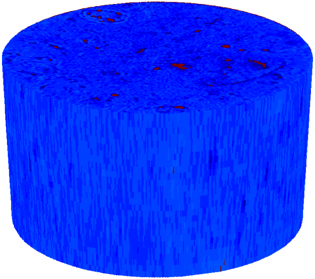
$$\frac{\|v^{ms} - v\|_2}{\|v\|_2} = 0.09$$

Example 4: Vuggy and fractured reservoir



Example 5: Core sample from Shell E&P

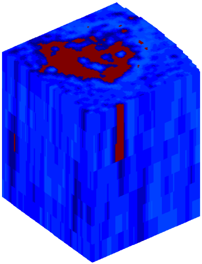
Full model:



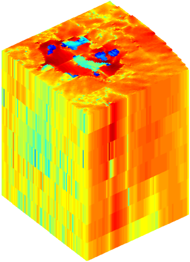
512 $\times$ 512 $\times$ 26 cells
3.449.654 active

Subsample:

Log10 permeability [Darcy]



Log10 MS flux [m/day]



85 $\times$ 85 $\times$ 8 cells, 55.192 active, 75 blocks
pressure boundary conditions

Concluding Remarks

Proof of concept for our MsMFE method:

- Multiphysics with different equations on the coarse and fine scales
- Can be used to simulate flow in carbonate reservoirs

Challenges/issues:

- Very high number of dofs
- Raviart–Thomas not exactly reproduced for homogeneous medium

Ideas to pursue:

- Triangular and unstructured grids
- Correction functions
- Use of (limited) global information