Multiscale Methods for Elliptic Problems in Porous Media Flow

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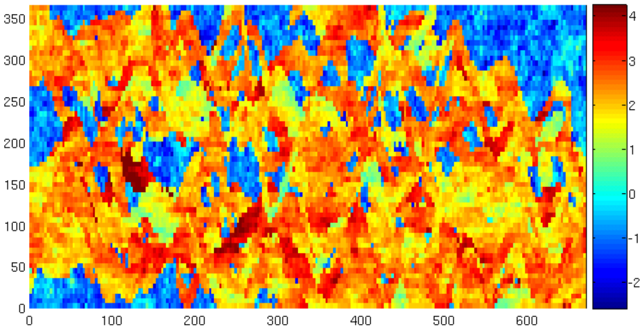
Outline

1. Introduction
2. Three Multiscale Methods for the Pressure Equation
   - Adaptive Local-Global Upscaling / Nested Gridding
   - Multiscale Mixed Finite Elements
   - Multiscale Finite Volumes
3. Comparison
   - Numerical Experiments
   - Computational Complexity
4. Conclusions

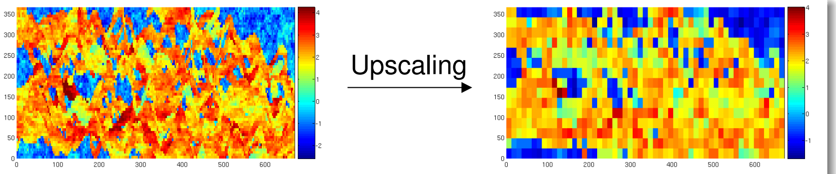
Geological Reservoir Description

- Geological reservoir models: $\mathcal{O}(10^6)$ – $\mathcal{O}(10^9)$ grid cells.
- Oscillating coefficients, $K_{\mathsf{max}}/K_{\mathsf{min}}$: $\mathcal{O}(10^6)$ – $\mathcal{O}(10^{12})$.



Simulation Models and Upscaling

- Geological models too large for standard simulators.
- Industry solution: Upscaling
- Simulation models: $\mathcal{O}(10^4) - \mathcal{O}(10^6)$ grid cells.
- Unfortunately: Fine-scale variations may be important.



Fractional flow formulation (no gravity or capillary forces):

Pressure (elliptic):

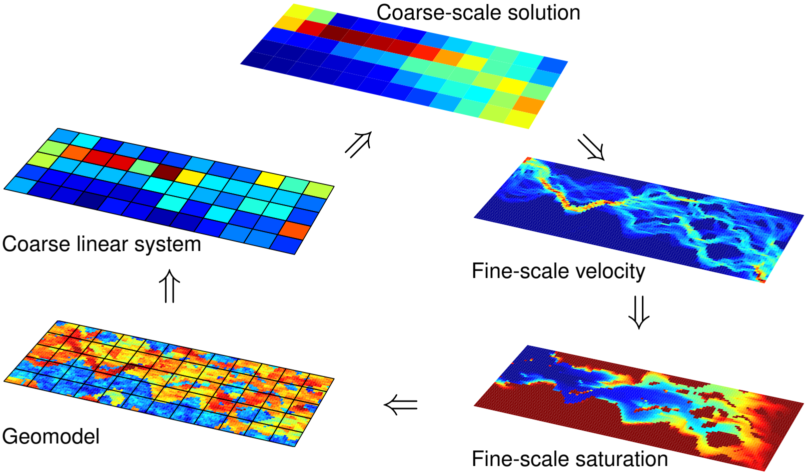
$$\begin{cases} v & = -K\lambda_t(S)\nabla p, \\ \nabla \cdot v & = q \end{cases}$$

Saturation (hyperbolic):

$$\phi\partial_t S + \nabla \cdot (v f(S)) = 0$$

Solution method: Operator splitting.

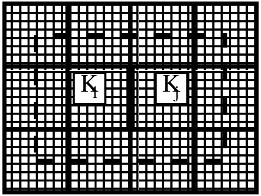
Multiscale Simulation



Adaptive Local-Global Upscaling / Nested Gridding

1. Upscale transmissibility:

$$-\nabla \cdot K\nabla p = 0 \quad \text{in} \quad \Omega_{lj}$$
$$p = Ip^* \quad \text{in} \quad \partial\Omega_{lj}$$



$$T^*_{lj} = \frac{\int_{\partial K_l \cap \partial K_j} v \cdot n_{lj}\, ds}{\int_{K_l} p\, dx - \int_{K_j} p\, dx}$$

2. Solve coarse-scale problem:

$$\sum_j T^*_{lj}(p_l - p_j) = \int_{K_l} q\, dx \quad \forall K_l$$

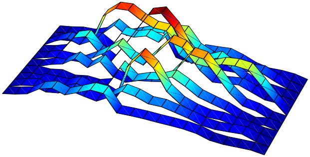
3. Construct fine-scale velocity:

$$v = -K\nabla p, \quad \nabla \cdot v = q \quad \text{in } K_l$$
$$v \cdot n = \frac{T_{ki}(v^* \cdot n_{lj})}{\sum_{\gamma_{ki} \subset \Gamma_{lj}} T_{ki}} \quad \text{on } \partial K_l$$

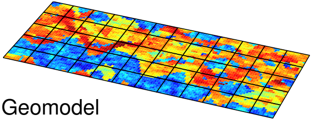
(Here i runs over the underlying fine grid)

Multiscale Mixed Finite Elements

Velocity basis functions ψ_{ij} $\Longrightarrow$ Coarse-grid approximation space

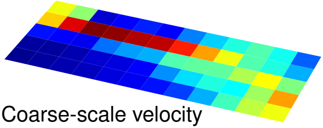


$\Uparrow$



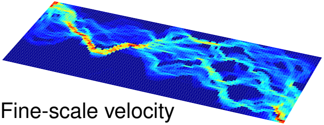
Geomodel

$\Downarrow$



Coarse-scale velocity

$\Downarrow$

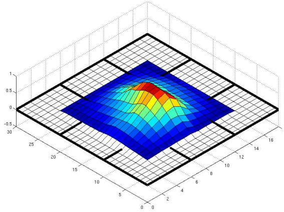


Fine-scale velocity

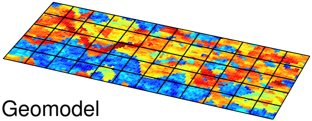
For the MsMFEM the fine-scale velocity field is a linear superposition of basis functions: $v = \sum_{ij} v^*_{ij}\psi_{ij}$.

Multiscale Finite-Volume Method

Pressure basis functions



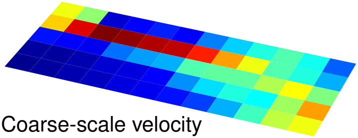
⇑



Geomodel

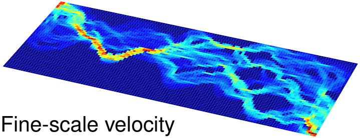
⇒ Mass balance equations on coarse grid

⇓



Coarse-scale velocity

⇓

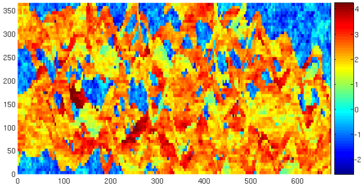


Fine-scale velocity

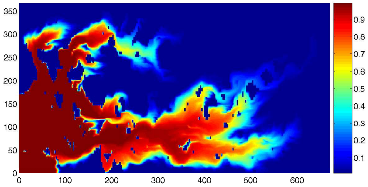
For the MsFVM the fine-scale pressure field is a linear superposition of basis functions: $p = \sum_i p_i^* \phi_i$.

Layer 85 from the 10th SPE Comparison Project

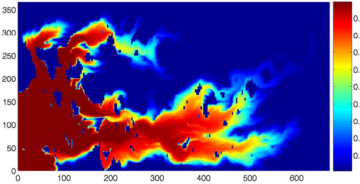
- Fine grid: 60×220
- Coarse grid: 10×22
- Cell aspect ratio: $dx/dy = 2$



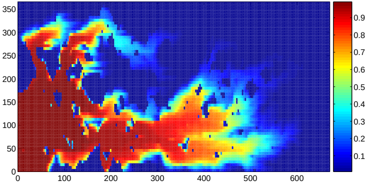
(a) $\log_{10} K$



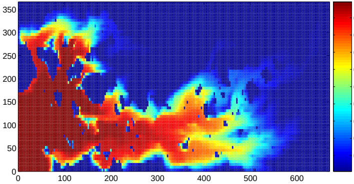
(b) Reference solution ($4\times$ grid)



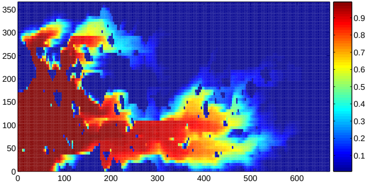
(a) Reference solution (4× grid)



(b) MsMFEM

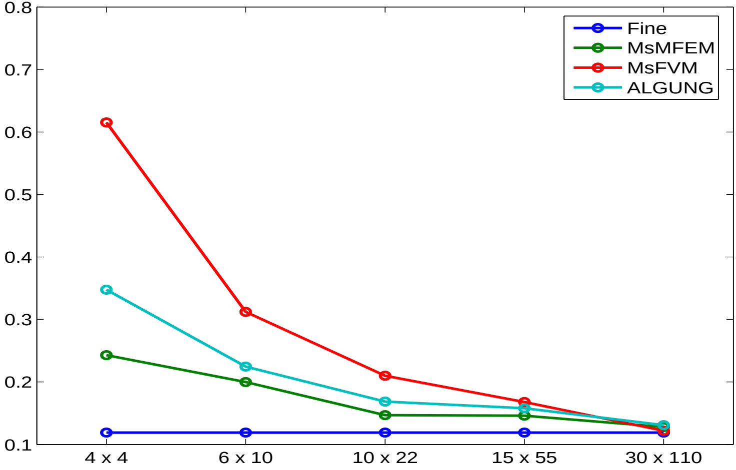


(c) MsFVM

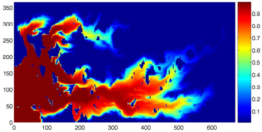


(d) ALGU-NG

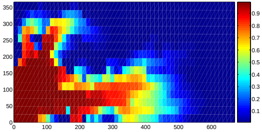
Numerical Experiments
Fluvial Reservoir – Fine Grid Solution



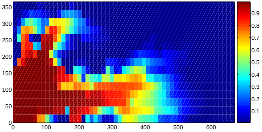
Saturation error as a function of coarse grid size.



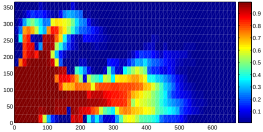
(a) Reference solution (4× grid)



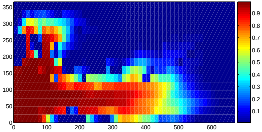
(b) MsMFEM



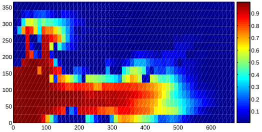
(c) MsFVM



(d) ALGU-NG

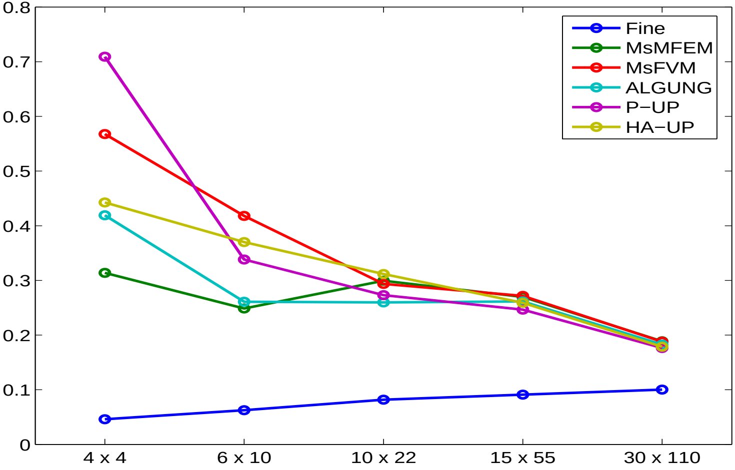


(e) Pressure Method

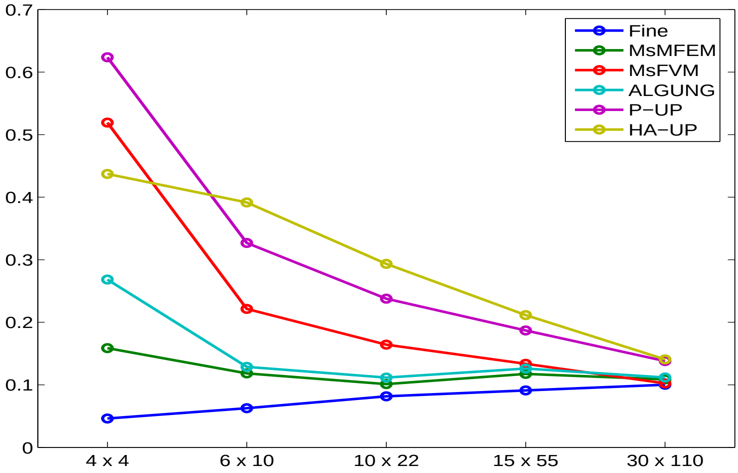


(f) Harmonic-Arithmethic Averaging

- Upscaled grid size: 15×55



- Saturation equation solved on the upscaled grid.
- Errors computed on the upscaled grid.



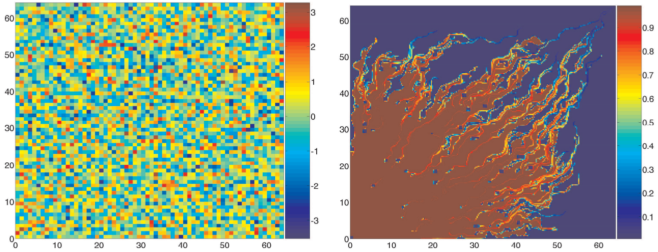
- Saturation equation solved on the fine grid.
- Errors computed on the upscaled grid.

Numerical Experiments

Log-Normal Permeability – Uncorrelated

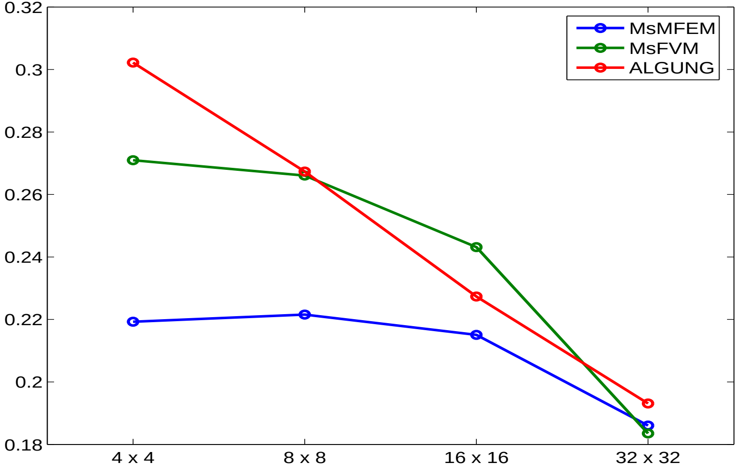
Uncorrelated Log-Normal Permeability

- 100 realizations
- Fine grid: 64×64
- Coarse grids: 4×4, 8×8, 16×16, and 32×32.

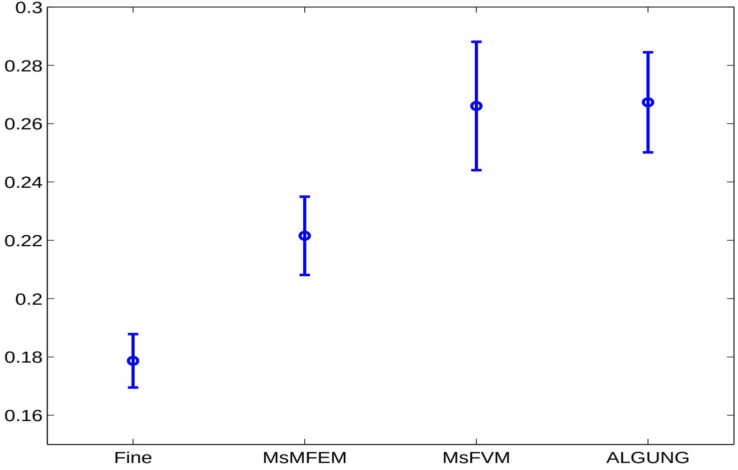


(a) Sample realization ($\log_{10} K$)   (b) Reference solution

Numerical Experiments
Log-Normal Permeability – Uncorrelated



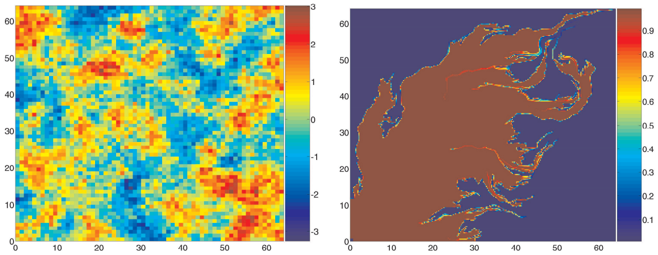
Mean saturation error as a function of coarse grid size.



Mean and standard deviation of the saturation error for the coarse grid of size 8×8.

Spacially Correlated Log-Normal Permeability

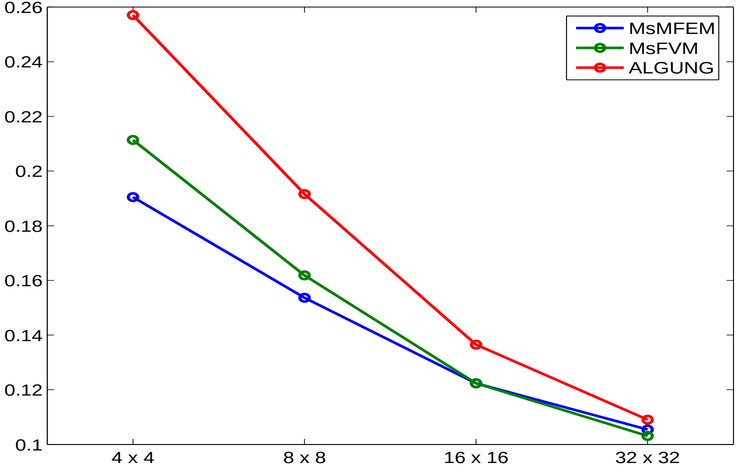
- 100 realizations
- Same grids as before
- Dimensionless correlation length 0.1 in each direction.



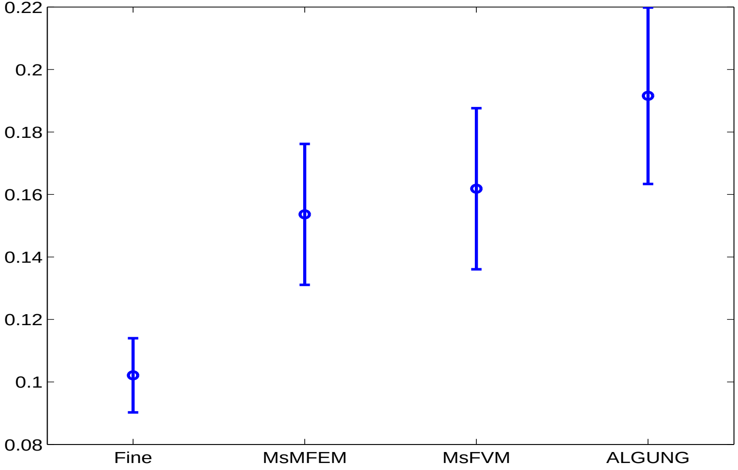
(a) Sample realization ($\log_{10} K$)   (b) Reference Solution

Numerical Experiments

Log-Normal Permeability – Spacially Correlated



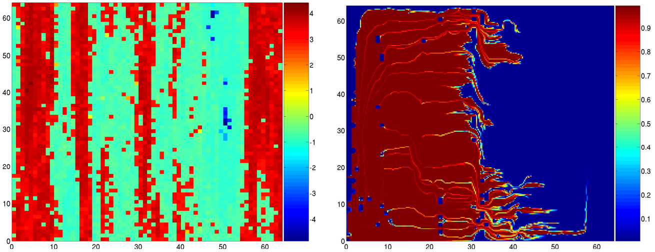
Mean saturation error as a function of coarse grid size.

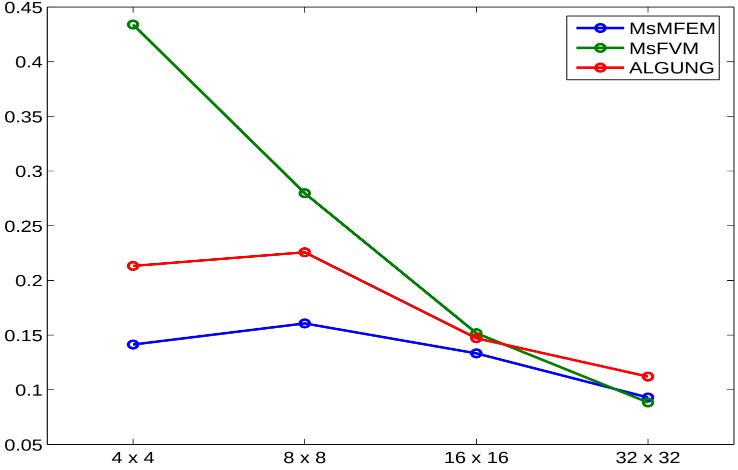


Mean and standard deviation of the saturation error for the coarse grid of size 8×8.

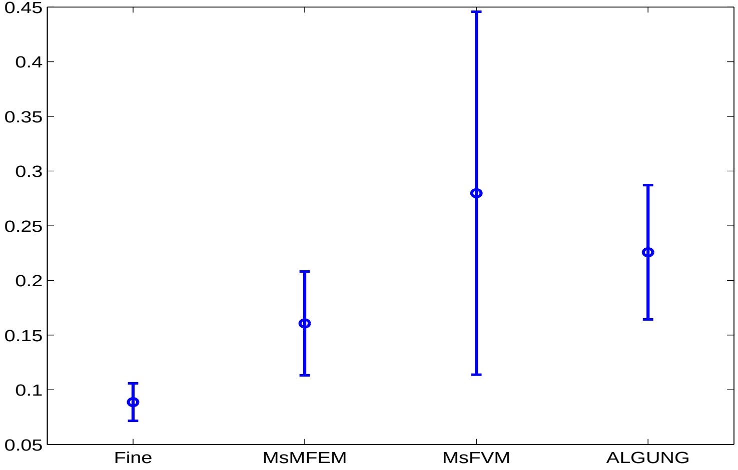
Vertical High-Permeability Channels:

- 100 realizations
- Same grids as before
- Dimensionless correlation length 10 in the vertical direction and 0.1 in the horizontal direction.
- Conditioning on artificial data to produce the channels.



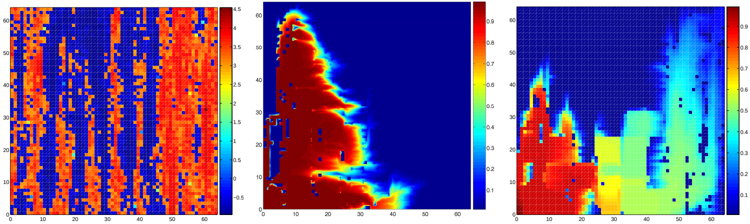


Mean saturation error as a function of coarse grid size.



Mean and standard deviation of the saturation error for the coarse grid of size 8×8.

One of the bad realizations for the MsFVM :

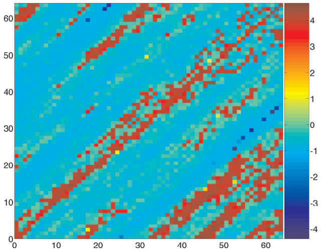


(a) $\log_{10} K$ (b) $4\times$ Reference (c) MsFVM

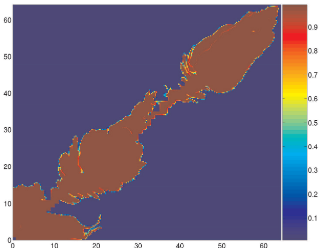
- Solution is smeared out inside coarse cells.
- We will return to this problem in a moment.

Diagonal High-Permeability Channels:

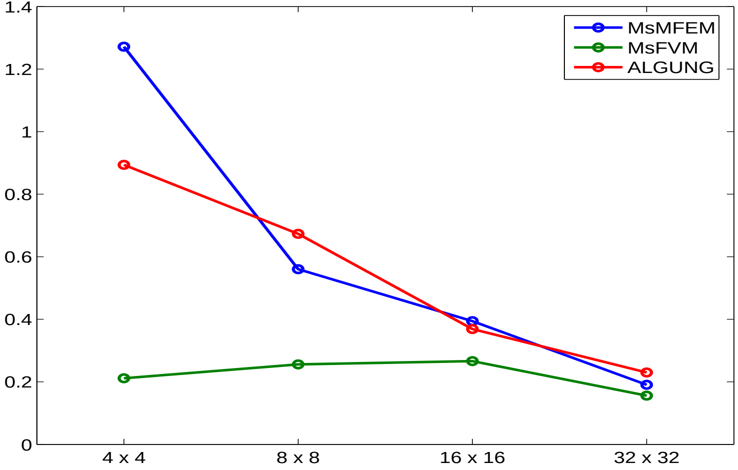
- 100 realizations
- Same grids as before
- Similar to previous case, but rotated 45°.



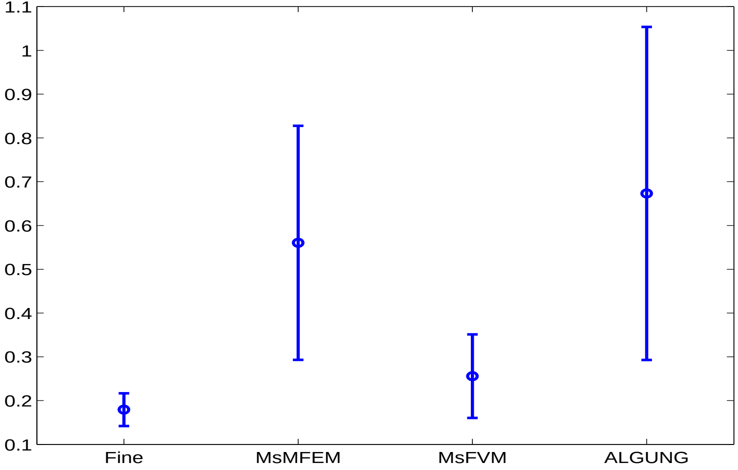
(a) Sample realization ($\log_{10} K$)



(b) Reference Solution

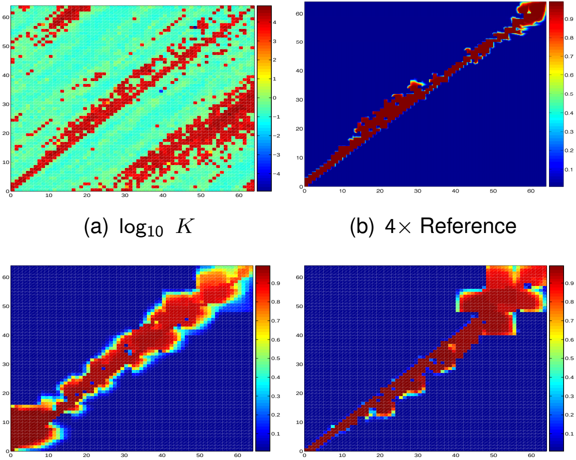


Mean saturation error as a function of coarse grid size.



Mean and standard deviation of the saturation error for the coarse grid of size 8×8.

One of the bad realizations for the MsMFEM and ALGU-NG :

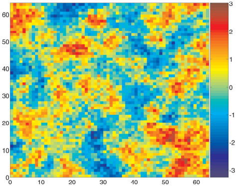


(a) $\log_{10} K$ (b) $4\times$ Reference

(c) MsMFEM (d) ALGU-NG

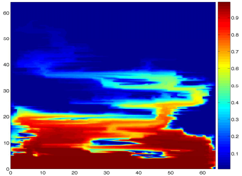
Comparison
Anisotropic Medium / High Aspect Ratio

Spacially correlated log-normal permeability:

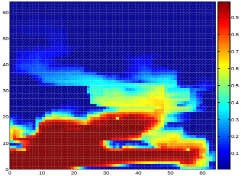


$\log_{10}\ K$

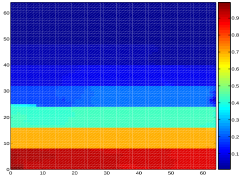
$K_x/K_y = 10^4$
or
$l_x/l_y = 10^{-2}$



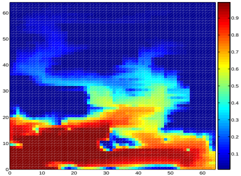
(a) Reference ($4 \times grid$)



(b) MsMFEM

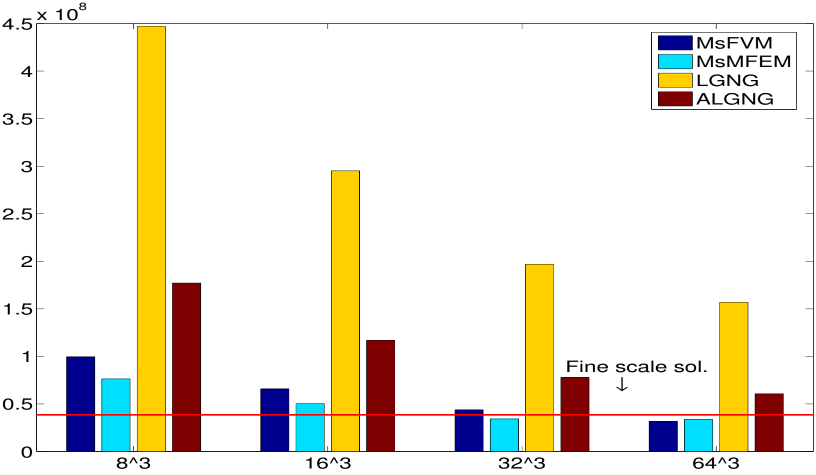


(c) MsFVM



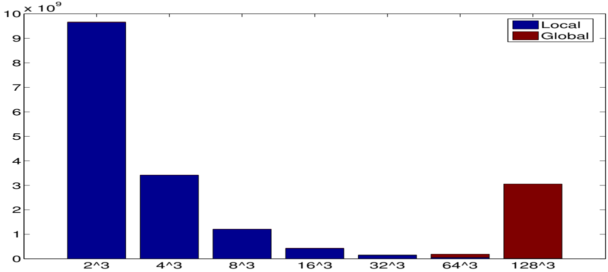
(d) ALGU-NG

Example: 3D (128x128x128), $\alpha = 1.2$ and $m = 3$



Direct solution more efficient, so why bother with multiscale?

- Full simulation: $\mathcal{O}(10^2)$ steps.
- Basis functions need not be recomputed



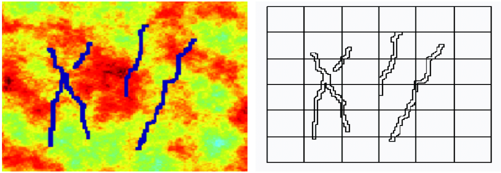
Also:

- Possible to solve very large problems
- Easy parallelization

A Few Words About Implementation

In our experience:

- MsFVM and ALGU-NG:
  Dual grid $\longrightarrow$ Special cases (along external boundaries and internal structures)
- MsMFEM :
  Coarse grid cell is union of fine grid cells $\longrightarrow$
  – Implementation straightforward given a fine grid method.
  – Method quite indepent of coarse grid cell geometry.



(a) Shale Barriers (b) MsMFEM coarse grid!

Conclusions

- All three methods: High accuracy on typical data.
- MsMFEM: Advantage for uncorrelated data and media where the trends are aligned with the grid.
- MsFVM: Advantage when the grid is not aligned with the main flow direction (multi-point stencil).
- MsFVM: Trouble for anisotropic media / high aspect ratios
- MsMFEM and MsFVM have similar computational complexity, ALGU-NG is less efficient