Integro-PDEs: Numerical methods, Analysis, and Applications to Finance.

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Geilo, 30.1.2006

eVita project 176877/V30

Type: Researcher Project

Tittel: Integro-PDEs: Numerical methods, Analysis, and Applications to Finance.

Briefly:

- Integro-partial differential equations (integro-PDEs) is a class of equations used in modern advanced models of the value of different types of contracts/"utilities" in Finance.

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- Many questions regarding how such models/equations can be computed numerically are unresolved/unanswered, both practical and theoretical questions.
- In this project we will focus on different challanges related to the numerical solution of such equations.

Participants and Budget

Participants:	Espen R. Jakobsen	NTNU	(Project leader)
	Kenneth H. Karlsen	CMA, UiO	
	PhD student	NTNU	
	PostDoc	NTNU/UiO	

Budget:

2006	2007	2008	2009
173k	1040k	1388k	867k

Financed: 1 PhD (3 years), 1 PostDoc (2 years), other funds.

Other funds:
- Guests
- Travels
- Equipment (computers/books)

Timeline

		PhD	PostDoc	Other funds
2006	4	*		*
2007	1	*		*
	2	*		*
	3	*	*	*
	4	*	*	*
2008	1	*	*	*
	2	*	*	*
	3	*	*	*
	4	*	*	*
2009	1	*	*	*
	2	*	*	*
	3	*		*
	4	*	4. year paid for by	
2010	1	*	IMF/NTNU	
	2	*		
	3	*		

Progress and Expected Benefits

Progress:

- Hired one PhD student (Simone Cifani)
- 2 travels and 4 guest-visits completed/planed.

Expected Benefits:

- New knowlegde in an active, popular, and interdisiplinary area of research.
- Research on a high international level.
- Publications in international journals.
- Education of sought-after candidates for the finance sector, a sector in rapid growth.
- Strengthening of international and national research networks.
- The project contributes to increased activities within "hovedsatsningsområdene" of NTNU/CMA.

Project Backgound

Overview:

1. The Portfolio Problem in Finance.
2. Mathematical model.
3. Solving the problem I. Integro-PDEs.
4. Solving the problem II. Numerics ...

1. The Portfolio Problem in Finance.

How to manage a portfolio consisting of a stock (hight risk and return) and a bankaccount (low risk and return) in a way that maximizes a given utility function. Typically: Want high return and low risk.

I.e. one seeks a stategy which at any given time tells you what proportion of the wealth to invest in the stock and what proportion to invest in the bank.

Background

2. Mathematical model.

The value of the stock and the bankaccount is modelled using stochastic differential equations:

$$dX_t = b(X_t, \theta_t)dt + \sigma(X_t, \theta_t)dW_t + \int_z j(X_t, \theta_t, z)\bar{\nu}(dz, ds), \quad t > 0.$$

$$X_0 = x.$$

[Brown terms: Old model, brown+blue: modern/accurate model]

The strategy θ_t yields the proportion of the wealth that is invested in the stock. The task is to find θ_t such that the expected utility is maximized:

$$\max_{\theta_\cdot} E\left[\int_0^t f(X_s, \theta_s)ds + g(X_t)\right] =: u(t, x).$$

The quantity $u(t, x)$ is called the value function of the problem.

Background

3. Solving the problem I. Integro-PDEs.

It can be shown that the valuefunction $u(t, x)$ satisfy an integro-PDE:

$$u_t + \sup_{\vartheta} \Big\{ -\mathcal{L}^{\vartheta} u - \mathcal{I}^{\vartheta} u - f(x, \vartheta) \Big\} = 0, \quad t > 0,$$
$$u(0, x) = g(x).$$

where

$$\mathcal{L}^{\vartheta} u(x) = \frac{1}{2} \operatorname{tr} \left[\sigma(x, \vartheta) \sigma(x, \vartheta)^T D^2 u(x) \right] + b(x, \vartheta) Du(x),$$
$$\mathcal{I}^{\vartheta} u(x) = \int_z \left[u(t, x + j(x, \vartheta, z)) - u(x) - \mathbf{1}_{|z|<1}\, j(x, \vartheta, z) Du(x) \right] \nu(dz).$$

If this equations can be solved, then the (almost) optimal stategy ϑ_t can typically be found using a simple iterative method.

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- Challanges:
  - The problem is degenerate.

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- The problems are understood/investigated only in very simple cases.
- Only "simple" numerical metods have been tested.
- There is a lack of theory.

Focus of the Project

Numerical metods

- Developement, testing, and analysis of:
  - + High order methods
  - + Metods using unstructured grids
  - + Choice/discretization of boundary conditions
- Develope a framework for error estimates.
- Analyse and solve numerically concrete models from finance: Portfolio problems with fixed transaction costs.

Related theory

- Study regularity of solutions of integro-PDEs. (Error analysis)
- Study boundary value problems for integro-PDEs. (Numercal boundary conditions)