Relationships and Similarities Between, LR-Splines and T-Splines

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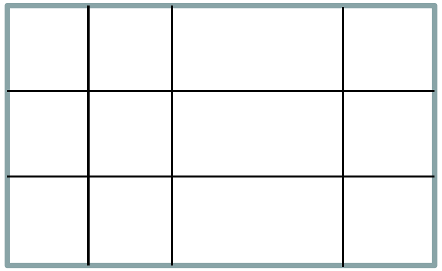
Structure of talk

- The LR-spline rules
- Outline the difference and similarities between T-splines and LR-splines
- LR-splines and Linear independence
- Overloading and the Peeling algorithm
- Some LR-spline theory (if time allows)

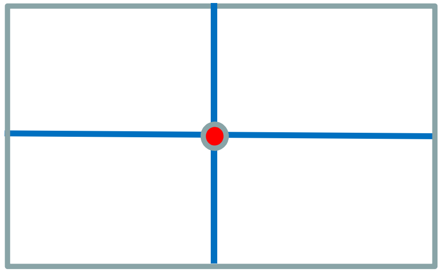
The LR-spline rules

- Given a d-variate tensor product B-spline basis $d \geq 2$.
- Incrementally refine the spline space by splitting the support of selected B-splines by inserting mesh-rectangles (knot-lines)
  - For each refinement: Perform additional refinements if some B-splines do not have minimal support .

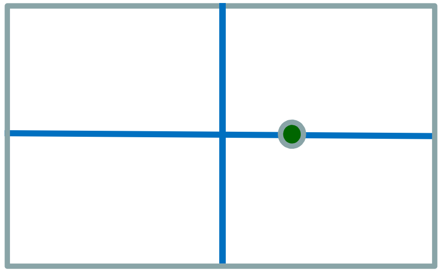
The LR-spline rules
geometric interpretation (2-variate)



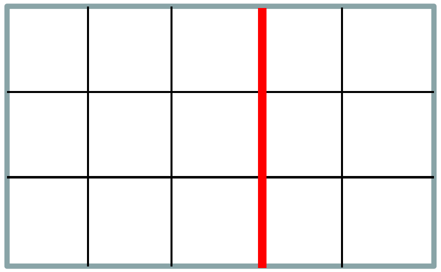
B-spline degree (3,2)
domain and knotlines



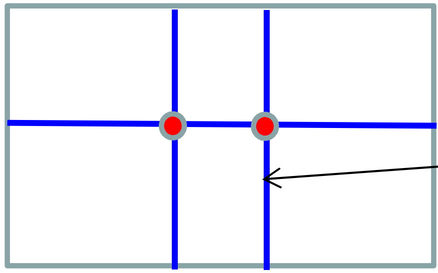
B-spline domain, Greville point and Greville frame



Specify refinement on Greville frame



Resulting knotline



Resulting Greville points and composite Greville frame

Further refinements can be anywhere on the composite Greville frame

Difference of LR-splines and T-splines

Spline space

- LR-splines are spline space centric, and refine the spline space by knot-line (mesh-rectangle) insertion
- T-splines are vertex grid centric, and anchor new vertices in the T-mesh (vertex mesh). The B-splines are inferred from the vertex grid according to the T-spline rules.

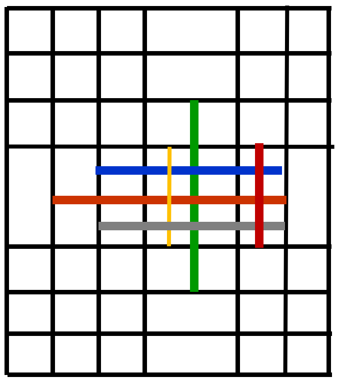
Minimal support B-splines

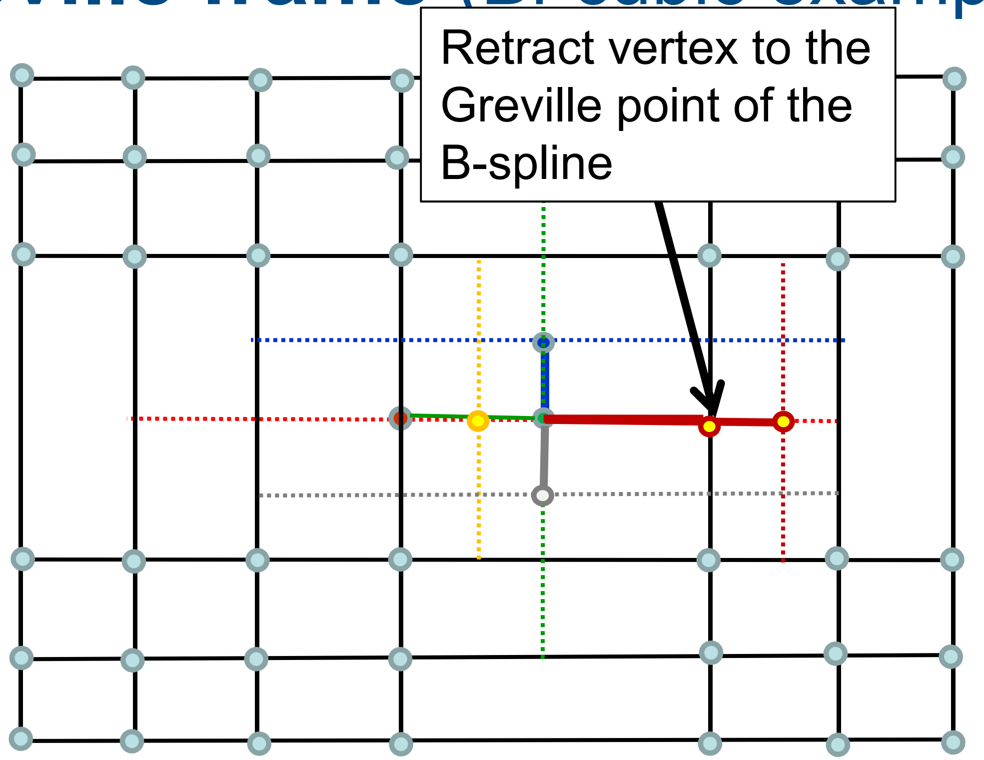
- The B-splines of LR-splines are all minimal support and their vertices can be anchored in the parametrization according to the Greville point.
- The B-splines of T-splines will not always be minimal support as the anchors define the Greville points of the B-splines

Difference of permitted refinements of LR-splines and T-splines

- As I understand T-splines a new vertex has to be inserted between two existing Greville points. The new vertex is anchored by a new knot in the middle interval(s) of an existing B-spline.
- LR-splines can insert a knot in any interval of an existing B-spline (anywhere on the Greville frame)
  - LR-splines refine by inserting new knots (meshrectangles, knotlines) thus offering a wider range of refinements
  - Near T-spline compatible LR-splines can be made by only allowing refinements between existing Greville points
    - However, such restricted refinements can also trigger further refinements that change Greville point locations or increase the number of B-splines
    - Further restrictions might be necessary to take heed of all T-spline rules.

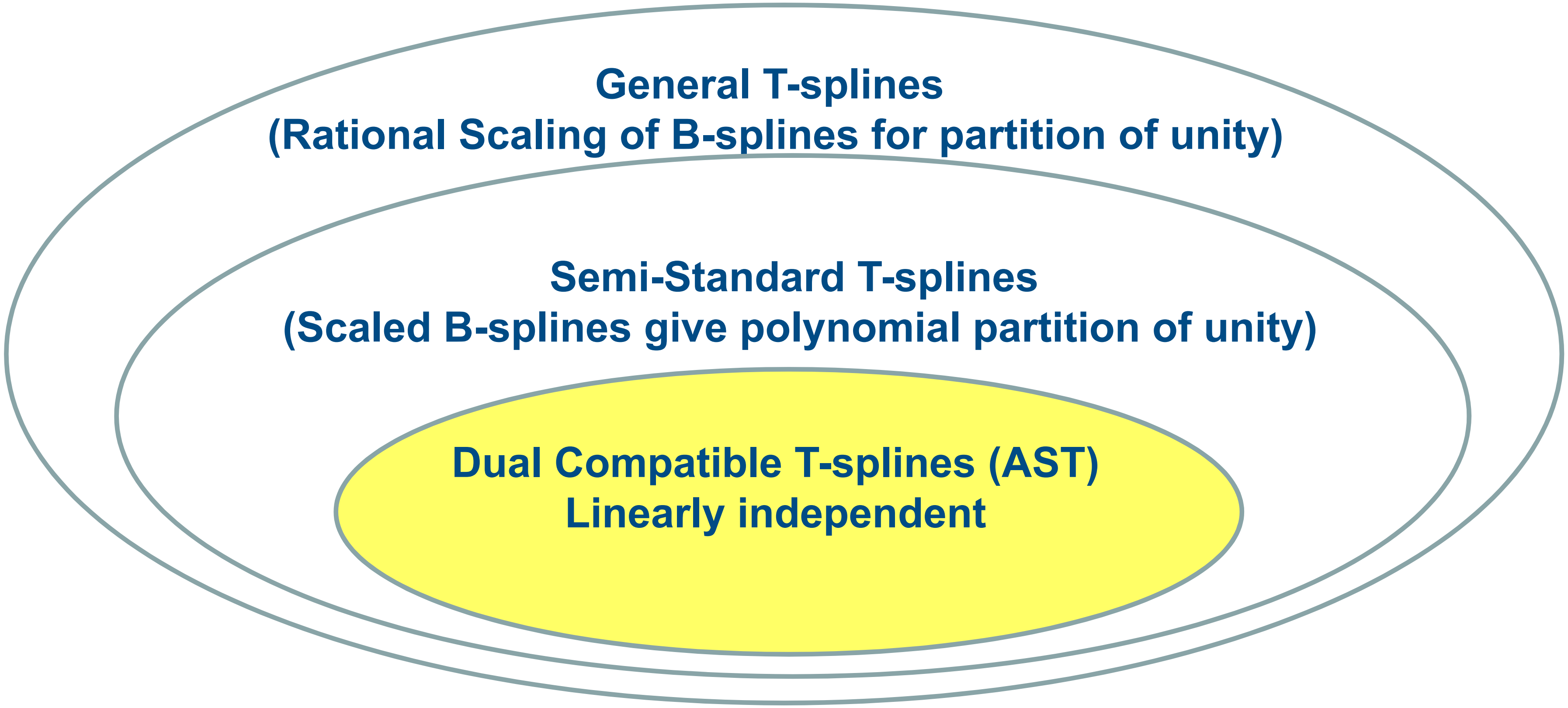
Specify LR-spline refinement in the composite Greville frame (Bi-cubic example)





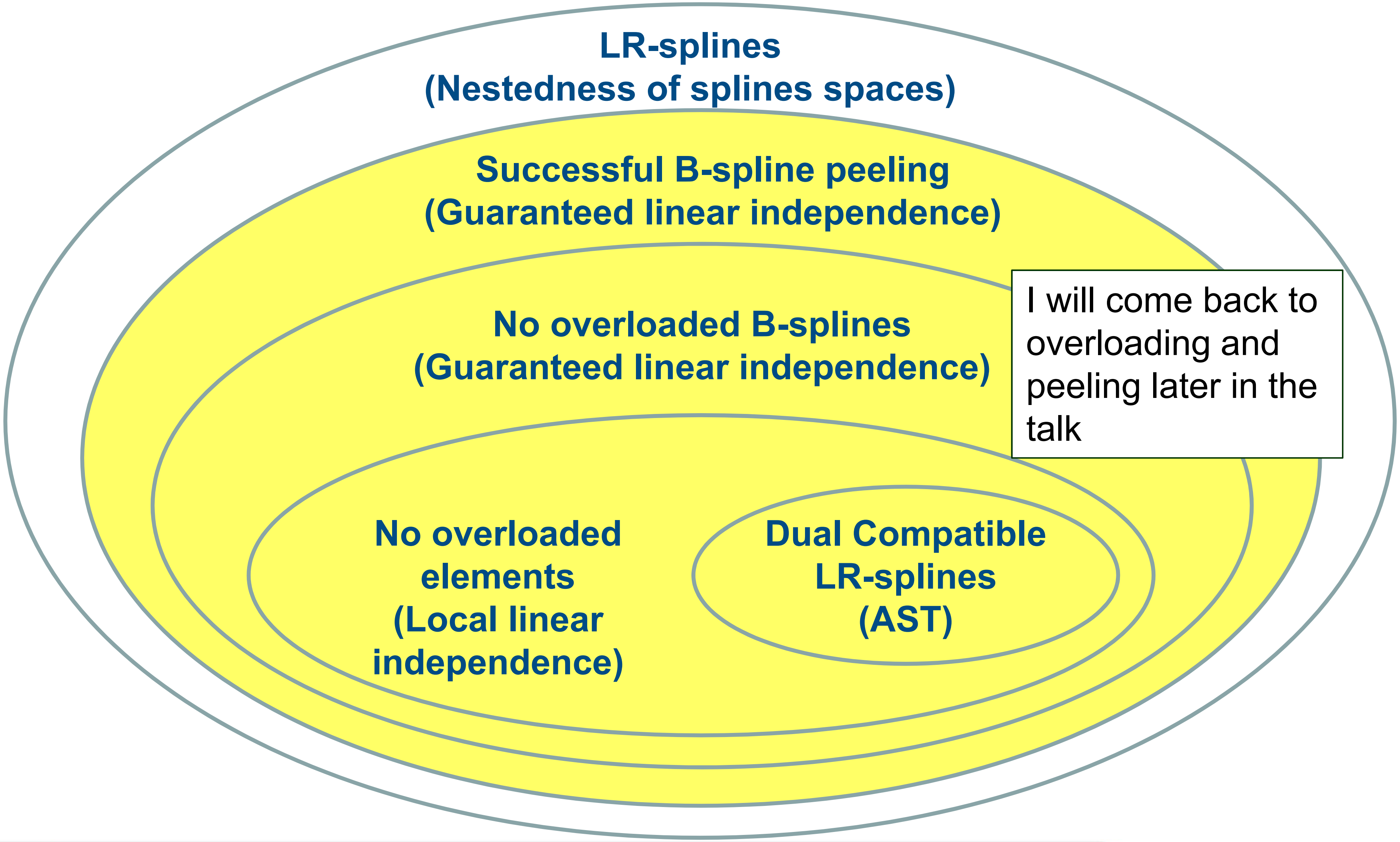
- The solid lines represent the LR-spline vertex mesh (Corresponding to the T-mesh or dual mesh of T-splines)
- Dotted lines added to visualize the additional lines of the Greville frame and suggest possible locations for refinement specification.
- When defining a vertex control mesh the dotted lines can be kept to give the mesh a proper structure, and allow refinement specification in the vertex mesh

T-splines and Linear Independence

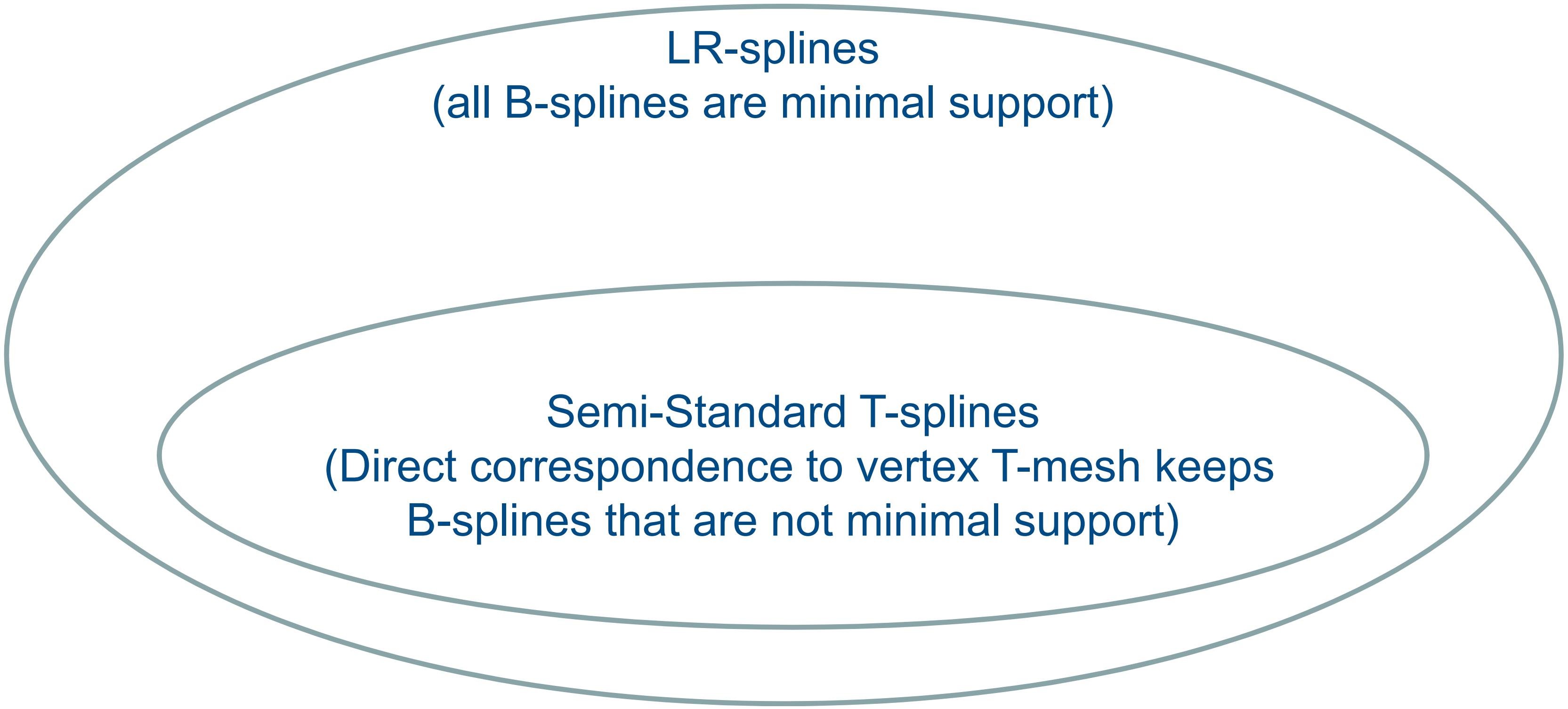


- Nestedness of splines spaces
  - General T-splines;: No
  - Semi-Standard T-splines: Yes

LR-splines and Linear Independence

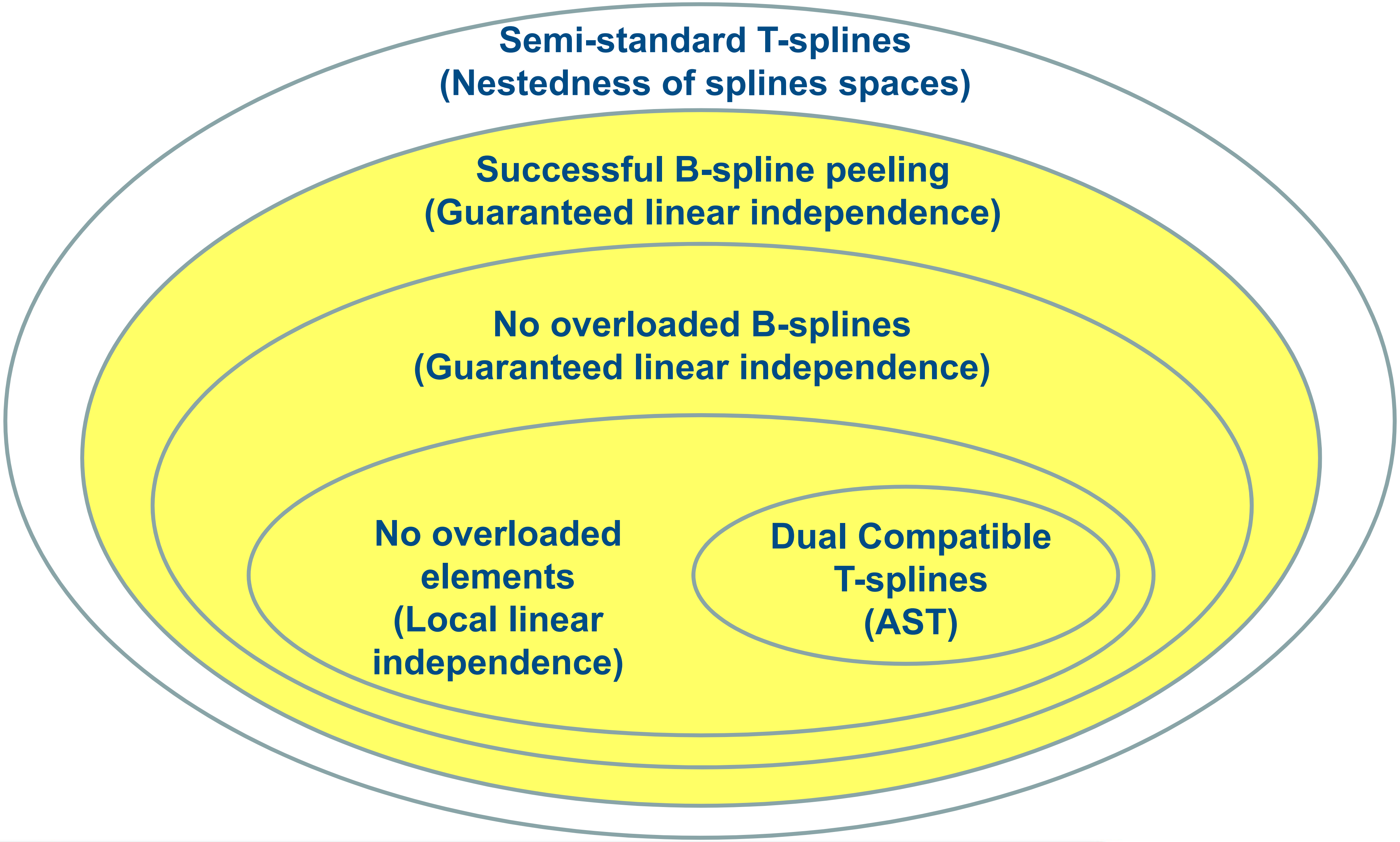


Splines spaces of Semi-Standard T-splines seem to be included in the spline spaces of LR-splines



- Linear independence based on no overloaded elements, no overloaded B-splines and peeling should also be applicable for semi-standard T-splines.

If the spline space of semi-standard T-spline ⊆ LR-splines, then



Overloading and peeling

Peeling for Ensuring Linear Independence

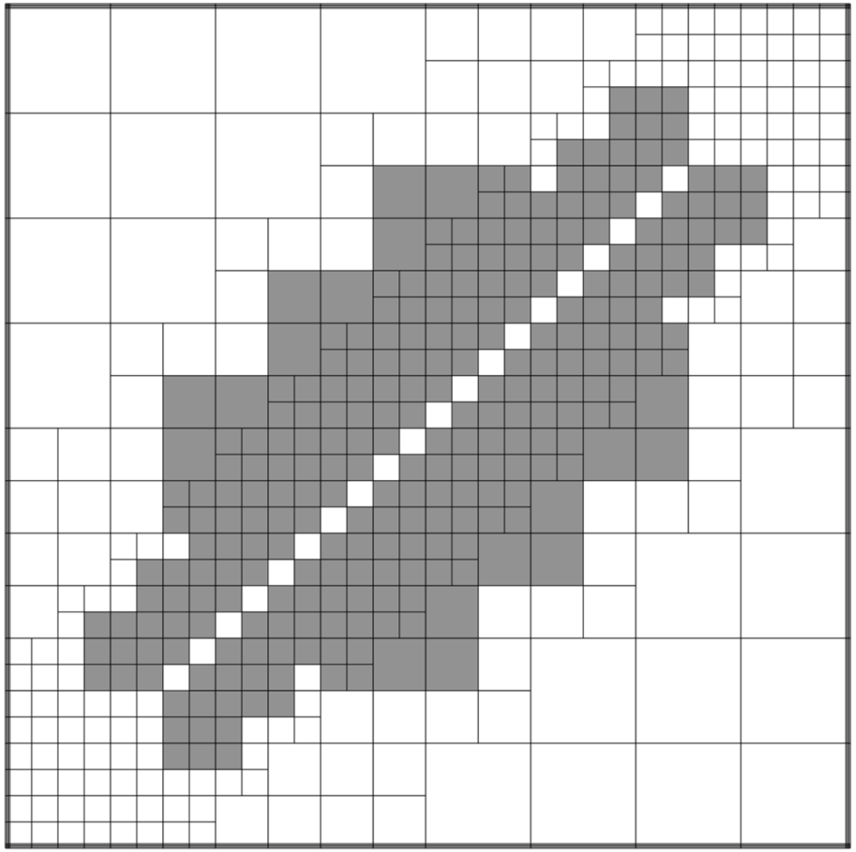
(Valid for LR-splines and assumed for Semi-Standard T-splines)

The refinement starts from a tensor product B-spline space with $(p_1 + 1)(p_2 + 1) \ldots (p_d + 1)$ B-splines covering an element spanning the polynomial space of degree $(p_1, p_2, \ldots, p_d)$ over the element.

- A refinement cannot reduce the polynomial space spanned over an element.
- An extra B-spline in a linear dependency relation can be removed without changing spanning properties over its elements
  - Before the removal of a B-spline there must consequently be more than $(p_1 + 1)(p_2 + 1) \ldots (p_d + 1)$ B-splines covering all elements of the removed B-spline.

Overloaded elements and B-splines

- We call an element overloaded if there are more than $(p_1 + 1)(p_2 + 1) \ldots (p_d + 1)$ B-splines covering the element.
- We call a B-spline overloaded if all its elements are overloaded.



The support of overloaded B-splines colored grey.

Illustration by:
Kjetil A. Johannessen,
SINTEF

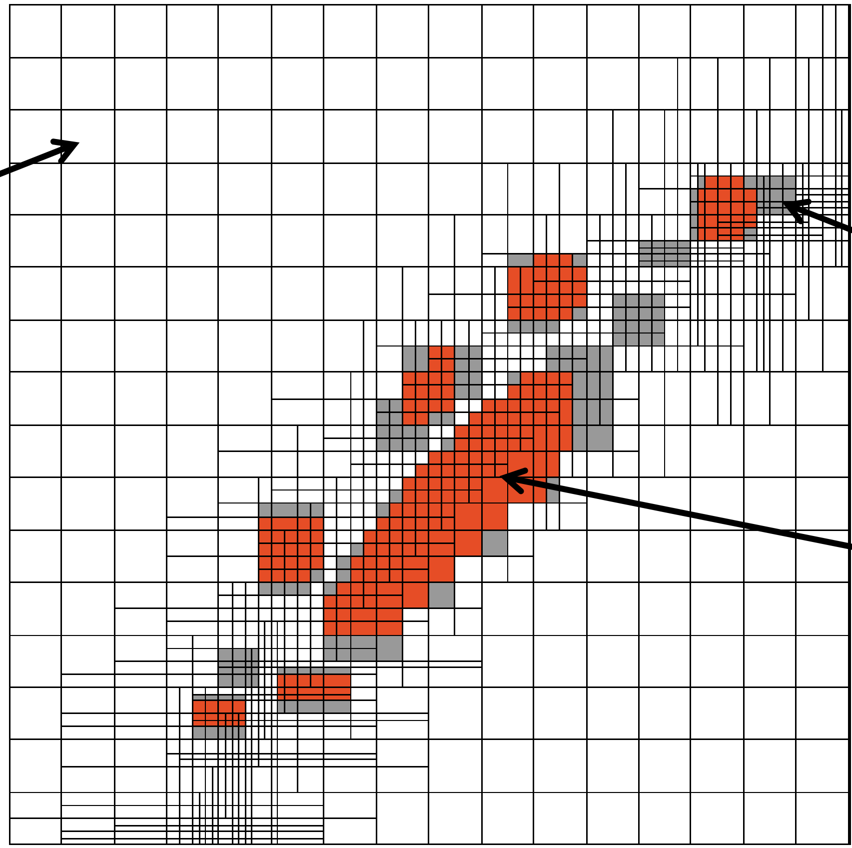
Observations

- If there is no overloaded B-spline then the B-splines are locally (and globally) linearly independent
  - All overloaded elements not part of an overloaded B-spline can be disregarded
- Only overloaded B-splines can occur in linear dependency relations
- A linear dependency relation has to include at least two* overloaded B-splines.
  - Elements with only one overloaded B-spline cannot be part of linear dependency relation. Thus overloaded B-spline having such an element cannot be part of linear dependency relation.

* The number is actually higher, at least: $2^l + 1$ in the l-variate case.

Example peeling algorithm for overloaded B-splines.

White: Elements covered by no overloaded B-spline



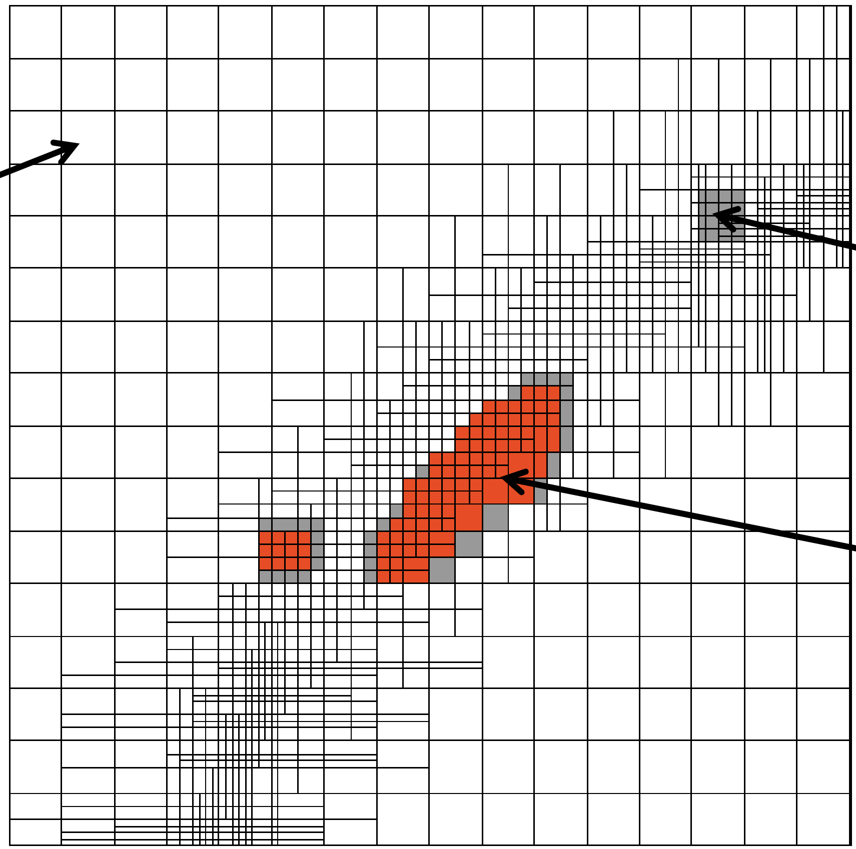
Grey: Elements covered by only one overloaded B-spline (can be removed)

Red: Elements covered by more than one overloaded B-spline

Illustration by: Kjetil A. Johannessen, NTNU

Example, Continued.

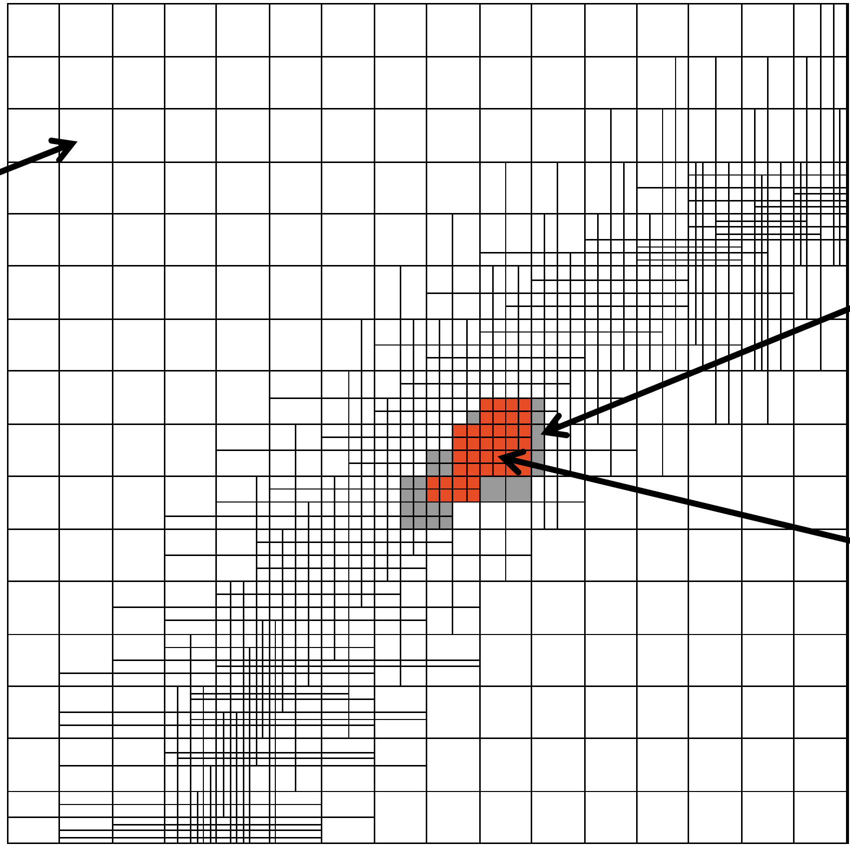
White: Elements covered by no overloaded B-spline



Grey: Elements covered by only one overloaded B-spline (can be removed)

Red: Elements covered by more than one overloaded B-spline

Example, Continued.



White: Elements covered by no overloaded B-spline

Grey: Elements covered by only one overloaded B-spline (can be removed)

Red: Elements covered by more than one overloaded B-spline

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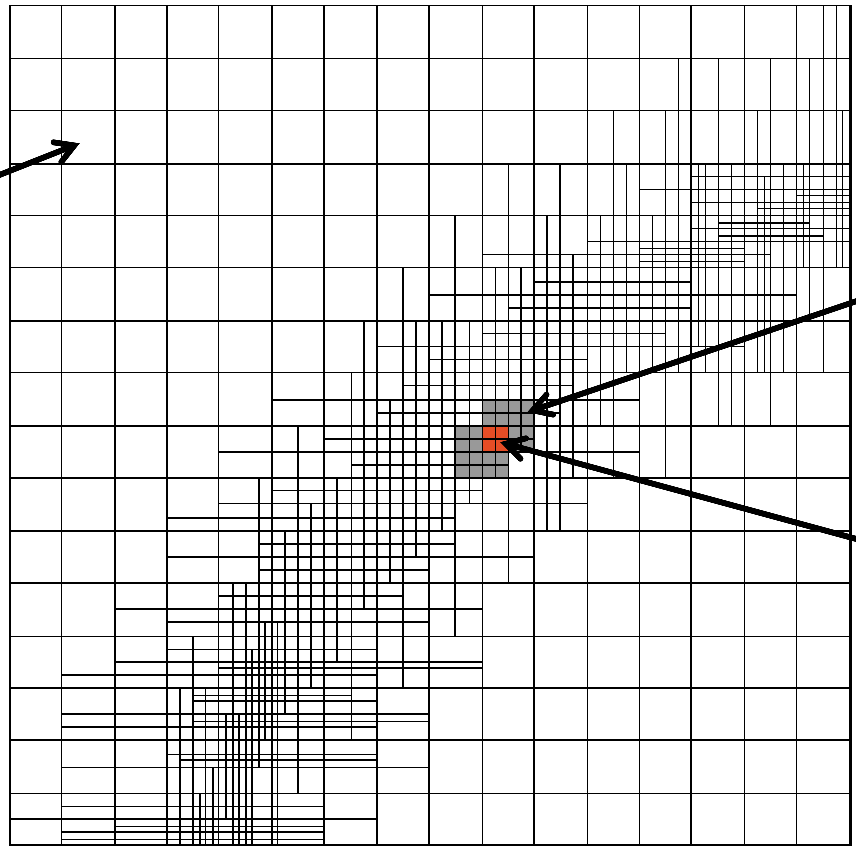
All B-splines remaining some elements that are overloaded only once. Linear dependency not possible.

White: Elements covered by no overloaded B-spline



Grey: Elements covered by only one overloaded B-spline (can be removed)

Red: Elements covered by more than one overloaded B-spline

Illustration by: Kjetil A. Johannessen, NTNU

No overloaded B-splines remaining. Linear dependency not possible.

White:
Elements covered by no overloaded B-spline

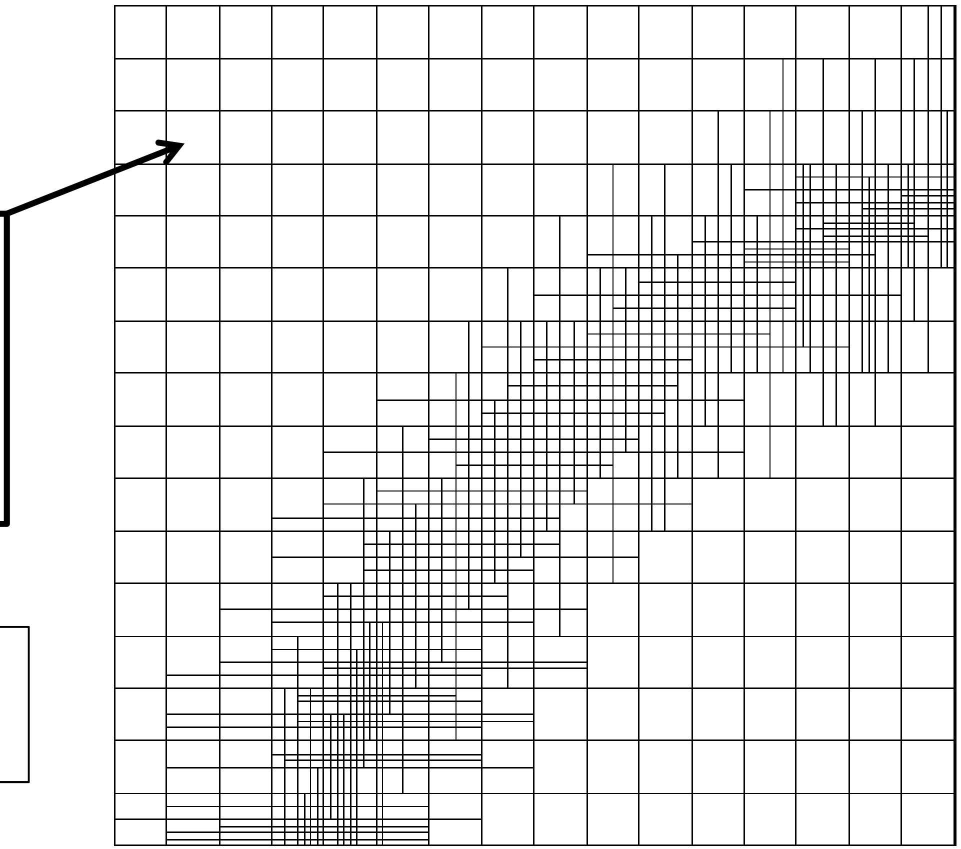


Illustration by:
Kjetil A. Johannessen, NTNU

Spline forest and LR-splines and

Mike Scott defined yesterday the following ingredients for the spline forest.

1. Simple nesting theory
   - LR-splines provide nested spline spaces by construction
2. Local linear independence of B-splines
   - LR-splines with no overloaded elements are locally linearly independent.

The approach could probably be extended to LR-splines where no B-spline is overloaded

Current work on LR-splines at SINTEF

- LR Splines extensions to the SINTEF GoTools C++ library. EU-project: TERRIFIC (2011-2014)
  - www.terrific-project.eu
- We work on IgA based on LR-splines
- We work on efficient LR-spline visualization on GPUs
- We address representation of geographic information using LR-splines in the EU-project IQmulus (2012-2016)
  - www.iqmulus.eu
- We will work on compact representation of 3-variate analysis results using LR-splines in the EU-project VeLaSSco (2014-2016)