Pipe networks: coupling constants in a junction for the isentropic Euler equations

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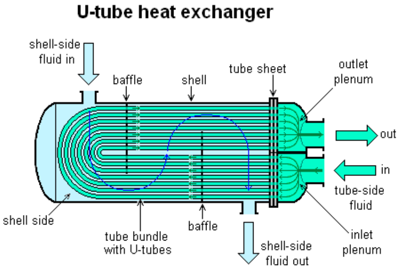
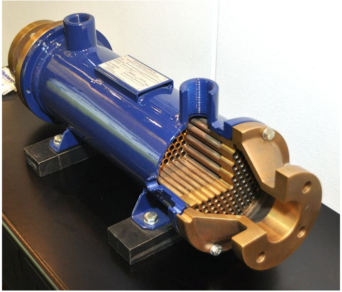
Introduction

- Heat exchangers with parallel channels
  - Pressure drop dependent on phase composition
  - May result in wrong flow distribution
  - May cause instabilities
- Need to model junctions dynamically
  - F.ex. main inlet pipe to tubes
  - Repartition of mass flow in each pipe





Shell-tube heat exchangers
Wikimedia commons

Overview

- Numerical modelling of flow in pipes
- The model: Isentropic Euler equations
- The Riemann problem
  - Mathematical notions
  - Coupling of the pipes through the generalised Riemann problem
  - The right coupling condition
- Physical interpretation
- Numerical examples
  - Entropy condition at the junction
  - Conservation of energy in junctions

Modelling of flow in pipes

- One-dimensional models

Modelling of flow in pipes

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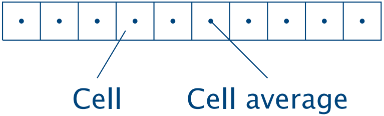
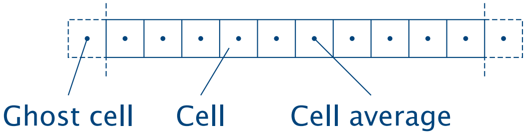
- Finite-volume method



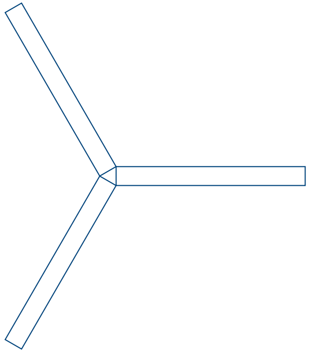
Modelling of flow in pipes

- One-dimensional models
- Finite-volume method
- Boundary conditions with ghost cells



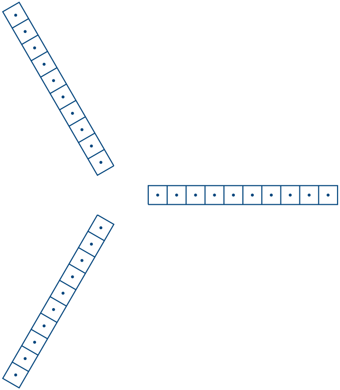
Junctions

- Several pipe connected together



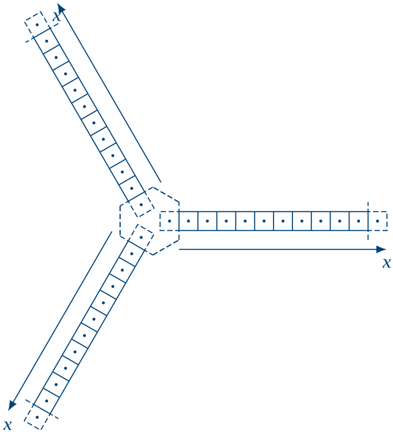
Junctions

- Several pipe connected together
- How to represent a junction?
  - Pipes: one-dimensional models
  - Junction: multi-dimensional flow



Junctions

- Several pipe connected together
- How to represent a junction?
  - Pipes: one-dimensional models
  - Junction: multi-dimensional flow
- Describe the junction with ghost cells
  - Solve the pipes as independent domains
  - Set the ghost cells
- The junction has no volume



The model

Isentropic Euler equations

- Conservation of mass: $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$
- Conservation of momentum: $$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}\left(\rho v^2 + p\right) = 0$$

Equation of state for isentropic flow

$$p = k\rho^{\gamma} \quad (\text{then, } s(x,t) = \text{const})$$

In quasilinear form $\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A}\frac{\partial \boldsymbol{U}}{\partial x} = 0$

$$\boldsymbol{U} = \begin{pmatrix} \rho \\ \rho v \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ a^2 - v^2 & 2v \end{pmatrix} \quad \text{where } a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho}$$

The Riemann problem

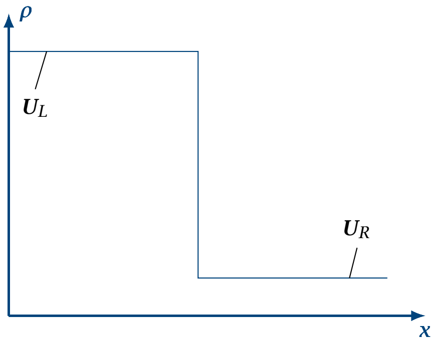
- Eigenvalues of the Jacobian

$$v - a \, , \; v + a$$

with eigenvectors

$$\begin{pmatrix} 1 \\ v - a \end{pmatrix} , \begin{pmatrix} 1 \\ v + a \end{pmatrix}$$

- Shock or rarefaction wave



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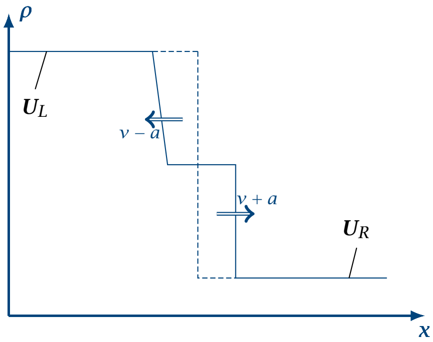

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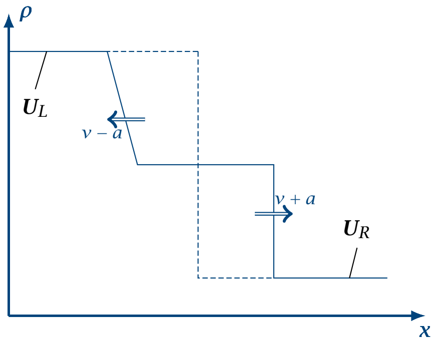

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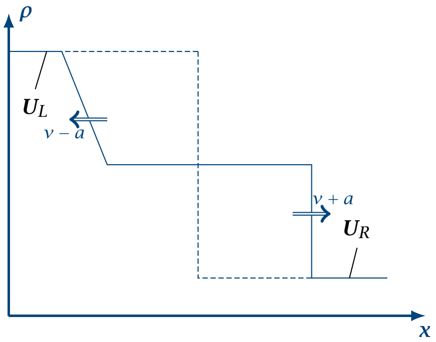

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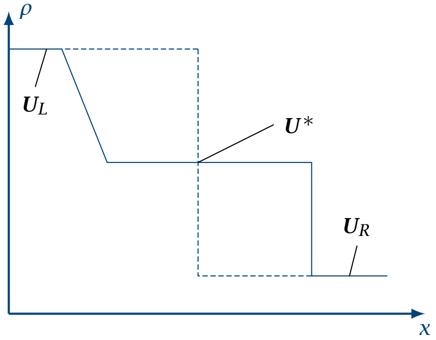
- Shock or rarefaction wave
- The star-state $\boldsymbol{U}^*$
  - Related to $\boldsymbol{U}_L$ and $\boldsymbol{U}_R$ through the wave of family 1 and 2, respectively



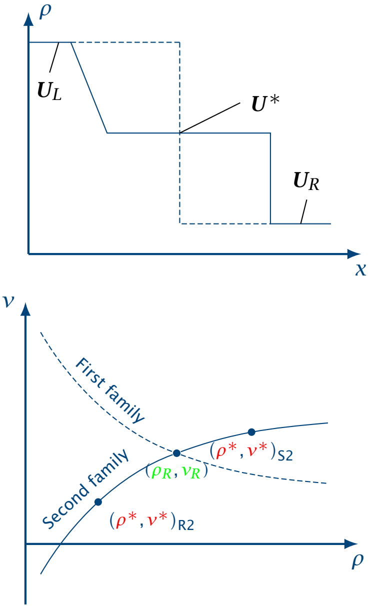
The Riemann problem

Equations for the waves of the second family

$$v^* (\rho^*; \rho_R, v_R)_{\mathrm{R2}} = v_R + \frac{2\sqrt{\gamma k}}{\gamma - 1} \left({\rho^*}^{\frac{\gamma-1}{2}} - {\rho_R}^{\frac{\gamma-1}{2}} \right), \quad 0 < \rho^* \leq \rho_R$$

$$v^* (\rho^*; \rho_R, v_R)_{\mathrm{S2}} = v_R + \sqrt{\frac{k (\rho^* - \rho_R) ({\rho^*}^{\gamma} - {\rho_R}^{\gamma})}{\rho^* \rho_R}}, \quad \rho^* > \rho_R$$

Hugoniot Locus: points connected by a curve separated by one wave



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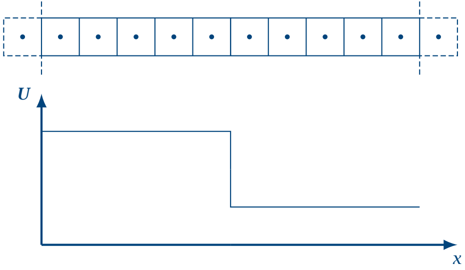
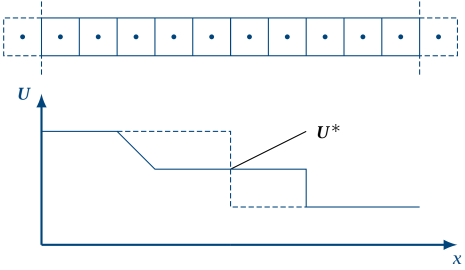


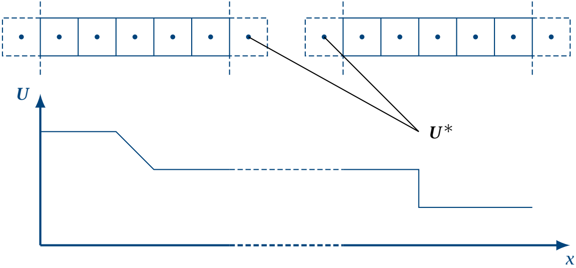
Pipe initialised with a Riemann problem

The Riemann problem



After evolution. At the initial discontinuity, the U^*-state

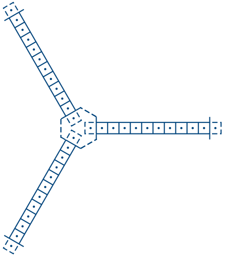
The Riemann problem



- Pipe cut in two at the initial discontinuity,
  - U^* as initial value in the boundary cells.
- The same waves propagate to the left and to the right as in the whole pipe.
- The two half-pipes are coupled using the U^*-state.

The generalised Riemann problem

- We can couple 2 pipes and get the same behaviour as if we had a single pipe.
  - U^*-state is the only information needed to couple them
- Can we find a U^*-state for more than 2 pipes?



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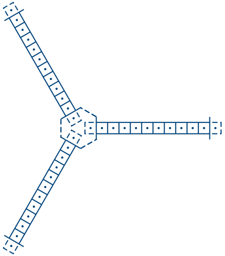
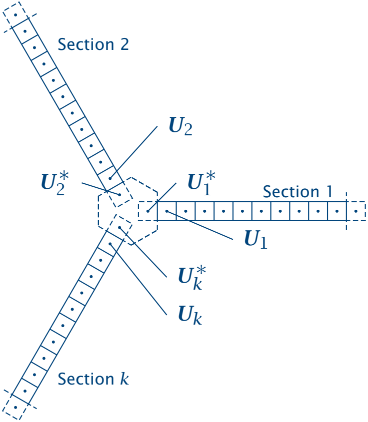
  - Yes, but slightly more complicated



The generalised Riemann problem

- Each pipe section has its own $\boldsymbol{U}_k^*$-state,
- following the conditions:



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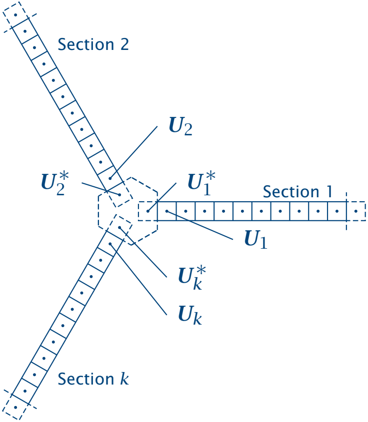
    - $\Rightarrow$ Wave equation (positive speed or stationary)



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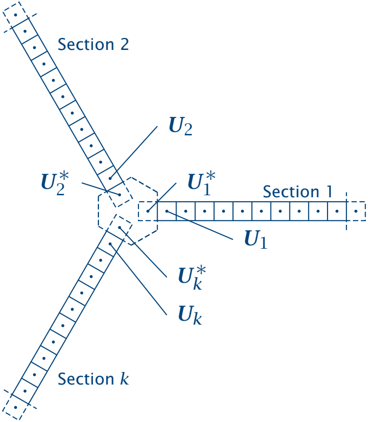
  - $\boldsymbol{U}_1^*, \ldots, \boldsymbol{U}_N^*$ are related together
    - $\Rightarrow$ Junction condition



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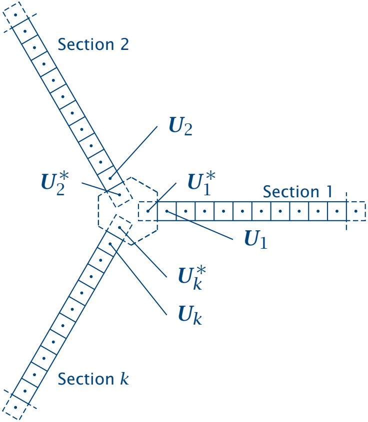
- Reminder: the junction has no volume



The coupling conditions

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  - It is a vector quantity
  - Conserved as a scalar in 1D-models
  - Conserved as a vector in 3D

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  - Conserved as a vector in 3D
  - Junctions are 3D objects

The coupling conditions

Momentum condition expressed as a coupling constant

$$\text{For all } k,\ \mathcal{H}\left(\rho_k^*, v_k^*\right) = \widetilde{\mathcal{H}}$$

- The quantity $\mathcal{H}\left(\rho_k^*, v_k^*\right)$, function of the $\boldsymbol{U}_k^*$-state,
- is equal to a unique $\widetilde{\mathcal{H}}$ *for all the pipe sections*.
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Momentum is a vector quantity

- If the flow derives from a scalar potential field, it is possible to return to a scalar coupling condition
- The Bernoulli invariant is the scalar potential
- It is the stagnation enthalpy $h + \frac{1}{2}v^2$

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In term of coupled quantity $\mathcal{H}\left(\rho, v\right)$

$$\mathcal{H}_{BI}(\rho, v) = h + \frac{1}{2}v^2 = \frac{k\gamma}{\gamma - 1}\rho^{\gamma-1} + \frac{1}{2}v^2$$

- The flow is virtually brought to a rest at the junction
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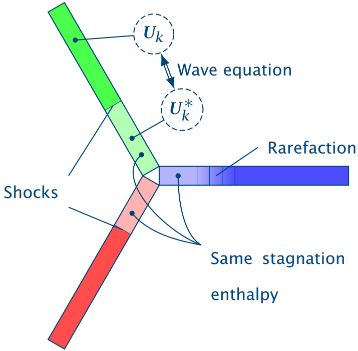
The coupling conditions

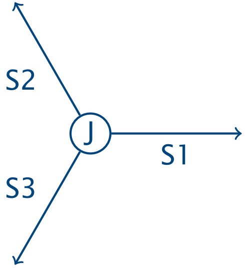
To summarise

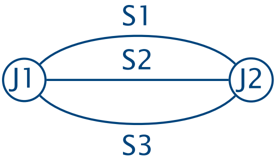
- Need to find *the* stagnation enthalpy (identical for all $\boldsymbol{U}_k^*$)
- Such that, in each pipe section k, $\boldsymbol{U}_k^*$ and $\boldsymbol{U}_k$ are related by the relevant wave equation
- One stagnation enthalpy, N different $\boldsymbol{U}_k^* = \left(\rho_k^*, u_k^*\right)$



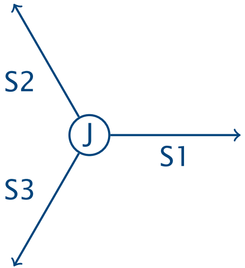
Numerical results

- Examples of why the Bernoulli-based coupling is right
  - Entropy condition at an isolated junction
  - Energy in a closed system
- Simulated with a Roe scheme





Entropy condition in a junction



- Three pipe sections
  - Junction at one end
  - Extrapolation at the other end (infinite pipe)
- Initialised with v_3 either $0\,\mathrm{m/s}$ or $50\,\mathrm{m/s}$.
- The junction reaches steady state

Initial conditions

	Pressure (bar)	Velocity (m/s)
Section 1	1	0
Section 2	1.5	0
Section 3	1.4	v_3

Entropy condition in a junction

Entropy condition

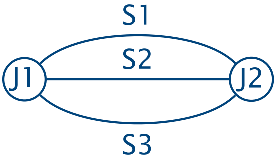
$$\sigma_{\mathrm{J}} = \sum_{k=1}^{N} A_k \rho_k^* v_k^* \left(h_k^* + \frac{1}{2} {v_k^*}^2 \right) \leq 0$$

Value of the entropy condition at steady state

	Equal pressure	Momentum flux	Stagnation enthalpy
$v_3 = 0\,\mathrm{m/s}$	$1.1 \times 10^5\,\mathrm{J/s}$	$-8.2 \times 10^4\,\mathrm{J/s}$	$\approx 0\,\mathrm{J/s}$
$v_3 = 50\,\mathrm{m/s}$	$-6.6 \times 10^4\,\mathrm{J/s}$	$9.8 \times 10^4\,\mathrm{J/s}$	$\approx 0\,\mathrm{J/s}$

Energy balance in a closed system

- Three pipe sections
  - Junctions at each end
- The system's energy content is followed
  - Should decrease, because shocks dissipate energy



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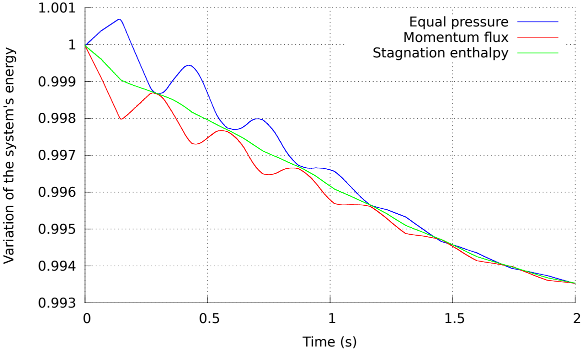

Evolution of the total energy content of the system

⇒ Wrong coupling constants in the junctions cause a non-physical production of energy.

Summary

- Using the wrong coupling quantity breaks the laws of physics
  - In particular, energy conservation
- Rather theoretical derivation, proved for isentropic Euler equations
- Coupling multiphase flow models with real thermodynamics
  - Physical interpretation hints that stagnation enthalpy should play a role
- Energy is a scalar quantity: same conservation principle as mass?

Proofs in:
Gunhild Allard Reigstad, *Mathematical modelling of fluid flows in pipe networks*, Doctoral theses at NTNU, 2014:120

Acknowledgment

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