# Fast Realization of the Modal Vector Fitting Method for Rational Modeling With Accurate Representation of Small Eigenvalues

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Abstract-Admittance-based rational modeling of multiport systems is prone to error magnification in applications with high-impedance terminations. This problem is overcome by the modal vector fitting method (MVF) which is formulated in terms of modal components with inverse least-squares weighting by the eigenvalue magnitude. A direct realization of MVF is very demanding in computation time and memory requirements. This paper overcomes the performance deficiency via three steps: 1) the required number of MVF iterations is reduced by precalculating an improved initial pole set via conventional vector fitting with inverse magnitude weighting; 2) the pole identification step is calculated in an efficient manner by solving for only the few essential unknowns while exploiting the sparse matrix structure; and 3) the residue identification step is calculated efficiently by a row-wise solution procedure that takes advantage of symmetry. The approach is demonstrated to give large savings for the modeling of a frequency-dependent network equivalent.

*Index Terms*—Electromagnetic transients, frequency-dependent network equivalent (FDNE), network equivalent, pole-residue model, rational model, simulation, state-space model, vector fitting.

# I. INTRODUCTION

**R** ATIONAL fitting is a powerful technique for wide-band modeling of linear devices and systems. The procedure takes into account the frequency dependent effects in a straightforward manner, and the obtained model can be easily included in general simulation tools such as EMTP, for the simulation of electromagnetic transients.

The fitting process can be carried out using the pole-relocating method known as vector fitting (VF) [1] with recent enhancements [2]–[4] and extensions [5]–[7]. It is both fast and robust, and it produces a model with guaranteed stable poles.

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The extracted model can be subjected to passivity enforcement (perturbation) in order to guarantee a stable time domain simulation [8]–[11].

Multiport systems are often modeled by (rational) pole-residue models, starting from an admittance based description in the frequency domain. In power systems, the admittance matrix Y of components and systems is often characterized by a large ratio between the largest and smallest eigenvalue. For instance, Y of a transmission line has at low frequencies large eigenvalues that correspond to short circuit currents and small eigenvalues that correspond to charging currents. Direct fitting of the matrix elements will often produce a model where the smallest eigenvalues of Y are inaccurately represented, thereby leading to inaccurate model behavior when applied with high-impedance terminations [6]. For the modeling of transmission lines, this problem was overcome in [12] by the Folded Line Equivalent (FLE) which decomposes the admittance matrix into two parts that represent short circuit and open circuit conditions. Unfortunately, this approach is not applicable to general components, for instance frequency-dependent network equivalents (FDNEs).

In order to overcome this problem, the modal vector fitting (MVF) approach was introduced [6]. MVF attempts to minimize the error of the modal components associated with  $\mathbf{Y}$ , rather than the matrix elements. These modal components are explicitly introduced in the VF formulation, thereby allowing to fit each component with relative error control. A disadvantage of the MVF approach is that the least squares (LS) matrix structures associated with the steps of pole-identification and residue-identification are much less favorable than in the VF formulation. This leads to a substantial increase in the computation time, limiting the MVF to relatively small problems.

In this paper, we show three steps which in total gives large savings in computation time and memory requirements compared to the original implementation of MVF. The number of iterations needed with MVF is reduced by computing an improved initial pole set via modal decomposition by a real transformation matrix. The computation time for the pole identification step is reduced by solving for only the few parameters of interest. The computation time for the residue identification step is reduced by decoupling the matrix rows of the LS problem while taking into account the symmetry of  $\mathbf{Y}$ . This fast realization of MVF is demonstrated for the modeling of a frequency dependent network equivalent (FDNE).

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## II. ADMITTANCE-BASED MODELING

## A. Pole-Residue Formulation

The modeling starts from the nodal admittance matrix  $\mathbf{Y}(s)$ , which relates voltages  $\mathbf{v}$  and currents  $\mathbf{i}$  at the terminals (ports) of the device or system.  $\mathbf{Y}$  is a symmetric matrix of dimension  $n \times n$ , where n is the number of terminals

$$\mathbf{i}(s) = \mathbf{Y}(s)\mathbf{v}(s). \tag{1}$$

The objective is to calculate a rational approximation (2) where terms **D** and **E** are possibly zero. The poles  $\{a_m\}$  and residue matrices  $\{\mathbf{R}_m\}$  are either real or come in complex conjugate pairs.  $\{\mathbf{R}_m\}$ , **D** and **E** are symmetrical matrices

$$\mathbf{Y}(s) \cong \mathbf{Y}_{rat}(s) = \sum_{m=1}^{N} \frac{\mathbf{R}_m}{s - a_m} + \mathbf{D} + s\mathbf{E}.$$
 (2)

### B. Vector Fitting

The rational approximation (2) can be calculated using variants of the pole relocating vector fitting (VF) method [1]. The elements of the upper (or lower) triangle of Y are stacked into a vector of elements, f(s). A common pole fitting for f is obtained by solving the least squares (LS) problem (3) with a set of initial poles  $\{a_m\}$ . The equation is normalized by requiring that the sum of the real part of  $\sigma$  be equal to the number of sample points (relaxed VF [2]). After solving (3), the new poles for f(s)are calculated as the zeros of  $\sigma$ , which are obtained by solving the eigenvalue problem (4). (A is a diagonal matrix holding the initial poles, b is a column of ones, and the row-vector  $c^T$  holds the residues of  $\sigma$ .) The new poles are reused as initial poles in an iterative procedure. Finally, the residues for the fitting of **f** are calculated by solving (3a) with  $\sigma$  equal to unity, and the rational model for  $\mathbf{f}$  is converted into the pole-residue model (2) for  $\mathbf{Y}$ by rearrangement of terms. In the pole and residue identification steps, the system matrix columns are scaled to unit length before solving, in order to improve the numerical conditioning

$$\sigma(s)\mathbf{f}(s) \cong \sum_{m=1}^{N} \frac{\mathbf{r}_m}{s - a_m} + \mathbf{d} + s\mathbf{e}$$
(3a)

$$\sigma(s) = \sum_{m=1}^{N} \frac{\tilde{r}_m}{s - a_m} + \tilde{d}$$
(3b)

$$\{a_m\} = eig(\mathbf{A} - \mathbf{b}\tilde{d}^{-1}\mathbf{c}^T).$$
(4)

## C. Error Magnification Problem

A direct application of VF to  $\mathbf{Y}$  will produce a model which behaves accurately when voltages are applied to the terminals. If, however, we apply currents to the terminals, the voltage response is given by the impedance matrix  $\mathbf{Z}$ , i.e., the inverse of  $\mathbf{Y}$ . If  $\mathbf{Y}$  has both large and small eigenvalues, it is likely that the small eigenvalues are corrupted by the fitting process since they are only weakly observable in  $\mathbf{Y}$ . By carrying out the matrix inversion (5), it is observed that the small eigenvalues of  $\mathbf{Y}$ become the large eigenvalues of  $\mathbf{Z}$ . Clearly, a catastrophic error magnification takes place if the small eigenvalues of  $\mathbf{Y}$  are inaccurately represented. It is noted that in an application with some open terminals, zero current is applied to these terminals and so a submatrix of  $\mathbf{Y}$  is effectively inverted in the simulation. Error magnifications will take place if this submatrix has a large eigenvalue ratio

$$\mathbf{v} = \mathbf{Z}\mathbf{i} = \mathbf{Y}^{-1}\mathbf{i} = \left(\mathbf{T}_{Y}\mathbf{\Lambda}_{Y}\mathbf{T}_{Y}^{-1}\right)^{-1}\mathbf{i} = \left(\mathbf{T}_{Y}\mathbf{\Lambda}_{Y}^{-1}\mathbf{T}_{Y}^{-1}\right)\mathbf{i}.$$
(5)

## D. Modal Vector Fitting

In order to overcome the error magnification problem, the modal vector fitting (MVF) [6] was introduced. This approach leads to a model where the eigenvalue contributions to  $\mathbf{Y}$  are fitted with a relative error criterion. That way, all eigenvalues are represented with good accuracy and so the matrix inversion (5) can be carried out without error magnification.

Y (data) is diagonalized into its eigenvectors and eigenvalues (6). Postmultiplying with the eigenvector matrix T gives for each eigenpair  $(\lambda_i, \mathbf{t}_i)$  the modal relation (7). Dividing with the eigenvalue magnitude (8) results in a LS problem where each modal contribution tends to be fitted with an error that is proportional to the size of the eigenvalue

$$\mathbf{Y}_{rat} \cong \mathbf{Y} = \mathbf{T} \mathbf{\Lambda} \mathbf{T}^{-1} \tag{6}$$

$$\mathbf{Y}_{rat}\mathbf{t}_i \cong \lambda_i \mathbf{t}_i, \ i = 1\dots n \tag{7}$$

$$\frac{1}{|\lambda_i|} (\mathbf{Y}_{rat} \mathbf{t}_i - \lambda_i \mathbf{t}_i) \cong 0, \ i = 1 \dots n.$$
(8)

The fitting problem (8) is combined with VF (3), leading to Modal VF (MVF), (9). After solving (9) with an initial pole set, a new pole set is calculated as the zeros of  $\sigma$  by (4), just like in VF. The pole relocation is repeated to convergence, and the residues for the final fitting are calculated by setting  $\sigma$  equal to unity in (9a)

$$\frac{\lambda_i}{|\lambda_i|}\sigma(s)\mathbf{t}_i \cong \frac{1}{|\lambda_i|} \left( \sum_{m=1}^N \frac{\mathbf{R}_m}{s - a_m} + \mathbf{D} + s\mathbf{E} \right) \mathbf{t}_i,$$
  
$$i = 1 \dots n \tag{9a}$$

$$\sigma(s) = \sum_{m=1}^{N} \frac{\tilde{r}_m}{s - a_m} + \tilde{d}.$$
(9b)

The actual solving of (9) is considerably more time consuming than that of VF (3), since the resulting system matrix is more dense. The differences in sparsity structure results from the multiplication with the vector  $\mathbf{t}_i$  on the right side of (9a). A solution to this problem is described in the subsequent sections.

# III. IMPROVING THE INITIAL POLE SET

In order to reduce the number of iterations needed by MVF, we first calculate an improved pole set using conventional VF.

The condition number  $\kappa$  of **Y** (ratio between largest and smallest singular value) is calculated by sweeping over the frequency band of interest. It is noted that the condition number is a precise measure of the eigenvalue ratio since the singular values are equal to the square-root of the eigenvalues of **Y**<sup>H</sup>**Y**.

**Y** is diagonalized at the frequency point  $s_0$  where  $\kappa$  is maximum (10)

$$\mathbf{Y}(s_0) = \mathbf{T}(s_0)\mathbf{\Lambda}(s_0)\mathbf{T}^{-1}(s_0).$$
(10)

The eigenvectors are rotated so as to minimize their imaginary parts in the least squares sense [13] and the imaginary parts are discarded. The obtained (real) transformation  $\mathbf{T}_0$  is applied to  $\mathbf{Y}(s)$  by (11), which gives a near diagonal matrix. The off-diagonal elements are discarded and the diagonal elements are stacked into a single vector  $\lambda_0(s)$ 

$$\boldsymbol{\lambda}_0(s) = diag\left(\mathbf{T}_0^{-1}\mathbf{Y}(s)\mathbf{T}_0\right). \tag{11}$$

The usage of a real and constant transformation matrix in (11)causes the elements of  $\lambda_0$  to be linear combinations of the elements of Y with real coefficients. As a consequence, the elements of  $\lambda_0$  contain the same poles as Y. At the same time,  $\lambda_0$ gives an excellent representation of the eigenvalues of  $\mathbf{Y}$  in the neighborhood of  $s_0$ , i.e., where the eigenvalue ratio is largest. (At other frequencies, the representation of the eigenvalues of Y is less good but this is not critical since the eigenvalue ratio is lower.) It follows that the poles associated with the modes of **Y** are well observable in  $\lambda_0$ , including the small modes. By subjecting the vector  $\lambda_0$  to VF with inverse magnitude weighting, the elements in  $\lambda_0$  tend to be fitted with an error that is proportional to the element magnitude and so the extracted poles are suitable for representing the modes of Y. Thus, application of VF to  $\lambda_0$  is a fast way of obtaining an improved initial pole set for MVF, thereby reducing the required number of iterations by MVF. In addition, the fitting of  $\lambda_0$  gives a good indication of the required order.

#### IV. POLE IDENTIFICATION

# A. Direct Realization

To see how the equations in MVF are built, consider the case of a 2 × 2 Y which has been fitted with a single pole-residue term with zero D and E terms. The matrix product  $\mathbf{Rt}_i$  in (9a) is rewritten as shown in (12), giving a new coefficient matrix where each row corresponds to the same row in R (and Y)

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} t_1 & t_2 & 0 & 0 \\ 0 & 0 & t_1 & t_2 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{21} & R_{22} \end{bmatrix}^T.$$
(12)

It follows that for each eigenpair  $(\lambda_i, \mathbf{t}_i)$ , the building of (9) leads to an equation of the form (13) where each row of  $\mathbf{Y}$  gives one row-partition.  $\mathbf{x}_1$  and  $\mathbf{x}_2$  hold the residues associated with the first and second row of  $\mathbf{R}$ , and  $\mathbf{x}_3$  holds the residues of  $\sigma$ . ( $\mathbf{A}_1$  and  $\mathbf{A}_2$  are in general different due to different rowscalings)

$$\begin{bmatrix} \mathbf{A}_1 & 0 & | & \mathbf{B}_1 \\ 0 & \mathbf{A}_2 & | & \mathbf{B}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \cong \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}.$$
(13)

Table I lists the block sizes when fitting a Y of dimension  $n \times n$  with N pole-residue terms and a nonzero D and E. It is

TABLE I BLOCK STRUCTURE

	MVF	VF
Number of left partition blocks	п	$n^2$
Left partition block size (columns)	n(N+2)	N+2
Right partition block size (columns)	N+2	N+2

seen (left column) that MVF produces n (large) blocks in the left partition of size n(N+2), whereas VF produces  $n^2$  (small) blocks of size N+2. Clearly, MVF produces many more fill-ins than VF, thereby requiring more memory and longer computation time.

# B. Utilization of Symmetry

In VF, the symmetry of **Y** is trivially preserved by fitting only the upper (or lower) triangle of **Y**, thus reducing the number of blocks in the left partition from  $n^2$  to n(n + 1)/2. This leads to a further reduction in computation time and memory requirements.

In the original implementation of MVF [6], symmetry was utilized for reducing the number of free variables. Unfortunately, this results in a coupling between the blocks as shown in (14). In the case of systems with many ports, this leads to a highly irregular system matrix and the solving for (9) becomes time consuming, even when using a sparse solver

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} t_1 & t_2 & 0 \\ 0 & t_1 & t_2 \end{bmatrix} [R_{11} \ R_{12} \ R_{22}]^T.$$
(14)

# C. Fast Realization

The block structure (13) appears also with the conventional VF. Based on an idea in [14], it was shown [4] that the block structure can be utilized for obtaining a very fast solution procedure for the poles (fast FV). This is possible since we are only interested in the free variables associated with  $\sigma$ , which appear in the right partition ( $\mathbf{x}_3$ ). The application to MVF is completely analogous and is described in what follows.

In the fast realization, each row-partition in (13) is initially considered independently. For the first partition we have

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_3 \end{bmatrix} \cong \mathbf{b}_1.$$
(15)

Applying QR-factorization  $(\mathbf{A} = \mathbf{QR})$  gives

$$\begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ 0 & \mathbf{R}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_3 \end{bmatrix} \cong \mathbf{Q}^T \begin{bmatrix} \mathbf{b}_1^1 \\ \mathbf{b}_1^2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1^1 \\ \mathbf{y}_1^2 \end{bmatrix}$$
(16)

where superscripts 1 and 2 denote upper and lower partition of the vector, respectively. Since we are only interested in  $\mathbf{x}_3$  (free variables of  $\sigma$ ) we get

$$\mathbf{R}_{22}\mathbf{x}_3 \cong \mathbf{y}_1^2. \tag{17}$$

Equation (17) is stacked for all row partitions in (13). When fitting a matrix  $\mathbf{Y}$  of n rows we thus get (18). In order to improve the numerical conditioning, the columns of the system

matrix (18) are scaled to unit length before solving for  $\mathbf{x}_3$  via QR-factorization

$$\begin{bmatrix} \mathbf{R}_{22,1} \\ \mathbf{R}_{22,2} \\ \vdots \\ \mathbf{R}_{22,n} \end{bmatrix} \mathbf{x}_{3} \cong \begin{bmatrix} \mathbf{y}_{1}^{2} \\ \mathbf{y}_{2}^{2} \\ \vdots \\ \mathbf{y}_{n}^{2} \end{bmatrix}.$$
(18)

When using relaxation of the nontriviality constraint of  $\sigma$  for improved convergence [2], all elements on the right side of (15) are zero, except for the last element in  $\mathbf{b}_1^n$ . This leads to further savings when forming the right side in (16).

# V. RESIDUE IDENTIFICATION

## A. Direct Realization

In the residue identification step, (9a) is solved with  $\sigma$  equal to unity. This gives

$$\frac{1}{|\lambda_i|} \left( \sum_{m=1}^N \frac{\mathbf{R}_m}{s - a_m} + \mathbf{D} + s\mathbf{E} \right) \mathbf{t}_i \cong \frac{\lambda_i}{|\lambda_i|} \mathbf{t}_i.$$
(19)

In the case of the  $2 \times 2$  example, this leads to the blockequation (20). The blocks  $A_1$  and  $A_2$  are identical to those in (13) but the right side is different

$$\begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \cong \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}.$$
 (20)

In the original implementation of MVF [6], the symmetry of  $\mathbf{Y}$  was again preserved by removing redundancy in the free variables. This reduces the size of (20), but an irregular, coupled matrix structure results; see (14). The coupling requires to solve for all free variables simultaneously and so the computation becomes time consuming.

## B. Fast Realization

In order to reduce computation time, we solve for the rows of **Y** independently while updating the right side to enforce symmetry. To see this, we consider the  $2 \times 2$  example with a single pole-residue term with **D** and **E** equal to zero. This gives for each  $\mathbf{t}_i$  in (19)

$$\frac{1}{|\lambda_i|} \begin{bmatrix} t_1 & t_2 & 0 & 0\\ 0 & 0 & t_1 & t_2 \end{bmatrix} [R_{11} \ R_{12} \ R_{21} \ R_{22}]^T \cong \begin{bmatrix} b_1\\ b_2 \end{bmatrix}.$$
(21)

We first solve for the first row, giving

$$\frac{1}{|\lambda_i|} [t_1 \quad t_2] [R_{11} \quad R_{12}]^T \cong [b_1].$$
(22)

When solving for the next row, we take advantage of the fact that some unknowns are known from the previous row(s)

$$\frac{1}{|\lambda_i|} [t_2] [R_{22}] \cong [b_2] - \frac{1}{|\lambda_i|} R_{12} t_1.$$
(23)

The generalization to an  $n \times n \mathbf{Y}$  with N pole-residue terms and nonzero  $\mathbf{D}$  and  $\mathbf{E}$  is straightforward. Each subsequent row has N + 2 fewer unknowns than the previous row, and the last row has only N + 2 unknowns. In addition to enforcing symmetry, this procedure reduces the computation time considerably.

TABLE II Computational Complexity

Fast MVF	$\sim n^4 N^3$
Fast VF	$\sim n^2 N^3$

## VI. COMPLEXITY ANALYSIS

In the pole identification step, the computation time is dominated by calculating the QR-factorization of the row-partitions (15). The computation time with QR factorization increases with the cube of the number of columns. Since the number of columns equals 2nN (ignoring terms **D** and **E**), and the factorization must be done for n rows, the computation cost is  $\sim n(nN)^3 = n^4N^3$ . In the residue identification step, we need to solve a linear equation with nN columns via QR factorization, n times. If we ignore the fact we take advantage of symmetry (23), we find again a computation cost of  $n^4N^3$ .

For comparison, with VF (with fast realization [4]), the pole identification step is dominated by the QR-factorization step of n(n + 1)/2 blocks of size N, giving a total computation cost of  $\sim n^2 N^3$ . The same computational cost occurs also with the residue identification step.

The computational complexity for MVF and VF is summarized in Table II. Clearly, MVF remains much slower than VF for situations with many terminals. Fortunately, most applications in power systems involves only three terminals, although six terminals is sometimes encountered. The FDNE example in Section VIII is a six-terminal case.

## VII. PASSIVITY ENFORCEMENT

The extracted pole-residue model may easily produce unstable results when included in a time domain simulation, due to passivity violations. We therefore subject the model to passivity enforcement using the Fast Modal Perturbation method (FMP) [11]. FMP is combined with robust iterations [11] and fast passivity assessment by a half-size singularity test matrix (STM) [15] in order to reach a guaranteed passive model. This is achieved by perturbing the eigenvalues of the residue matrices and the **D**-term in (2). In addition, the **E**-term is enforced to be positive real. Usage of FMP has the special advantage that it is formulated in terms of the modes of **Y** in the LS part of the constrained optimization problem. FMP is therefore a counterpart to MVF, allowing to correct passivity violations without corrupting the small eigenvalues of **Y**.

## VIII. EXAMPLE: FDNE MODELING

#### A. Distribution System

Fig. 1 shows a three-phase, 24 kV distribution system. A frequency-dependent network equivalent (FDNE) is to be established in the frequency range 1 Hz–51 kHz with respect to the two buses A and B, in phase-coordinates. Each overhead line and underground cable is modeled by an exact PI-equivalent in the frequency domain, taking into account skin effect in conductors and earth. The system admittance matrix is established and reduced to buses A and B, thereby giving a six-by-six admittance matrix **Y**. The elements of **Y** are shown in Fig. 2.



Fig. 1. Three-phase distribution system. All distances in kilometers.



Fig. 2. Elements of nodal admittance matrix Y (six-port).

## B. Improving the Initial Pole Set

The initial poles for VF are taken as N = 40 poles that are complex conjugate with weak attenuation (24), and with imaginary parts linearly spaced between 1 Hz and 51 kHz

$$a_m = -\alpha_m + j\beta_m, \ a_{m+1} = -\alpha_m - j\beta_m \quad (24a)$$
$$\{\alpha_m\} = 0.01\{\beta_m\}. \quad (24b)$$

In order to obtain an improved pole set, the frequency point  $s_0$  is identified where the condition number of **Y** is maximum, see Fig. 3. At this frequency (1 Hz), **Y** is diagonalized and eigenvalues  $\lambda_0$  are identified by (10) and (11) assuming a constant transformation matrix. These eigenvalues are subjected to rational fitting with a common pole set using 10 VF iterations with relative error control by inverse magnitude weighting. The result (Fig. 4) shows that the eigenvalues have been fitted with a high relative accuracy.

# C. Fast Pole Identification, Fast Residue Identification

The modified poles are used as an improved initial pole set in MVF. Fig. 5 shows the representation of the eigenvalues of  $\mathbf{Y}$  after two MVF iterations. It is observed that the eigenvalues have been fitted with a high relative accuracy.

One might think that the model refinement by MVF is not needed since the modeling via a constant transformation matrix



Fig. 3. Condition number of Y.



Fig. 4. Fitting of  $\lambda_0$  (10 VF iterations, N = 40).



Fig. 5. Fitting of eigenvalues of Y (2 MVF iterations).

is often adequate. This is however not the case in the given example. Fig. 6 compares the eigenvalues of **Y** with those  $(\lambda_0)$  of



Fig. 6. Eigenvalues of **Y** versus  $\lambda_0$ .



Fig. 7. Eigenvalues of G. Passivity enforcement by FMP.

the model with a constant transformation matrix. The agreement is seen to be very good at low frequencies but poor at higher frequencies. This result is a consequence of the transformation matrix being frequency dependent.

#### D. Passivity Enforcement

The identified model was found to have passivity violations at high frequencies, outside the fitting band. This can be observed as negative eigenvalues in  $\mathbf{G} = \operatorname{Re}{\{\mathbf{Y}\}}$ , see Fig. 7. It is observed that application of FMP enforces all eigenvalues to be positive, thereby giving a passive model. The passivity enforcement does however not corrupt the in-band behavior, see the eigenvalue plot in Fig. 8. The fitting errors are only slightly higher than in the original model (Fig. 5).

# IX. TIMING RESULTS

Table III shows timing results for a single MVF iteration using Matlab on a personal computer with a 1.3 GHz Pentium processor. The given example has 40 pole-residue terms, one



Fig. 8. Eigenvalues of Y. Passivity enforcement by FMP.

 TABLE III

 TIME CONSUMPTION (SINGLE ITERATION) (IN SECONDS)

	MVF	FMVF	Ratio
Pole identification	80.2	10.4	7.7
Residue identification	73.8	2.7	27.3
Total	154.0	13.1	11.8

TABLE IV TOTAL MODELING TIME FOR MVF (IN SECONDS)

Calculate improved pole set	1.9
(10 FVF iterations)	
FMVF: (2 iterations)	28.8

**D**-term and 250 frequency samples. The table compares the result by the original realization of MVF [6] and the fast realization presented in this work (FMVF), excluding the time needed for building the equations. It is seen that the fast realization reduces the computation time by order of one magnitude.

Table IV shows the total modeling time, including overhead costs. Calculating the improved pole set via fast VF [4] by 10 iterations required 1.9 s. Two pole identification steps by FMVF and one residue identification step required a further 28.8 s.

# X. RESULT BY DIRECT FITTING APPROACH

## A. Application of Vector Fitting to Matrix Elements

It is useful to compare the result by MVF with that by a conventional approach. Fig. 9 shows the fitting result after application of 10 VF iterations to the elements of  $\mathbf{Y}$ , followed by passivity enforcement by FRP [11]. The fitting used 40 pole-residue terms and a common weighting equal to the inverse of the matrix 2-norm. It is seen that all elements of  $\mathbf{Y}$  have are fitted with a high level of accuracy.

However, Fig. 10 shows that the small eigenvalues of  $\mathbf{Y}$  are corrupted. This happens because the small eigenvalues are only weakly observable in the elements of  $\mathbf{Y}$  and so they become ignored in the fitting process.



Fig. 9. Rational fitting of Y. After passivity enforcement by FRP.



Fig. 10. Eigenvalues of Y. After passivity enforcement by FRP.



Fig. 11. Voltage application with four open terminals.

## B. Error Magnification

The inaccurate representation of the small eigenvalues of  $\mathbf{Y}$  may lead to catastrophic error magnifications as was explained in Section II-C. As an example we will calculate the open circuit voltage on terminals 3–6 when applying a voltage to terminals 1–2, see Fig. 11.

The nodal admittance matrix  $\mathbf{Y}$  of the model is partitioned as shown in (25), where subscripts *a* and *b*, respectively, represent terminals 1–2 and 3–6. The voltage response on terminals 3–6 is now directly given by (26)

$$\begin{bmatrix} \mathbf{i}_a \\ \mathbf{i}_b \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_b \end{bmatrix}$$
(25)

$$\mathbf{v}_B = - \mathbf{B}^{-1} \mathbf{C} \mathbf{v}_A.$$



Fig. 12. Open circuit response.



Fig. 13. System energization.

Fig. 12 compares the calculated voltage responses when the model has been calculated by either MVF or by the direct fitting approach using VF. The identification (x, 1) denotes the response on terminal x when applying one volt on terminal one and zero volt on terminal 2. Similarly, the identification (x, 2)denotes the response on terminal x when applying one volt on terminal two and zero volt on terminal 1. The direct fitting of Y by VF gives a highly inaccurate result for the voltage responses (3,1), (6,1), (3,2), and (6,2). From Fig. 11, these responses are seen to represent the voltage on the unconnected conductor, which at low frequencies is governed by capacitive charging currents. These charging currents approach zero at low frequencies and correspond to small eigenvalues of Y. It is further seen that all voltage responses are well reproduced when the modeling is done via MVF since all eigenvalues are now represented with high relative accuracy.

# XI. TIME DOMAIN RESULTS

The FDNE model is applied in a time domain simulation where bus A (Fig. 1) is energized from a three-phase voltage source behind a short circuit inductance, see Fig. 13. At t =5 ms opens the breaker which is connected to terminal A3. The circuit simulation is done using an EMTP-like program implemented in Matlab, based on trapezoidal integration [16].

Fig. 14 shows the simulated voltage response at terminals B1 and B3, when the modeling is based on either MVF or direct fitting by VF. In addition, the simulation result by PSCAD

(26)



Fig. 14. Time domain simulation.



Fig. 15. Time domain simulation. Extended time scale.

is shown when each overhead line and cable has been modeled by the highly accurate Universal Line Model (ULM) [17], known as "phase domain model" in PSCAD. It is seen that the PSCAD simulation gives a result which closely matches those by the FDNE approaches. However, after the breaker opens at t = 5 ms, the trapped charge voltage on the third conductor (B3) becomes highly incorrect as simulated by the FDNE extracted via VF. This is clearly seen in Fig. 15, where the simulation time has been extended to 40 ms. On the other hand, the FDNE model that was obtained via MVF remains accurate. It is noted that this simulation example is closely related to the example in Section X-B, where the direct modeling approach was demonstrated to be unable to represent the voltage response on the third conductor.

## XII. DISCUSSION

# A. Computational Efficiency

The fast realization of MVF was demonstrated to greatly reduce the computation time for one example of FDNE modeling. Here, the fast realization reduced the computation time of each



Fig. 16. RMS-error as function of iteration count.



Fig. 17. RMS-error as function of iteration count. Expanded view.

MVF iteration by an order of magnitude. In addition, pre-calculation of an improved initial pole set via conventional VF reduced the number of iterations needed by MVF. The modeling of a six-port case with 40 pole-residue terms and 2 MVF iterations required about 30 s which is still about 10 times slower than a modeling by VF. Nevertheless, MVF can now be applied to medium scale examples without excessive computation time.

The significance of precalculating an improved initial pole set can be better appreciated by considering the (weighted) RMSerror of (9a) with  $\sigma = 1$ 

$$RMS = \frac{1}{\sqrt{n^2 N_s}} \sum_{i=1}^{n} \left\| \frac{1}{|\lambda_i|} \left( \sum_{m=1}^{N} \frac{\mathbf{R}_m}{s - a_m} + \mathbf{D} \right) \mathbf{t}_i - \frac{\lambda_i}{|\lambda_i|} \mathbf{t}_i \right\|_2.$$
(27)

Fig. 16 shows the rms-error as function of the number of iterations by MVF where iteration "0" denotes the result without pole relocation, i.e., only residue calculation. The expanded view in Fig. 17 shows that one needs about seven MVF iterations to reach the same error achieved with pole improvement and a single MVF iteration. For the given example, one could even skip the MVF pole identification steps altogether since the iterations do not reduce the fitting errors appreciably. The total modeling time would then become 1.9 + 2.7 = 4.6 s (Tables III and IV). Clearly, the pole improvement step greatly contributes to reducing the required number of MVF iterations and thus the total computation time.

In the case of FDNEs, the eigenvalue ratio will normally be highest at DC. The use of frequency sweeping (Section III) for locating the frequency  $s_0$  to be used for diagonalization (11) can therefore be avoided, thereby saving computation time. In other applications, the highest ratio may occur at a non-dc frequency, thereby requiring identification of  $s_0$  via frequency sweeping.

# B. Robustness

In the steps of pole identification (Section IV) and residue identification (Section V), the resulting linear systems are solved using QR-decomposition with rank revealing column pivoting. This approach, which is implemented via the back-slash (" $\$ ") operator in Matlab, gives excellent results also in situations when the problem is not of full numerical rank.

Usage of Normal Equations with Gaussian elimination is an alternative way of improving the computational efficiency of MVF. However, that procedure is numerically less robust and may lead to inaccurate results.

# C. Passivity

The main advantage of MVF is that it produces a model where the small eigenvalues of  $\mathbf{Y}$  are accurately represented, thereby avoiding the error magnification phenomenon that may occur in situations with high-impedance terminations. In practice one can often avoid the error magnification problem also with conventional VF by using a sufficiently high fitting order, thereby capturing the small eigenvalues buried in  $\mathbf{Y}$ . Unfortunately, such an approach often leads to large out-of-band passivity violations that cannot be removed without corrupting the in-band behavior.

The fast modal perturbation method (FMP) [11] is the counterpart to MVF for passivity enforcement. It has the capability of correcting passivity violations without corrupting small eigenvalues of  $\mathbf{Y}$ . Thus, MVF and FMP form together an excellent way of modeling multiport systems for use with high-impedance terminal conditions as they produce a model where the modal components of  $\mathbf{Y}$  are represented with a relative accuracy. This property was demonstrated in Section XI to prevent the occurrence of error magnifications in a time domain simulation. The computation time by FMP is usually much smaller that of one MVF iteration, since FMP perturbs only a reduced set of free variables (the residue matrix eigenvalues).

## XIII. CONCLUSIONS

This paper has introduced a fast realization of the modal vector fitting method (MVF) for rational modeling of multiport systems with relative error control of the eigenvalues of the admittance matrix  $\mathbf{Y}$ .

 The required number of iterations is kept as low as possible. This is achieved by diagonalizing the nodal admittance matrix via a constant real transformation matrix. The eigenvalues are fitted using conventional vector fitting with inverse magnitude weighting, thereby obtaining an improved pole set with a small computational cost.

- 2) In MVF, the pole identification step is made faster and less demanding in memory by calculating only the free variables associated with  $\sigma$  while exploiting the matrix structure. This permits to establish a compact system matrix from the solution of several sub-problems.
- The final residue identification step is made fast by independently solving for the rows of Y. A further reduction in computation time is achieved by utilization of symmetry.

The fast MVF realization was demonstrated for the modeling of a six-terminal 40th order FDNE. The extracted model allowed to simulate a trapped charge, closely matching the result by a detailed PSCAD model. A direct modeling approach using conventional vector fitting gave a highly incorrect result.

#### REFERENCES

- B. Gustavsen and A. Semlyen, "Rational approximation of frequency domain responses by vector fitting," *IEEE Trans. Power Del.*, vol. 14, no. 3, pp. 1052–1061, Jul. 1999.
- [2] B. Gustavsen, "Improving the pole relocating properties of vector fitting," *IEEE Trans. Power Del.*, vol. 21, no. 3, pp. 1587–1592, Jul. 2006.
- [3] D. Deschrijver, B. Haegeman, and T. Dhaene, "Orthonormal vector fitting: A robust macromodeling tool for rational approximation of frequency domain responses," *IEEE Trans. Adv. Packag.*, vol. 30, no. 2, pp. 216–225, May 2007.
- [4] D. Deschrijver, M. Mrozowski, T. Dhaene, and D. De Zutter, "Macromodeling of multiport systems using a fast implementation of the vector fitting method," *IEEE Microw. Wireless Compon. Lett.*, vol. 18, no. 6, pp. 383–385, Jun. 2008.
- [5] S. Grivet-Talocia, "Package macromodeling via time-domain vector fitting," *IEEE Microw. Wireless Compon. Lett.*, vol. 13, no. 11, pp. 472–474, Nov. 2003.
- [6] B. Gustavsen and C. Heitz, "Modal vector fitting: A tool for generating rational models of high accuracy with arbitrary terminal conditions," *IEEE Trans. Adv. Packag.*, vol. 31, no. 4, pp. 664–672, Nov. 2008.
- [7] N. Wong and C.-U. Lei, "IIR approximation of FIR filters via discretetime vector fitting," *IEEE Trans. Signal Process.*, vol. 56, no. 3, pp. 1296–1302, Mar. 2008.
- [8] B. Gustavsen and A. Semlyen, "Enforcing passivity for admittance matrices approximated by rational functions," *IEEE Trans. Power Syst.*, vol. 16, no. 1, pp. 97–104, Feb. 2001.
- [9] D. Saraswat, R. Achar, and M. S. Nakhla, "Enforcing passivity for rational function based macromodels of tabulated data," in *Proc. Electrical Performance of Electronic Packaging*, 2003, pp. 295–298.
- [10] S. Grivet-Talocia, "Passivity enforcement via perturbation of Hamiltonian matrices," *IEEE Trans. Circuits Syst. I*, vol. 51, no. 9, pp. 1755–1769, Sep. 2004.
- [11] B. Gustavsen, "Fast passivity enforcement for pole-residue models by perturbation of residue matrix eigenvalues," *IEEE Trans. Power Del.*, vol. 23, no. 4, pp. 2278–2285, Oct. 2008.
- [12] B. Gustavsen and A. Semlyen, "Admittance-based modeling of transmission lines by a folded line equivalent," *IEEE Trans. Power Del.*, vol. 24, no. 1, pp. 231–239, Jan. 2009.
- [13] V. Brandwajn, "Modification of user's instructions for 'MARTI SETUP'," *EMTP Newslett.*, vol. 3, no. 1, pp. 76–80, Aug. 1982.
- [14] D. Traina, G. Macchiarella, and T. K. Sarkar, "Robust formulations of the Cauchy method for microwave duplexers modeling," *IEEE Trans. Microw. Theory Tech.*, vol. 55, no. 5, pp. 974–982, May 2007.
- [15] A. Semlyen and B. Gustavsen, "A half-size singularity test matrix for fast and reliable passivity assessment of rational models," *IEEE Trans. Power Del.*, vol. 24, no. 1, pp. 345–351, Jan. 2009.
- [16] B. Gustavsen and O. Mo, "Interfacing convolution based linear models to an electromagnetic transients program," in *Proc. Int. Conf. Power Systems Transients*, 2007, pp. 1–6.
- [17] A. Morched, B. Gustavsen, and M. Tartibi, "A universal model for accurate calculation of electromagnetic transients on overhead lines and underground cables," *IEEE Trans. Power Del.*, vol. 14, no. 3, pp. 1032–1038, Jul. 1999.
- [18] P. Triverio, S. Grivet-Talocia, M. S. Nakhla, F. G. Canavero, and R. Achar, "Stability, causality, and passivity in electrical interconnect models," *IEEE Trans. Adv. Packag.*, vol. 30, no. 4, pp. 795–808, Nov. 2007.

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