AN IMPROVED ROE SOLVER FOR THE DRIFT-FLUX TWO-PHASE FLOW MODEL

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ABSTRACT
This paper concerns the numerical solution of a hyperbolic system of conservation laws describing two-phase flow in a pipeline. The selected model is a one dimensional drift-flux model consisting of two mass conservation equations, one momentum conservation equation and one slip function relating the velocities of each phase. The approximate Riemann solver of Roe (1981) is used due to its robustness and relative simplicity, and an improved Roe solver compared with the one shown in Flåtten and Munkejord (2006) is presented. Along with the model, some relevant numerical examples are presented to illustrate the accuracy and robustness of the method.

Keywords: Multiphase pipe flow, Drift-flux model, Roe scheme.

NOMENCLATURE

Greek Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>αg</td>
<td>Volume fraction</td>
<td>[–]</td>
</tr>
<tr>
<td>β</td>
<td>Wave strength</td>
<td>[–]</td>
</tr>
<tr>
<td>Δx</td>
<td>Grid length</td>
<td>[m]</td>
</tr>
<tr>
<td>ε</td>
<td>Minimum gas volume fraction in the moving Gauss curve</td>
<td>[–]</td>
</tr>
<tr>
<td>Φ</td>
<td>Slip relation</td>
<td>[m/s]</td>
</tr>
<tr>
<td>η</td>
<td>Dynamic viscosity</td>
<td>[Pa.s]</td>
</tr>
<tr>
<td>κ</td>
<td>Compressibility parameter</td>
<td>[m³/kg.s²]</td>
</tr>
<tr>
<td>λ</td>
<td>Eigenvector of the flux Jacobi matrix</td>
<td>[m/s]</td>
</tr>
<tr>
<td>μ</td>
<td>Position of initial maximum point in the moving Gauss curve</td>
<td>[m]</td>
</tr>
<tr>
<td>μg</td>
<td>Partial derivative of slip relation with respect to gas volumetric mass</td>
<td>[m³/kg.s]</td>
</tr>
<tr>
<td>μl</td>
<td>Partial derivative of slip relation with respect to liquid volumetric mass</td>
<td>[m³/kg.s]</td>
</tr>
<tr>
<td>μv</td>
<td>Partial derivative of slip relation with respect to gas phase velocity</td>
<td>[–]</td>
</tr>
<tr>
<td>θ</td>
<td>Measure of the smoothness of a characteristic component of the solution</td>
<td>[–]</td>
</tr>
<tr>
<td>ρg</td>
<td>Mass density</td>
<td>[kg/m³]</td>
</tr>
<tr>
<td>ρl</td>
<td>Pseudo mass</td>
<td>[kg/m³]</td>
</tr>
<tr>
<td>σ</td>
<td>Deviation in the moving Gauss curve</td>
<td>[m]</td>
</tr>
<tr>
<td>ζ</td>
<td>Partial derivative of slip relation with respect to liquid phase velocity</td>
<td>[–]</td>
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</table>

Latin Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>A(q)</td>
<td>Flux Jacobi matrix</td>
</tr>
<tr>
<td>Δ</td>
<td>Linearised Roe matrix</td>
</tr>
<tr>
<td>Q</td>
<td>Vector of fluxes</td>
</tr>
<tr>
<td>P</td>
<td>Vector containing the conserved variables</td>
</tr>
<tr>
<td>R</td>
<td>Right eigenvector</td>
</tr>
<tr>
<td>s(q)</td>
<td>Vector of sources</td>
</tr>
<tr>
<td>W</td>
<td>Wave</td>
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Sub/superscripts

<table>
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<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>0</td>
<td>Indicator of reference state</td>
</tr>
<tr>
<td>g</td>
<td>Gas phase</td>
</tr>
<tr>
<td>i</td>
<td>Cell index</td>
</tr>
<tr>
<td>k</td>
<td>Indicator of phase, l=liquid, g=gas</td>
</tr>
<tr>
<td>ℓ</td>
<td>Liquid phase</td>
</tr>
<tr>
<td>p</td>
<td>Wave number</td>
</tr>
<tr>
<td>R</td>
<td>Grid on the right hand side of a grid interface</td>
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</table>

INTRODUCTION

The selected pipe flow model for two phase flows has many applications, including oil and gas transport, nuclear engineering, CO₂ capture and storage and the modelling of heat exchangers. The numerical solution is obtained using an approximate Riemann solver of Roe (1981). This is a convenient upwind finite volume method due to its robustness and relative simplicity. The solver is also easily extended to second-order accuracy for smooth solutions through the wave-limiting approach of LeVeque (2007). The use of a finite volume method ensures that physically conserved variables are also numerically conserved.

The parameter vector approach suggested by Roe (1981) to obtain the solver requires a certain level of algebraic sim-
plicity of the equation system. The slip relation and the thermodynamic closure relations in a drift-flux model generally have a complex structure which makes the approach unfeasible.

Toumi and Caruge (1998) used a weak formulation of the approximate Riemann solver of Roe in order to overcome this challenge. In this approach the Jacobian matrix is made dependent on a smooth path linking the left and right states of a grid interface in addition to the states themselves. The Roe solver was applied on a three-dimensional drift-flux model. Romate (1997) established a matrix satisfying the Roe conditions by a numerical approach. Based on an intermediate condition dependent on the left and right states, the Jacobian matrix was identified and its eigenvalues and eigenvectors were calculated. The Jacobian matrix was then represented by the eigenvector matrix, its inverse and the matrix containing the eigenvalues of the Jacobian matrix along its diagonal. By modifying the diagonal matrix, the Roe conditions could be satisfied.

In a previous work (Flåtten and Munkejord, 2006), an alternative approach for constructing an analytical Roe solver for the drift-flux model was presented. Herein, the problem was independent approach for constructing an analytical Roe solver for the drift-flux model. Along with the improved model, numerical examples relevant for industrial challenges related to multiphase pipeline transport are presented. These illustrate the accuracy and robustness of the method.

THE DRIFT-FLUX MODEL

The drift-flux model consists of two equations for conservation of mass, one for each phase, and one equation for the conservation of total momentum as shown in equations (1) to (4).

\[
\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = s(q)
\]

\[
q = \begin{bmatrix}
\rho_\alpha v_\alpha \\
\rho_\mu v_\mu \\
\rho_\ell v_\ell \\
\end{bmatrix}
\]

\[
f(q) = \begin{bmatrix}
\rho_\alpha v_\alpha v_\alpha \\
\rho_\mu v_\mu v_\mu \\
\rho_\ell v_\ell v_\ell \\
\end{bmatrix}
\]

\[
s(q) = \begin{bmatrix}
m_\ell \\
l_\ell \\
-F_w \\
\end{bmatrix}
\]

The volume fractions satisfy:

\[
\alpha_\ell + \alpha_\mu = 1
\]

Thermodynamic submodel

The drift-flux model presented above is based on the assumption of isentropic and isothermal flow. Hence dynamic mass and energy transfer between the phases are neglected. A result of this assumption is that the pressure may be found as:

\[
p = p(\rho_\ell) = p(\rho_\mu)
\]

Thus, the thermodynamic model relates the phase density to the common pressure according to equation (7).

\[
\rho_k = \rho_{k,0} + \frac{p - p_{k,0}}{c_k^2}
\]

The variables \(p_{k,0}\) and \(c_k^2\) are defined in equation (8) and (9) respectively.

\[
p_{k,0} = p(\rho_{k,0})
\]

\[
c_k^2 = \frac{\partial p}{\partial \rho_k}(\rho_{k,0})
\]

For convenience, the model is implemented in the form of equation (10), where the variable \(p_k^0\) is defined by equation (11).

\[
p_k = c_k^2 (p_k - p_k^0)
\]

\[
p_k^0 = \rho_{k,0} - \frac{p_{k,0}}{c_k^2}
\]

Hydrodynamic submodel

In addition to the equations (1) to (4), an equation relating the liquid and gas velocities to each other is needed. The slip relation is defined as \(\Phi = v_g - v_\ell\), and in general it is presented on the form:

\[
v_g - v_\ell = \Phi(m_g, m_\ell, v_g)
\]

In the present work, two different slip relations are implemented, the no slip relation, eq. (13), and the Zuber-Findlay slip relation, eq. (14).

\[
\Phi = 0
\]

\[
\Phi = \frac{(K-1)v_g + S}{K \alpha_\ell}
\]

The Zuber-Findlay slip relation is valid for slug and bubbly flow regimes, and \(K\) and \(S\) are flow dependent constants. The models are implemented because of their simplicity. However, as the slip relation for the various flow regimes may be far more complex, use of the analytic expression for the slip relation is avoided in the derived Roe averages used in this work.

THE ROE NUMERICAL SCHEME

If the flux functions are smooth in all independent variables such that the partial derivatives exists, the equation system in equation (1) may be written in a quasi-linear form as in equation (15).

\[
\frac{\partial q}{\partial t} + A(q) \frac{\partial q}{\partial x} = s(q)
\]

The Roe scheme is based upon a replacement of the Jacobian matrix \(A\), with a matrix \(\tilde{A}\) containing averaged values for each grid interface. Hence the non-linear system is approximated by a locally linearised system:

\[
\frac{\partial \tilde{q}}{\partial t} + \tilde{A}_{-1/2} \frac{\partial \tilde{q}}{\partial x} = \tilde{s}(q)
\]

In Flåtten and Munkejord (2006), the Jacobian matrix, \(A\), was derived as:

\[
A = \frac{1}{\hat{\rho}} \begin{bmatrix}
m_\ell m_\ell v_\ell + \zeta m_\ell v_\ell & m_\ell m_\ell v_\ell - m_\ell v_\ell & m_\ell \\-(m_\ell m_\ell v_\ell + \zeta m_\ell v_\ell) & m_\ell v_\ell - m_\ell m_\ell v_\ell & \zeta m_\ell \\a_{31} & a_{32} & a_{33} \end{bmatrix}
\]

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\]

\[
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m_\ell m_\ell v_\ell + \zeta m_\ell v_\ell & m_\ell m_\ell v_\ell - m_\ell v_\ell & m_\ell \\-(m_\ell m_\ell v_\ell + \zeta m_\ell v_\ell) & m_\ell v_\ell - m_\ell m_\ell v_\ell & \zeta m_\ell \\a_{31} & a_{32} & a_{33} \end{bmatrix}
\]
where
\[ a_{31} = \kappa \hat{p} \rho \dot{t} + 2m_g m_t \mu_g (v_g - v_t) + (\zeta m_t - m_g) v_g^2 \]
\[ -2\zeta m_t v_g v_t \]
\[ a_{32} = \kappa \hat{p} \rho \dot{g} + 2m_g m_t \mu_g (v_g - v_t) \]
\[ - (\zeta m_t - m_g) v_g^2 - 2m_g v_g v_t \]
\[ a_{33} = 2(m_g v_g + \zeta m_t) \]
(18)
(19)
(20)

The variables \( \mu_g, \mu_t \) and \( \mu_s \) are partial derivatives of the slip function with respect to volumetric mass, liquid volumetric mass and gas phase velocity respectively. \( \zeta \) is the partial derivative of the gas velocity with respect to liquid velocity. The pseudo mass \( \hat{p} \) is defined as:
\[ \hat{p} = m_g + \zeta m_t \]
(21)
\[ \kappa \]
\[ \frac{1}{(\partial_c p_c) \rho_c (\partial_c) + (\partial_c p_c) \rho_c \alpha e} \]
(22)

In accordance with the Jacobian matrix, the linearised matrix, \( \hat{A} \), is defined as:
\[ \hat{A} = \frac{1}{\hat{p}} \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \hat{m_g} \\ -\hat{a}_{11} & -\hat{a}_{12} & L \hat{m_t} \\ \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} \end{bmatrix} \]
(23)
\[ \hat{a}_{11} = m_g \hat{m_t} \hat{m_g} + \zeta \hat{m_t} \hat{v_g} \\ \hat{a}_{12} = m_g \hat{m_t} \hat{m_t} - m_t \hat{v_g} \\ \hat{a}_{31} = \kappa \hat{p} \hat{p} \hat{m_t} + 2m_g m_t \mu_g \hat{(v_g - v_t)} + (\hat{\zeta} m_t - m_g) \hat{v_g}^2 \\ -2\zeta \hat{m_t} \hat{v_g} \hat{v_t} \\ \hat{a}_{32} = \kappa \hat{p} \hat{p} \hat{m_t} + 2m_g m_t \mu_t \hat{(v_g - v_t)} - (\hat{\zeta} m_t - m_g) \hat{v_t}^2 \\ -2m_g \hat{v_g} \hat{v_t} \\ \hat{a}_{33} = 2(m_g \hat{v_g} + \hat{\zeta} m_t) \hat{v_t} \]
(24)
(25)
(26)
(27)
(28)

The \( \hat{A}_{1/2} \) matrix has to fulfill three conditions:

**Condition 1** \( \hat{A}_{1/2} \) must be diagonalisable and have real eigenvalues

**Condition 2** \( \hat{A}_{1/2} \rightarrow f'(q) \) as \( Q_{1/2} \rightarrow q \)

**Condition 3** \( \hat{A}_{1/2}(Q_{ci} - Q_{ci-1}) = f(Q_{ci}) - f(Q_{ci-1}) \)

Condition 1 ensures that system (16) is hyperbolic. Condition 2 is required in order for the method to be consistent with the original conservation law. The last condition is proposed based on a desire of having \( \psi / p \) as an eigenvector of \( \hat{A}_{1/2} \) if the states \( Q_{ci-1} \) and \( Q_{ci} \) are connected by a single wave \( \psi / p = Q_{ci} - Q_{ci-1} \) in the true Riemann solution. For the particular expressions for fluctuations selected in this work (see eq. (67)), it will also guarantee that the numerical method is conservative. (LeVeque, 2007)

According to Theorem 2 in Flätten and Munkejord (2006), the Roe matrix system can be divided into sub-systems in order to simplify the derivation of the averaged variables. The selected sub-systems are:

**Equations for conservation of mass**

\[ \hat{A}_m = \frac{1}{\rho} \begin{bmatrix} m_g \mu_t & \zeta \hat{m_t} v_g \\ -m_g \mu_t & \hat{m_t} v_t \end{bmatrix} \]
(29)

where
\[ f_m(q) = \begin{bmatrix} m_g v_g \\ m_t v_t \end{bmatrix} \]
(30)

with the corresponding equation for Roe condition 3:
\[ \hat{A}_m (Q^R - Q^L) = f_m(Q^R) - f_m(Q^L) \]
(31)

In equation (31), the condition in cell \( i \) is labelled with \( R \) as it is on the right hand side of the cell interface. Similarly the condition in cell \( (i-1) \) is labelled with \( L \). This notation will be used in the rest of this section.

**Equations for conservation of momentum**

**Gas momentum convection**

\[ \hat{A}_g = \frac{1}{\tilde{\rho}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
(32)

\[ f_g(q) = \begin{bmatrix} 0 \\ 0 \\ m_g \hat{v}_g \end{bmatrix} \]
(33)

where
\[ \tilde{\rho}_{g,31} = 2m_g \hat{m_t} \hat{v}_g \hat{\mu}_g + (\zeta \hat{m_t} - m_g) \hat{v}_g^2 \]
(34)
\[ \tilde{\rho}_{g,32} = 2m_g \hat{m_t} \hat{v}_g \hat{\mu}_t - 2m_g \hat{v}_g \hat{v}_t \]
(35)

The Roe condition number 3 for the gas momentum conservation equations is expressed as:
\[ \hat{A}_g (Q^R - Q^L) = f_g(Q^R) - f_g(Q^L) \]
(36)

**Liquid momentum convection**

\[ \hat{A}_l = \frac{1}{\rho} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
(37)

\[ f_l(q) = \begin{bmatrix} 0 \\ 0 \\ m_t v_t \end{bmatrix} \]
(38)

where
\[ \tilde{\rho}_{l,31} = -(2m_g \hat{m_t} \hat{v}_g \hat{\mu}_g + 2\zeta \hat{m_t} \hat{v}_g \hat{v}_t) \]
(39)
\[ \tilde{\rho}_{l,32} = -(2m_g \hat{m_t} \hat{v}_g \hat{\mu}_t + (\zeta \hat{m_t} - m_g) \hat{v}_t^2) \]
(40)

The 3rd Roe condition expressed as:
\[ \hat{A}_l (Q^R - Q^L) = f_l(Q^R) - f_l(Q^L) \]
(41)

**Pressure terms**

\[ \hat{A}_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
(42)

\[ f_p(q) = \begin{bmatrix} 0 \\ 0 \\ \hat{\rho} \end{bmatrix} \]
(43)

with Roe condition number 3 expressed as:
\[ \hat{A}_p (Q^R - Q^L) = f_p(Q^R) - f_p(Q^L) \]
(44)
Derivation of averaged parameters

Derivation of \( \hat{\alpha}_k, \hat{\alpha}_z, \hat{\rho}_f, \) and \( \hat{\rho}_g \)

The averaged volume fractions, \( \hat{\alpha}_k, \hat{\alpha}_z, \) and densities, \( \hat{\rho}_f \) and \( \hat{\rho}_g \), are found from the Roe condition 3 applied on the pressure sub-system, eq. (44). As in Flåtten and Munkejord (2006), \( \tilde{k} \) is set equal to

\[
\tilde{k} = \frac{1}{(\hat{\alpha}_f \hat{\rho}_g + (\hat{\alpha}_f \hat{\rho}_f) \hat{\rho}_f) \hat{\alpha}_f} \tag{45}
\]

This leads to the equation

\[
\hat{\rho}_f (m^R_f - m^L_f) + \hat{\rho}_g (m^R_g - m^L_g) = p^R - p^L \tag{46}
\]

For \( \tilde{\alpha}_f \hat{\rho}_f \) the expression suggested by Flåtten and Munkejord (2006) is used:

\[
\tilde{\alpha}_f \hat{\rho}_f = \left\{ \begin{array}{ll}
\frac{\rho_f^R - \rho_f^L}{p_f^R - p_f^L} & p_f^R \neq p_f^L \\
(\tilde{\alpha}_f \hat{\rho}_f)^L & p_f^R = p_f^L
\end{array} \right. \tag{47}
\]

Entering the expression for \( \tilde{\alpha}_f \hat{\rho}_f \) into equation (46), the requirement for the averaged variables can be expressed as:

\[
\hat{\rho}_f (m^R_f - m^L_f) + \hat{\rho}_g (m^R_g - m^L_g) + \hat{\rho}_z \hat{\alpha}_z (\rho_z^R - \rho_z^L) + \hat{\rho}_f \hat{\alpha}_f (\rho_f^R - \rho_f^L) \tag{48}
\]

This equation is satisfied by arithmetic averages for volume fractions and densities:

\[
\hat{\alpha}_f = \frac{1}{2}(\alpha_f^L + \alpha_f^R) \tag{49}
\]

\[
\hat{\alpha}_z = \frac{1}{2}(\alpha_z^L + \alpha_z^R) \tag{50}
\]

\[
\hat{\rho}_f = \frac{1}{2}(\rho_f^L + \rho_f^R) \tag{51}
\]

\[
\hat{\rho}_g = \frac{1}{2}(\rho_g^L + \rho_g^R) \tag{52}
\]

Here, it was used that:

\[
m_k = \rho_f \alpha_k \tag{53}
\]

Derivation of \( \hat{\mu}_g, \hat{\mu}_l, \) and \( \hat{\nu}_v \)

In Flåtten and Munkejord (2006), the derivation of averaged volumetric masses and velocities from the set of equations in (31) resulted in the criteria:

\[
\mu_g (m^R_g - m^L_g) + \mu_l (m^R_l - m^L_l) + \mu_v (v^R_v - v^L_v) = \Phi^R - \Phi^L \tag{54}
\]

In the Roe scheme presented here we will use this as a starting point. Hence the averages derived in Flåtten and Munkejord (2006) may be kept:

\[
\hat{\mu}_g = \begin{cases} 
\Phi(m_g^R, m_g^L, v_g^L) - \Phi(m_g^R, m_g^L, v_g^L) & \text{for } m_g^L \neq m_g^R \\
\mu_g(m_g^L, m_g^L, v_g^L) & \text{for } m_g^L = m_g^R
\end{cases} \tag{55}
\]

\[
\hat{\mu}_l = \begin{cases} 
\Phi(m_l^R, m_l^L, v_l^L) - \Phi(m_l^R, m_l^L, v_l^L) & \text{for } m_l^L \neq m_l^R \\
\mu_l(m_l^R, m_l^L, v_l^L) & \text{for } m_l^L = m_l^R
\end{cases} \tag{56}
\]

\[
\hat{\mu}_v = \begin{cases} 
\Phi(m_v^R, m_v^L, v_v^L) - \Phi(m_v^R, m_v^L, v_v^L) & \text{for } v_v^L \neq v_v^L \\
\mu_v(m_v^R, m_v^L, v_v^L) & \text{for } v_v^L = v_v^L
\end{cases} \tag{57}
\]

Derivation of \( \hat{\mu}_g, \hat{\mu}_l, \) and \( \hat{\nu}_v \)

Inserting (54) into the first row of the equation system (31) gives the following condition:

\[
\hat{\mu}_g \hat{\mu}_l \left( \hat{\zeta} (v_g^R - v_g^L) - (v_l^R - v_l^L) \right) + \hat{\mu}_v (m_g^R - m_g^L) - \hat{\nu}_v (m_l^R - m_l^L) + \hat{\nu}_g (m_g^R - m_g^L) \left( \hat{v}_g^R - \hat{v}_g^L \right) \tag{58}
\]

A second equation is found by adding the equations for Roe condition 3 for the gas and liquid momentum, eq. (36) and eq. (41), and using equation (54):

\[
2 \hat{\mu}_g \hat{\mu}_l (v_g^R - v_g^L) \left( \hat{\zeta} (v_g^R - v_g^L) - (v_l^R - v_l^L) \right) + \left( (\hat{\zeta} \hat{\mu}_l - \hat{\mu}_g) \hat{v}_g^2 - 2 \hat{\zeta} \hat{\mu}_l \hat{v}_g \hat{v}_l \right) (m_g^R - m_g^L) - \left( (\hat{\zeta} \hat{\mu}_l - \hat{\mu}_g) \hat{v}_l^2 + 2 \hat{\mu}_g \hat{v}_g \hat{v}_l \right) (m_l^R - m_l^L) + (\hat{\mu}_g + \hat{\mu}_l) \left( m_g^R (v_g^R)^2 + m_g^L (v_g^L)^2 - m_l^R (v_l^R)^2 - m_l^L (v_l^L)^2 \right) \tag{59}
\]

By inserting (58) into (59), the equation may be simplified to:

\[
\hat{v}_g^2 (m_g^R - m_g^L) - 2 \hat{v}_g (m_g^R (v_g^R) - m_g^L (v_g^L)) + m_l^R (v_l^R)^2 - m_l^L (v_l^L)^2 = 0 \tag{60}
\]

The equation may be satisfied by the averages:

\[
\hat{v}_g = \sqrt{m_g^R v_g^L + m_g^R v_g^R} \tag{61}
\]

and

\[
\hat{v}_l = \sqrt{m_l^R v_l^L + m_l^R v_l^R} \tag{62}
\]

By reformulating equation (58) to:

\[
\hat{\mu}_l \hat{\mu}_l \left( m_l^R (v_l^R - v_l^L) + \hat{v}_l (m_l^R - m_l^L) - (m_g^R (v_g^R) - m_g^L (v_g^L)) \right) - \hat{\mu}_g \hat{\mu}_l (m_l^R (v_l^R - v_l^L) + \hat{v}_l (m_l^R - m_l^L)) - (m_l^R (v_l^R - m_l^L v_l^L)) = 0 \tag{63}
\]

it is seen that the conservation of mass is satisfied by the averages:

\[
\hat{m}_g = \sqrt{m_g^R m_g^L}, \tag{64}
\]

\[
\hat{m}_l = \sqrt{m_l^R m_l^L} \tag{65}
\]

when \( \hat{v}_g \) and \( \hat{v}_l \) are given by eq. (61) and (62).

**NUMERICAL ALGORITHM**

The numerical algorithm is similar to the one selected in Flåtten and Munkejord (2006). It is based on Godunov’s method, applied on non-linear systems and with the use of limiters in
order to increase the accuracy of smooth solutions (LeVeque, 2007):

\[
\begin{align*}
Q_{i+1}^n &= Q_i^p - \frac{\Delta t}{\Delta x} \left( \phi^+ \Delta Q_i^{i+1/2} + \phi^- \Delta Q_i^{i-1/2} \right) \\
&\quad - \frac{\Delta t}{\Delta x} \left( \tilde{F}_{i+1/2} - \tilde{F}_{i-1/2} \right) + \Delta t s_i Q_i^n
\end{align*}
\]

(66)

The source term has here been added on the right hand side of the equation for simplicity reasons. This will make the overall algorithm first order accurate. Using e.g. a Strang splitting algorithm to handle this term would increase the overall order of accuracy to two.

The fluctuations \( \phi^\pm \Delta Q_{i-1/2} \) are found as:

\[
\phi^\pm \Delta Q_{i-1/2} = \sum_{p=1}^{m} (s^p_{i-1/2})^\pm \mathbf{W}_i^{p,1/2}
\]

(67)

where

\[
(s^+_{i-1/2}) = \max(0,s^p_{i-1/2})
\]

(68)

\[
(s^-_{i-1/2}) = \min(0,s^p_{i-1/2})
\]

(69)

\[
s^p_{i-1/2} = \lambda^p_{i-1/2}
\]

(70)

As mentioned earlier, the selected fluctuations will give a conservative method due to condition 3. (LeVeque, 2007)

The waves, \( \mathbf{W}_{i-1/2} \), may be calculated from the eigenvectors of the Roe matrix \( \mathbf{A}_{i-1/2} \):

\[
\mathbf{Q}_i - \mathbf{Q}_{i-1} = \sum_{p=1}^{m} \mathbf{W}_i^{p,1/2} = \sum_{p=1}^{m} \beta_i^{p,1/2} \mathbf{W}_i^{p,1/2}
\]

(71)

\[
\beta_{i-1/2} = \tilde{\mathbf{R}}_{i-1/2} (\mathbf{Q}_i - \mathbf{Q}_{i-1})
\]

(72)

The correction flux is defined in equation (73) with the limited wave \( \tilde{\mathbf{W}}_{i-1/2} \) defined as in equation (74).

\[
\tilde{F}_{i-1/2} = \frac{1}{2} \sum_{p=1}^{m} \left| s^p_{i-1/2} \right| \left( 1 - \frac{\Delta t}{\Delta x} |s^p_{i-1/2}| \right) \tilde{\mathbf{W}}_{i-1/2}^{p,1/2}
\]

(73)

\[
\tilde{\mathbf{W}}_{i-1/2} = \phi(\theta_{i-1/2}) \mathbf{W}_{i-1/2}
\]

(74)

\[
\theta_{i-1/2} = \frac{\mathbf{W}_{i-1/2}^{p,1/2}}{\mathbf{W}_{i-1/2}^{p,1/2} - \mathbf{W}_{i-1/2}^{p,1/2}}
\]

(75)

where

\[
I = \begin{cases} 
  i - 1, & s^p_{i-1/2} \geq 0 \\
  i + 1, & s^p_{i-1/2} < 0
\end{cases}
\]

(76)

The monotonized central-difference limiter (MC limiter), eq. (77), was chosen.

\[
\phi(\theta) = \max(0,\min((1+\theta)/2,2,2\theta))
\]

(77)

**NUMERICAL RESULTS**

The numerical results from three different cases are presented. All cases are simulated with a CFL number of 0.5. (CFL = \( \max_j |\lambda_j| \Delta t / \Delta x \), \( \lambda_j \) is the \( j \)-th eigenvalue of the Jacobian matrix, \( \mathbf{A} \), \( \Delta t \) is the length of the time step and \( \Delta x \) is the size of the grid cells.) The phase velocities are related by the no-slip relation in the two first cases. In the third case, the Zuber-Findlay slip relation is used. This is also the only case where wall friction is included. In the two first cases the friction is neglected, e.g. \( F_w = 0 \).

<table>
<thead>
<tr>
<th>Quantity Symbol</th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volumetric mass:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Phase 1</td>
<td>( m_1 )</td>
<td>3.17123</td>
</tr>
<tr>
<td>Phase 2</td>
<td>( m_2 )</td>
<td>3.38324</td>
</tr>
<tr>
<td>Total volumetric momentum</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case: A Riemann problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sonic speeds, ( c_k )</td>
</tr>
<tr>
<td>Case 1: ( c_1 = c_2 )</td>
</tr>
<tr>
<td>Case 2: ( c_1 &lt; c_2 )</td>
</tr>
<tr>
<td>Case 2: ( c_1 &gt; c_2 )</td>
</tr>
</tbody>
</table>

**Figure 1: Riemann problem - Phase 1 volumetric mass**

The results at \( t = 0.06s \) are presented in figure 1 to figure 4. The discontinuity is initially positioned at \( x = 0m \) and for clarity, the figure only shows the section of the tube containing waves. The simulation is run with 1000 grid cells. Compared to the results in Banda et al. (2010), the waves have similar shapes and the results seem reasonable. However, the position of the wave-fronts and the amplitudes of the momentum waves differ. The largest differences are seen for the two cases \( \alpha_1^2 > \alpha_2^2 \) and \( \alpha_1^2 < \alpha_2^2 \). For these cases, the pressure levels presented in the article are wrong compared to the published initial data. It seems therefore that the authors of the article have presented results for the two cases that do not correspond with the data they provide. This is most likely the reason for the discrepancies that are observed.
Case: Moving Gauss curve - Investigation of model accuracy

The convergence order of smooth solutions has been verified using a constructed test case found in Munkejord et al. (2009). The initial volume fraction profile in a 12 meter long tube is according to equation (78), and it is moving along the tube with the speed of the liquid and vapour.

\[
\alpha_{g,0} = (1 - 2e)^{\exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)} + e \tag{78}
\]

![Graph showing the initial conditions for the moving Gauss curve case](image1)

Figure 2: Riemann problem - Phase 2 volumetric mass

![Graph showing the total volumetric momentum](image2)

Figure 3: Riemann problem - Total volumetric momentum

![Graph showing the pressure](image3)

Figure 4: Riemann problem - Pressure

Table 3: Initial data for the moving Gauss curve

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol (unit)</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gas volume fraction</td>
<td>(\alpha_{g})</td>
<td>(\alpha_{g,0})</td>
</tr>
<tr>
<td>Pressure</td>
<td>(p) (kPa)</td>
<td>100</td>
</tr>
<tr>
<td>Gas velocity</td>
<td>(v_g) (m/s)</td>
<td>100</td>
</tr>
<tr>
<td>Liquid velocity</td>
<td>(v_l) (m/s)</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 4: Equation of state parameters for the moving Gauss curve case

<table>
<thead>
<tr>
<th>Gas (g)</th>
<th>(\rho_g^k) (kg/m^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquid (l)</td>
<td>1000</td>
</tr>
</tbody>
</table>

where \(e = 1 \times 10^{-12}\), \(\mu = 6\) m and \(\sigma = 0.42\) m.

The simulation is run for 0.03 seconds with periodic boundary conditions. As seen in table 3, the fluid velocity in the pipe is 100 m/s. Hence at the end of the simulation time, the analytical solution is a volume fraction profile which is symmetric around \(x = 9\) m. This is shown as the analytical result in figure 5 and figure 6. The parameters presented in table 4 were used in the equation of state for the simulation.

The results presented in figure 5 and figure 6 illustrate the differences in the results introduced by the MC limiter. The main difference is the accuracy obtained when using the MC limiter, even for a fairly low number of grid cells. This is the main reason for the focus on the very small region in figure 6. The figures also show that while the original simulation is symmetric around \(x = 9\) m, the results from the simulation with limiter are nonsymmetrical. This is due to the nature of the limiter method (Munkejord et al., 2009).

The convergence order is calculated by finding the error in the calculated gas volume fraction as (Munkejord et al., 2009):

\[
\|\Delta \varepsilon (\alpha_g, \Delta x)\|_1 = \Delta x \sum_{\forall j} |\alpha_{g,j} - \alpha_{g,ref,j}| \tag{79}
\]

The error for two simulations with grid size \(\Delta x_1\) and \(\Delta x_2\) are then compared in order to determine the convergence order (Munkejord, 2005):

\[
n = \frac{\ln(\|\Delta \varepsilon (\alpha_g, \Delta x_2)\|_1)}{\ln(\|\Delta \varepsilon (\alpha_g, \Delta x_1)\|_1)} \tag{80}
\]

Table 5 shows the estimated errors and convergence order for selected grid sizes. As expected, the numerical scheme with limiter approaches an order of 2, while the scheme without limiter approaches an order of 1.

Table 5: Convergence order calculated from simulation results

<table>
<thead>
<tr>
<th>(\Delta x)</th>
<th>Without limiter</th>
<th>With MC limiter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>|\Delta \varepsilon (\alpha_g)|_1</td>
<td>|\Delta \varepsilon (\alpha_g)|_1</td>
</tr>
<tr>
<td>0.0015</td>
<td>1.109 \times 10^{-1}</td>
<td>1.571 \times 10^{-3}</td>
</tr>
<tr>
<td>0.0075</td>
<td>5.852 \times 10^{-2}</td>
<td>4.077 \times 10^{-4}</td>
</tr>
<tr>
<td>0.00375</td>
<td>3.011 \times 10^{-2}</td>
<td>1.028 \times 10^{-4}</td>
</tr>
<tr>
<td>0.001875</td>
<td>1.528 \times 10^{-2}</td>
<td>2.598 \times 10^{-5}</td>
</tr>
<tr>
<td>0.0009375</td>
<td>7.695 \times 10^{-3}</td>
<td>6.525 \times 10^{-6}</td>
</tr>
</tbody>
</table>
Case: Pipe flow
In the pipe flow case, a pipe which is 1000m long is initially filled with stagnant liquid at a pressure of 1bar. A small gas fraction of $\alpha_g = 1 \times 10^{-5}$ flows with a velocity corresponding to the slip relation (14). The relation is used with $K$ as a constant of value 1.0. $S$ is expressed as a function of the volume fraction:

$$S = S(\alpha_g) = \frac{1}{2} \sqrt{1 - \alpha_g}$$  \hspace{1cm} (81)

At the outlet boundary, the pressure is kept constant at 1bar. At the pipe inlet, the flow rate of gas and liquid are varied:

- **Inlet liquid flow rate**: increased linearly from zero to 12.0kg/s from $t = 0s$ to $t = 10s$, kept constant from $t = 10s$ to $t = 175s$.

- **Inlet gas flow rate**: increased linearly to 0.08kg/s from $t = 0s$ to $t = 10s$, decreased linearly to $1 \times 10^{-4}$kg/s from $t = 50s$ to $t = 70s$ and kept constant for the rest of the simulation until $t = 175s$.

The equation of state parameters used in the case are found in table 6. In this case, wall friction is also included. This is modelled as

$$F_w = \frac{32\nu_m \eta_m}{d^2}$$  \hspace{1cm} (82)

At the pipe inlet, the flow rate of gas and liquid are varied:

- **Inlet liquid flow rate**: increased linearly from zero to 12.0kg/s from $t = 0s$ to $t = 10s$, kept constant from $t = 10s$ to $t = 175s$.

- **Inlet gas flow rate**: increased linearly to 0.08kg/s from $t = 0s$ to $t = 10s$, decreased linearly to $1 \times 10^{-4}$kg/s from $t = 50s$ to $t = 70s$ and kept constant for the rest of the simulation until $t = 175s$.

The results correspond well with the ones presented in Flåtten and Munkejord (2006).

### Table 6: Equation of state parameters for the pipe flow problem

<table>
<thead>
<tr>
<th></th>
<th>$c_w$ (m/s)</th>
<th>$\rho_l^0$ (kg/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gas (g)</td>
<td>$\sqrt{10^5}$</td>
<td>0</td>
</tr>
<tr>
<td>Liquid (l)</td>
<td>1000</td>
<td>999.9</td>
</tr>
</tbody>
</table>

where $\nu_m$ is the mixture velocity defined as

$$\nu_m = \alpha_g \nu_g + \alpha_l \nu_l$$  \hspace{1cm} (83)

$\eta_m$ is the dynamic mixture viscosity defined as

$$\eta_m = \alpha_g \eta_g + \alpha_l \eta_l$$  \hspace{1cm} (84)

d is the tube diameter, set to 0.1m. The constants $\eta_g$ and $\eta_l$ are set to $\eta_g = 5 \times 10^{-6}$Pa.s, $\eta_l = 5 \times 10^{-2}$Pa.s. Results for the time $t = 175s$ is presented in figure 7 to figure 10. The results correspond well with the ones presented in Flåtten and Munkejord (2006).

**Figure 7: Pipe flow case - Pressure**

**Figure 8: Pipe flow case - Gas volume fraction**

### CONCLUSION

A simplified analytical Roe scheme for a drift-flux, two-phase flow model is derived. The work is based on a previous work, Flåtten and Munkejord (2006), where the simplification is to introduce only one set of averaged velocities.
for each phase. The robustness of the scheme, and the possibility of extending to second order accuracy for smooth solutions by introducing wave-limiters are illustrated by the three different numerical examples presented.

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REFERENCES

Figure 9: Pipe flow case - Gas velocity

Figure 10: Pipe flow case - Liquid velocity