A Multiscale Mixed Finite-Element Solver for Compressible Black-Oil Flow

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Two-level methods for equations:
- with a near-elliptic behavior
- with strongly heterogeneous coefficients
- without scale separations

Aim:
- describe global flow patterns on coarse grid
- accurately account for fine-scale structures

*Provide a mechanism to recover approximate fine-scale solutions*
The Multiscale Mixed Finite Element (MsMFE) Method

The algorithm in a nutshell

1) Generate coarse grid (automatically)

- 9 different coarse blocks
- 44,927 cells
- 148 blocks

Solve flow problem for all pairs of blocks
The Multiscale Mixed Finite Element (MsMFE) Method
The algorithm in a nutshell

1) Generate coarse grid (automatically)

2) Detect all adjacent blocks

- 44,927 cells
- 148 blocks

3) Compute basis functions

- Solve flow problem for all pairs of blocks

4) Build global solution

- Basis functions: building blocks for global solution
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Solve flow problem for all pairs of blocks

Basis functions: building blocks for global solution
The Mixed Finite Element (MsMFE) Method
Computation of multiscale basis functions

Each cell $\Omega_i$: pressure basis $\phi_i$
Each face $\Gamma_{ij}$: velocity basis $\psi_{ij}$

\[ \vec{\psi}_{ij} = -\lambda K \nabla \phi_{ij} \]

\[ \nabla \cdot \vec{\psi}_{ij} = \begin{cases} w_i(x), & x \in \Omega_i \\ -w_j(x), & x \in \Omega_j \\ 0, & \text{otherwise} \end{cases} \]
The Mixed Finite Element (MsMFE) Method

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\end{cases}$$
The Mixed Finite Element (MsMFE) Method

Interpretation of the weight function

The weight function distributes $\nabla \cdot v$ on the coarse blocks:

$$(\nabla \cdot v)|_{\Omega_i} = \sum_j \nabla \cdot (v_{ij} \psi_{ij}) = w_i \sum_j v_{ij}$$

$$= w_i \int_{\partial \Omega_i} v \cdot n \, ds = w_i \int_{\Omega_i} \nabla \cdot v \, dx$$

Different roles:

Incompressible flow: $\nabla \cdot v = q$
Compressible flow: $\nabla \cdot v = q - c_t \partial_t p - \sum_j c_j v_j \cdot \nabla p$
The Mixed Finite Element (MsMFE) Method

Choice of weight function, \( w_i = \frac{\theta(x)}{\int_{\Omega_i} \theta(x) \, dx} \)

Incompressible flow:

\[
\int_{\Omega_i} q \, dx = 0, \quad \theta(x) = \text{trace}(K(x))
\]

\[
\int_{\Omega_i} q \, dx \neq 0, \quad \theta(x) = q(x)
\]

Compressible flow:

\( \theta \propto q \): compressibility effects concentrated where \( q \neq 0 \)

\( \theta \propto K \): \( \nabla \cdot v \) over/underestimated for high/low \( K \)

Another choice motivated by physics:

\( \theta(x) = \phi(x) \)

Motivation:

\( \partial p / \partial t \propto \phi \)
The Mixed Finite Element (MsMFE) Method

Choice of weight function, \( w_i = \theta(x) / \int_{\Omega_i} \theta(x) \, dx \)

**Incompressible flow:**

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\int_{\Omega_i} q \, dx = 0, \quad \theta(x) = \text{trace}(K(x))
\]

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**Compressible flow:**

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Another choice motivated by physics:

\[
\theta(x) = \phi(x), \quad \text{Motivation: } c_t \frac{\partial p}{\partial t} \propto \phi
\]
The Mixed Finite Element (MsMFE) Method
Key to efficiency: reuse of computations

Computational cost consists of:

- **basis functions** (fine grid)
- **global problem** (coarse grid)

High efficiency for multiphase flows:

- Elliptic decomposition
- Reuse basis functions
- Easy to parallelize

**Example: 128^3 grid**

<table>
<thead>
<tr>
<th># operations versus upscaling factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basis functions</td>
</tr>
<tr>
<td>8x8x8</td>
</tr>
</tbody>
</table>

Fine scale solution (AMG) $O(n^{1.2})$
The Mixed Finite Element (MsMFEM) Method
Recap from 2007 SPE RSS: million-cell models in minutes

SPE 10, Model 2:

Water-cut curves at the four producers

grid: $60 \times 220 \times 85$
Coarse grid: $5 \times 11 \times 17$
2000 days production
25 time steps

multiscale + streamlines: 142 sec on a 2.4 GHz PC

- upscaling/downscaling, - multiscale, - fine grid
MsMFE for Complex Grids

Challenges posed by grids from real-life models

Unstructured grids:

(Very) high aspect ratios:

800 × 800 × 0.25 m

Skewed and degenerate cells:

Non-matching cells:
MsMFE for Complex Grids
Applicable to general unstructured grids

Coarse blocks: (arbitrary) connected collection of cells
→ fully automated coarsening strategies

Coarse blocks: logically Cartesian in index space
Coarse blocks: (arbitrary) connected collection of cells

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MsMFE for Complex Grids
Fine-grid formulation

Discretization using a mimetic method (Brezzi et al):

\[
\begin{align*}
\mathbf{u}_E &= \lambda \mathbf{T}_E (p_E - \pi_E), \\
\mathbf{T}_E &= |E|^{-1} \mathbf{N}_E \mathbf{K}_E \mathbf{N}_E^T + \tilde{\mathbf{T}}_E
\end{align*}
\]

- \(\mathbf{N}_E\): face normals
- \(\mathbf{X}_E\): vector from face to cell centroids
- \(\tilde{\mathbf{T}}_E\): arbitrarily such that \(\tilde{\mathbf{T}}_E \mathbf{X}_E = 0\)

Key features:

- Applicable for general polyhedral cells
- Non-conforming grids treated as conforming polyhedral
- Generic implementation for all grid types
- Monotonicity as for MPFA
MsMFE for Complex Grids

Example: single phase, homogeneous $K$, linear pressure drop
MsMFE for Compressible Black-Oil Models
Fine-grid formulation

Pressure equation:

\[ c \frac{\partial p}{\partial t} + \nabla \cdot \vec{u} - \zeta \vec{u} \cdot \mathbf{K}^{-1} \vec{u} = q, \quad \vec{u} = -\mathbf{K} \lambda \nabla p \]

Time-discretization and linearization:

\[ c_{\nu-1} \frac{p_{\nu} - p_{\nu-1}}{\Delta t} + \nabla \cdot \vec{u}_{\nu} - \zeta_{\nu-1} \vec{u}_{\nu-1} \cdot \mathbf{K}^{-1} \vec{u}_{\nu} = q \]

Hybrid system:

\[
\begin{bmatrix}
B & C & D \\
C^T - V_{\nu-1}^T & P_{\nu-1} & 0 \\
D^T & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\vec{u}_{\nu} \\
-p_{\nu} \\
\pi_{\nu}
\end{bmatrix}
= \begin{bmatrix}
0 \\
P_{\nu-1}p_{\nu-1}^{n-1} + q
\end{bmatrix}
\]
MsMFE for Compressible Black-Oil Models

Coarse-grid formulation

\[
\begin{bmatrix}
\Psi^T B \Psi & \Psi^T C I & \Psi^T D J \\
\tilde{C}^T & I^T P I & 0 \\
J^T D^T \Psi & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
u \\
-p \\
\pi \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
I^T P p_f^n \\
0 \\
\end{bmatrix}
\]

\(\Psi\) – velocity basis functions
\(\Phi\) – pressure basis functions
\(I\) – prolongation from blocks to cells
\(J\) – prolongation from block faces to cell faces
\(\tilde{C} = \Psi^T (C - V) I - D_\lambda \Phi^T P I\)

New feature: fine-scale pressure

\[u_f \approx \Psi u, \quad p_f \approx I p + \Phi D_\lambda u, \quad D_\lambda = \text{diag}(\lambda_i^0/\lambda_i)\]
MsMFE for Compressible Black-Oil Models

Example 1: tracer transport in gas (Lunati & Jenny 2006)

constant $K$

$$p(0, t) = 1 \text{ bar, } p(x, 0) = 10 \text{ bar, coarse grid: 5 blocks, fine grid: 100 cells}$$

lognormal $K$
MsMFE for Compressible Black-Oil Models

Example 1: tracer transport in gas (Lunati & Jenny 2006)

\[ p(0, t) = 1 \text{ bar}, \quad p(x, 0) = 10 \text{ bar}, \quad \text{coarse grid: 5 blocks, fine grid: 100 cells} \]

Remedy: correction functions (Lunati, Jenny et al; Nordbotten)
Approximate residual equation by

\[ \hat{u} = \sum_{\Omega_i \subseteq \Omega} \hat{u}_i, \quad \hat{p} = \sum_{\Omega_i \subseteq \Omega} \hat{p}_i, \]

such that \( u \approx u_{\text{ms}} + \hat{u} \) and \( p \approx p_{\text{ms}} + \hat{p} \).

Local problems:

\((\hat{u}_i, \hat{p}_i)\) solves residual equation locally in \( \hat{\Omega}_i \) such that

- Zero right-hand-side in \( \hat{\Omega}_i \setminus \Omega_i \)
- Zero flux BCs on \( \partial \hat{\Omega}_i \)
MsMFE for Compressible Black-Oil Models

Example 1: tracer transport in gas (Lunati & Jenny 2006)

Non-overlapping correction:
MsMFE for Compressible Black-Oil Models

Example 1: tracer transport in gas (Lunati & Jenny 2006)

Overlapping $O(H/2)$ correction:
MsMFE for Compressible Black-Oil Models

Example 2: block with a single fault


1000 m³/day water injected into compressible oil at 205 bar (p_{bh} of 200 bar).
Conclusions and Outlook

The MsMFE method:
- is flexible with respect to grids
- allows automated coarsening
- requires correction functions for compressible flow

Future research:
- adaptivity of basis/correction functions
- parallelization
- error estimation (via VMS framework)