A Discontinuous Galerkin Method for Computing Flow in Porous Media

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Outline

Motivation

Aim: Construct a fast method to compute flow in porous media
Method: Discontinuous Galerkin Method (DGM)

- reservoir flow
- groundwater flow

The Time-Of-Flight Equation

- Fluids flow with velocity **v** obtained from Darcy's law,

$$\mathbf{v} = -\frac{\mathbf{K}}{\mu}\nabla p$$

- The time-of-flight of a particle along a streamline, Ψ:

$$T(x) = \int_{\Psi} \frac{ds}{|\mathbf{v}(\mathbf{x}(s))|}$$

- The time-of-flight is the solution of a boundary value problem:

$$\mathbf{v}(x) \cdot \nabla T = 1, \quad T = 0 \text{ on } \Gamma^{+}$$

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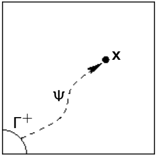

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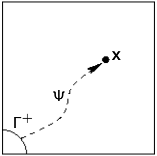

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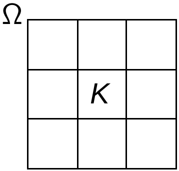
Solution Space

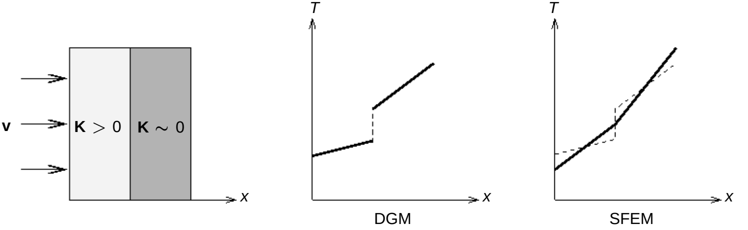


- Space for approximate solution T_h:

$$V_h^{(n)} = \{\varphi : \varphi|_K \in \mathbb{Q}^{(n-1)}\},$$

where $\mathbb{Q}^n = \mathrm{span}\{x^p y^q : 0 \le p, q \le n\}$

- No continuity across inter-element boundaries



Variational Formulation

For all elements K, and for all $\varphi \in C_\infty(K)$:

$$\mathbf{v} \cdot \nabla T = 1$$

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For all elements K, and for all $\varphi \in C_\infty(K)$:

$$\mathbf{v} \cdot \nabla T \varphi = 1\varphi$$

Variational Formulation

For all elements K, and for all $\varphi \in C_\infty(K)$:

$$\int_K \mathbf{v} \cdot \nabla T \varphi \, dxdy = \int_K \varphi \, dxdy$$

Variational Formulation

For all elements K, and for all $\varphi \in C_\infty(K)$:

$$\int_{\partial K} T \ \varphi \ \mathbf{v} \cdot \mathbf{n}_K ds - \int_K T \ \mathbf{v} \cdot \nabla \varphi \ dxdy = \int_K \varphi \ dxdy$$

Variational Formulation

For all elements K, and for all $\varphi_h \in V_h$:

$$\int_{\partial K} T_h \varphi_h \mathbf{v} \cdot \mathbf{n}_K \, ds - \int_K T_h \mathbf{v} \cdot \nabla \varphi_h \, dxdy = \int_K \varphi_h \, dxdy$$

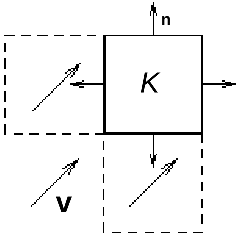
Variational Formulation

For all elements K, and for all $\varphi_h \in V_h$:

$$\int_{\partial K} \hat{f}(T_h, T_h^{ext}, \mathbf{v} \cdot \mathbf{n}_K)\varphi_h ds - \int_K T_h \mathbf{v} \cdot \nabla \varphi_h \, dxdy = \int_K \varphi_h \, dxdy$$

Numerical Flux Function

- The numerical flux function depends only on the values of T_h at the discontinuities



The numerical flux function:

$$\hat{f}(T_h, T_h^{ext}, \mathbf{v} \cdot \mathbf{n}_K) = T_h \max(\mathbf{v} \cdot \mathbf{n}_K, 0) + T_h^{ext} \min(\mathbf{v} \cdot \mathbf{n}_K, 0)$$

Solution Procedure

$$\int_{\partial K} \hat{f}(T_h, T_h^{ext}, \mathbf{v}\cdot\mathbf{n}_K)\varphi_h ds - \int_K T_h \mathbf{v}\cdot\nabla\varphi_h dxdy = \int_K \varphi_h dxdy$$

$$F_K(T) \quad - \quad R_K T_K \quad = \quad B_K$$

Solution Procedure

- The upwind flux can be written

$$F_K(T) = F_K^+ T_K + F_K^- T_{\Omega\setminus K},$$

where F_K^+ approximates the flux out of each element and F_K^- the flux entering from neighbour elements
- The system may then be written as

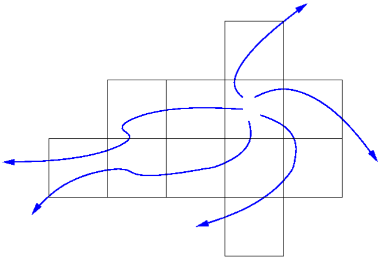
$$F_K^+ T_K \quad - \quad R_K T_K \quad = \quad B_K \quad - \quad F_K^- T_{\Omega\setminus K}$$

Reordering

- An elementwise solution is possible by exploiting the causality of the equation
- This sequence can be computed before solving the resulting system (using a depth-first search)
- Reduction in runtime:

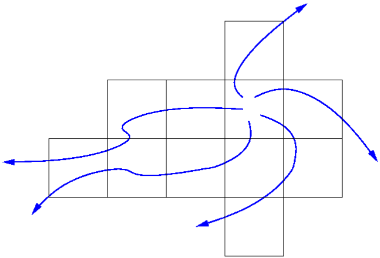
$$Nm \times Nm \text{ system} \longrightarrow N \text{ systems of size } m \times m$$

Elementwise solution



A few grid cells and streamlines...

Elementwise solution



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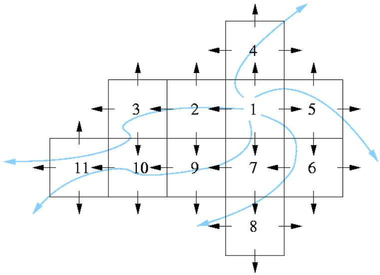


and the corresponding fluxes and a possible sequence of operations

L_2-errors and convergence rates

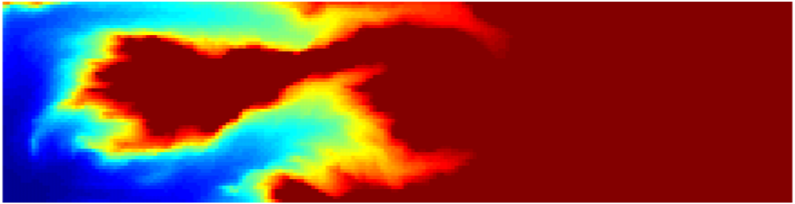
Ex: Linear rotation, $\mathbf{v} = (y, -x)$:

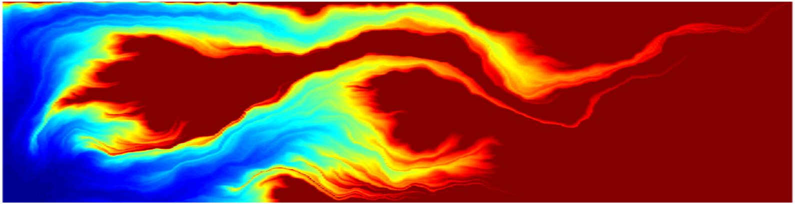
Table: L_2-errors and the convergence rates in a smooth domain.

N	1. order		2. order		3. order		4. order	
10	3.36e-03		3.13e-05		1.74e-07		2.77e-09	
20	1.52e-03	1.15	7.42e-06	2.08	2.24e-08	2.96	1.45e-10	4.25
40	8.01e-04	0.92	1.95e-06	1.93	2.90e-09	2.95	9.58e-12	3.92
80	4.14e-04	0.95	5.02e-07	1.96	3.69e-10	2.97	6.22e-13	3.94
160	2.05e-04	1.01	1.25e-07	2.01	4.60e-11	3.01	3.84e-14	4.02
320	1.02e-04	1.01	3.10e-08	2.01	5.73e-12	3.00	2.39e-15	4.01

Top Layer in SPE 10

$$n = 1$$

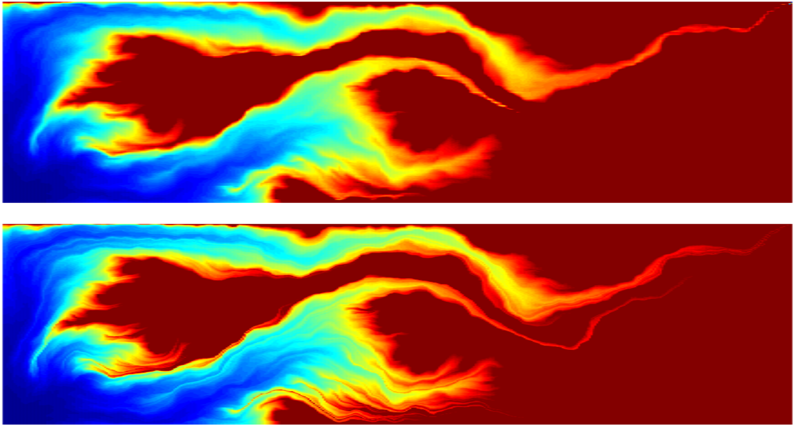




Comparison of DGM with a reference solution

Top Layer in SPE 10

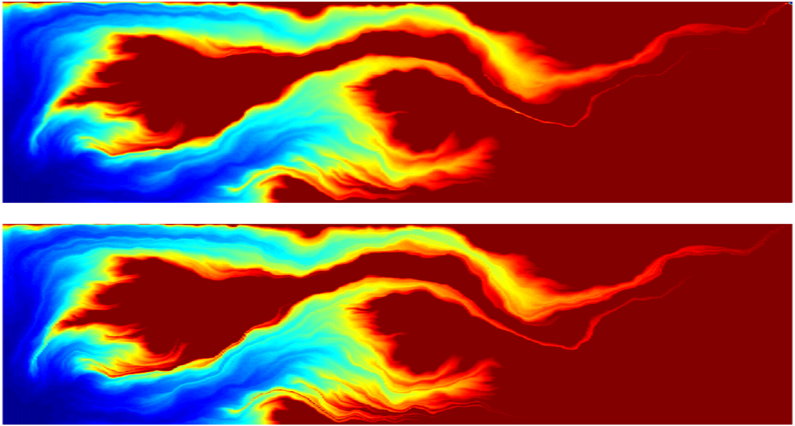
$$n = 2$$



Comparison of DGM with a reference solution

Top Layer in SPE 10

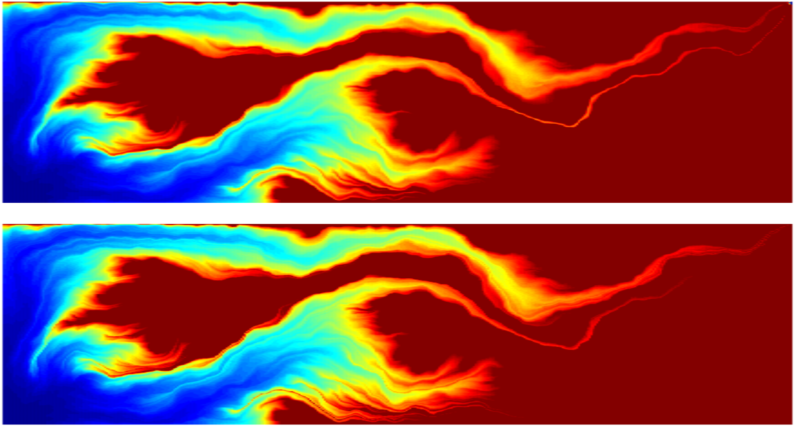
$n = 3$



Comparison of DGM with a reference solution

Top Layer in SPE 10

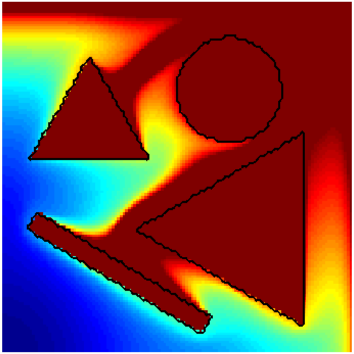
$n = 4$

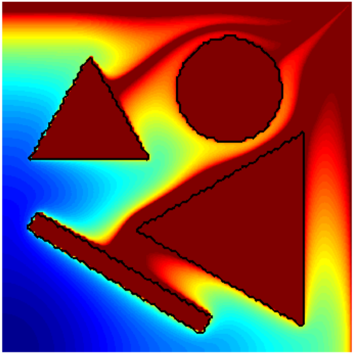


Comparison of DGM with a reference solution

Flow Around Strong Discontinuities

$$n = 1$$



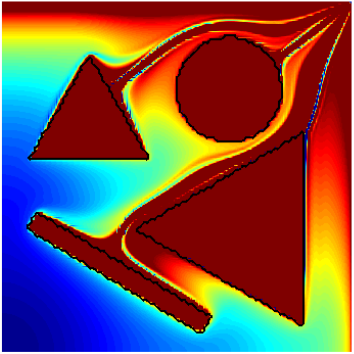


TOF using DGM

Reference solution

Flow Around Strong Discontinuities

$$n = 2$$



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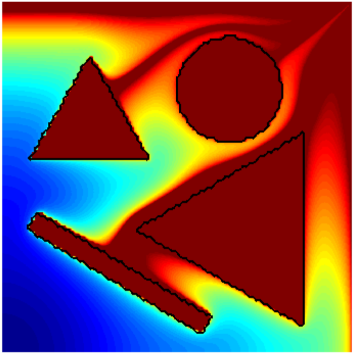

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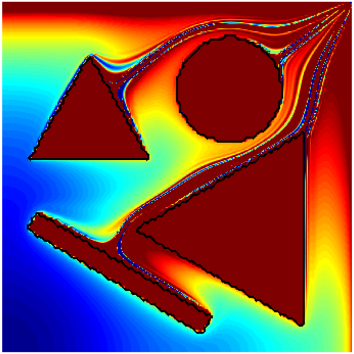

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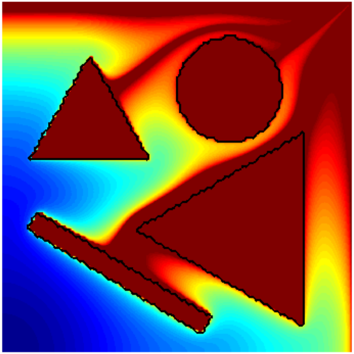


Reference solution

Tracer Flow

Linear transport equation:

$$\partial_t c + \nabla \cdot (\mathbf{v}c) = 0$$

Tracer Flow

Stationary distribution of tracers:

$$\nabla \cdot (\mathbf{v}c) = 0$$

Tracer Flow

Stationary distribution of tracers:

$$c\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla c = 0$$

Tracer Flow

Stationary distribution of tracers:

$$\mathbf{v} \cdot \nabla c = 0$$

Tracer Flow

Stationary distribution of tracers:

$$\mathbf{v} \cdot \nabla c = 0$$

Time-of-flight equation:

$$\mathbf{v} \cdot \nabla T = 1$$

Tracer Flow

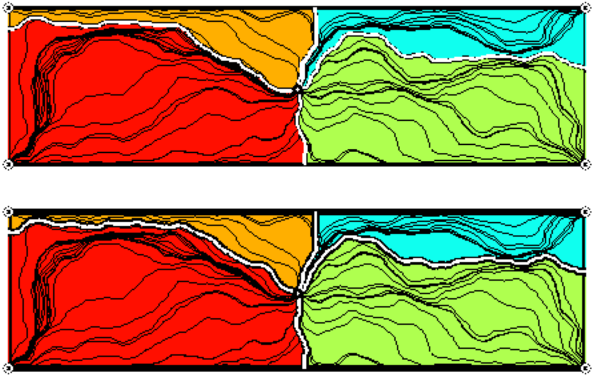
Stationary distribution of tracers:

$$\mathbf{v} \cdot \nabla c = 0$$

The linear equations for element K are

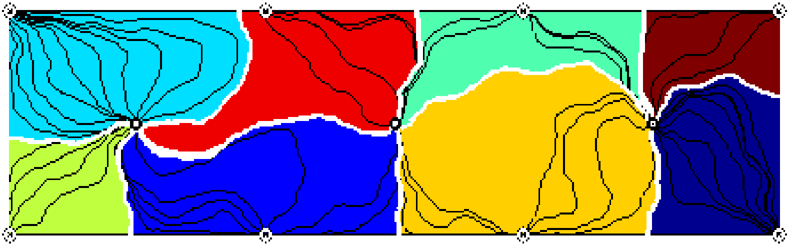
$$F_K^+ C_{i,K} - R_K C_{i,K} = -F_K^- C_{i,\Omega\setminus K}, \quad i = 1, \ldots, n$$

Top layer in SPE 10



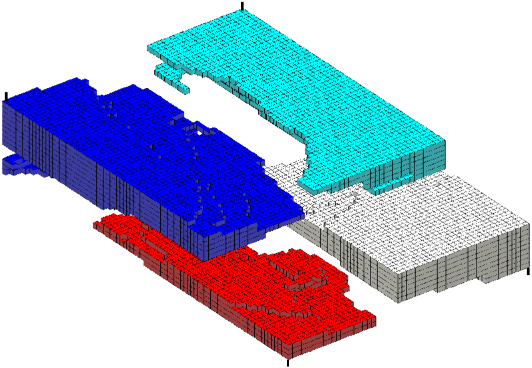
Comparison of the approximate tracer distribution using 1. and 5. order DGM

Top layer in SPE 10



Order 1 - Piecewise constant polynomials

3D: 15 layers of SPE 10



Implicit DG Solution

- Consider flow of two or more phases

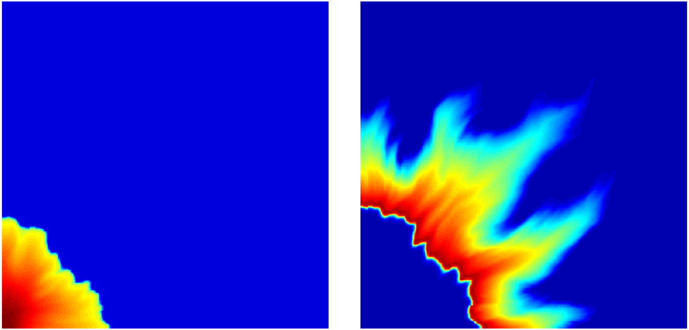
$$S_t + \nabla \cdot \big(\mathbf{v} F(S)\big) = 0$$

  where F has positive characteristics
- Using product rule and semi-discretization

$$S^{n+1} + \Delta t\, \mathbf{v} \cdot \nabla F(S^{n+1}) = S^n - \Delta t\, F(S^n) \nabla \cdot \mathbf{v}$$

- Discretization by DGM
- Reordering as for $\mathbf{v} \cdot \nabla T = 1 \longrightarrow$ elementwise solution of N *nonlinear* $m \times m$ systems
- For large models: reordered dG + domain decomposition

WAG Injection (3-Phase Flow)



Water ($t = 0.075$)   Gas ($t = 0.075$)

2nd order dG method with minmod postprocessing

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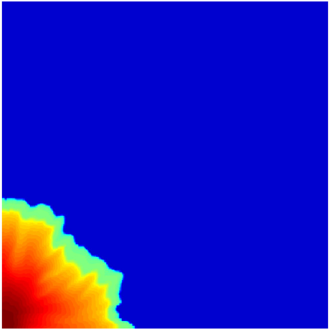
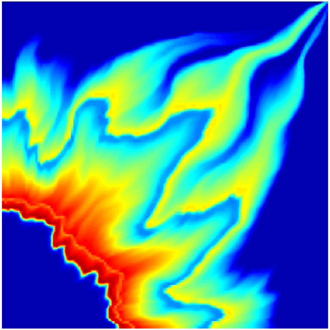


Water ($t = 0.125$)



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2nd order dG method with minmod postprocessing

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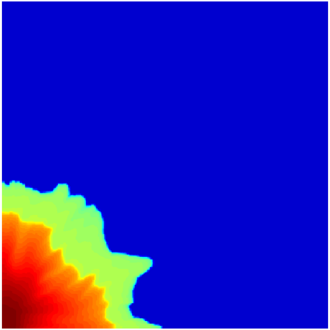
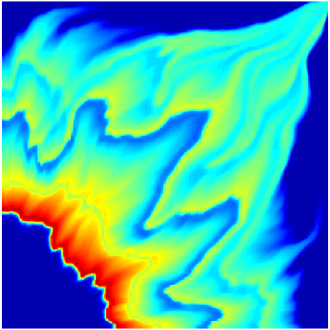


Water ($t = 0.175$)



Gas ($t = 0.175$)

2nd order dG method with minmod postprocessing

Summary

Summary

- Higher-order discontinuous Galerkin methods are implemented
- Fast elementwise solution strategy
- Runtime of the methods are $\mathcal{O}(N)$ for N unknowns
- Effective approximation of stationary tracer distribution
- Promising results for multiphase flow