Multiscale Methods for Elliptic Problems in Porous Media Flow

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Outline

1. Introduction

2. Three Multiscale Methods for the Pressure Equation
   - Adaptive Local-Global Upscaling / Nested Gridding
   - Multiscale Mixed Finite Elements
   - Multiscale Finite Volumes

3. Comparison
   - Numerical Experiments
   - Computational Complexity

4. Conclusions
1 Introduction

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   • Numerical Experiments
   • Computational Complexity

4 Conclusions
Geological Reservoir Description

- Geological reservoir models: \( O(10^6) - O(10^9) \) grid cells.
- Oscillating coefficients, \( K_{\text{max}}/K_{\text{min}} : O(10^6) - O(10^{12}) \).
Simulation Models and Upscaling

- Geological models too large for standard simulators.
- Industry solution: Upscaling
- Simulation models: $\mathcal{O}(10^4) - \mathcal{O}(10^6)$ grid cells.
- Unfortunately: Fine-scale variations may be important.
Fractional flow formulation (no gravity or capillary forces):

Pressure (elliptic):
\[
\begin{align*}
    \nabla \cdot v &= q \\
    v &= -K \lambda_t(S) \nabla p,
\end{align*}
\]

Saturation (hyperbolic):
\[
\phi \partial_t S + \nabla \cdot (v f(S)) = 0
\]

Solution method: Operator splitting.
Multiscale Simulation

Geomodel $\Rightarrow$ Coarse-scale solution $\Rightarrow$ Fine-scale velocity $\Downarrow$

Coarse linear system $\Uparrow$

Fine-scale saturation $\Leftarrow$
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1. Upscale transmissibility:

\[- \nabla \cdot K \nabla p = 0 \quad \text{in} \quad \Omega_{lj}\]

\[p = Ip^* \quad \text{in} \quad \partial\Omega_{lj}\]

\[T_{lj}^* = \frac{\int_{\partial K_l \cap \partial K_j} v \cdot n_{lj} \, ds}{\int_{K_l} p \, dx - \int_{K_j} p \, dx}\]

2. Solve coarse-scale problem:

\[\sum_j T_{lj}^*(p_l - p_j) = \int_{K_l} q \, dx \quad \forall K_l\]

3. Construct fine-scale velocity:

\[v = -K \nabla p, \quad \nabla \cdot v = q \quad \text{in} \quad K_l\]

\[v \cdot n = \frac{T_{ki}(v^* \cdot n_{lj})}{\sum_{\gamma_{ki} \subset \Gamma_{lj}} T_{ki}} \quad \text{on} \quad \partial K_i\]

(Here \(i\) runs over the underlying fine grid)
For the MsMFEM the fine-scale velocity field is a linear superposition of basis functions: \( v = \sum_{ij} v^*_{ij} \psi_{ij}. \)
For the MsFVM the fine-scale pressure field is a linear superposition of basis functions: \( p = \sum_i p_i^* \phi_i \).
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Numerical Experiments
Fluvial Reservoir – Fine Grid Solution

Layer 85 from the 10th SPE Comparison Project

- Fine grid: $60 \times 220$
- Coarse grid: $10 \times 22$
- Cell aspect ratio: $dx/dy = 2$

(a) $\log_{10} K$

(b) Reference solution ($4 \times$ grid)
Numerical Experiments
Fluvial Reservoir – Fine Grid Solution

(a) Reference solution (4× grid)

(b) MsMFEM

(c) MsFVM

(d) ALGU-NG
Saturation error as a function of coarse grid size.
Numerical Experiments
Fluvial Reservoir – Coarse Grid Solution

(a) Reference solution (4 × grid)
(b) MsMFEM
(c) MsFVM
(d) ALGU-NG
(e) Pressure Method
(f) Harmonic-Arithmetic Averaging

Upscaled grid size: 15 × 55
Numerical Experiments
Fluvial Reservoir – Coarse Grid Solution

Saturation equation solved on the upscaled grid.
Errors computed on the upscaled grid.
Numerical Experiments
Fluvial Reservoir – Coarse Grid Solution

- Saturation equation solved on the fine grid.
- Errors computed on the upscaled grid.
Uncorrelated Log-Normal Permeability

- 100 realizations
- Fine grid: $64 \times 64$
- Coarse grids: $4 \times 4$, $8 \times 8$, $16 \times 16$, and $32 \times 32$.

(a) Sample realization ($\log_{10} K$)  
(b) Reference solution
Numerical Experiments
Log-Normal Permeability – Uncorrelated

Mean saturation error as a function of coarse grid size.
Mean and standard deviation of the saturation error for the coarse grid of size $8 \times 8$. 
Spacially Correlated Log-Normal Permeability

- 100 realizations
- Same grids as before
- Dimensionless correlation length 0.1 in each direction.

(a) Sample realization ($\log_{10} K$)  
(b) Reference Solution
Mean saturation error as a function of coarse grid size.
Mean and standard deviation of the saturation error for the coarse grid of size $8 \times 8$. 
Numerical Experiments
Vertical Channels

Vertical High-Permeability Channels:

- 100 realizations
- Same grids as before
- Dimensionless correlation length 10 in the vertical direction and 0.1 in the horizontal direction.
- Conditioning on artificial data to produce the channels.
Mean saturation error as a function of coarse grid size.
Mean and standard deviation of the saturation error for the coarse grid of size $8 \times 8$. 
One of the bad realizations for the MsFVM:

- Solution is smeared out inside coarse cells.
- We will return to this problem in a moment.
Numerical Experiments
Diagonal Channels

Diagonal High-Permeability Channels:

- 100 realizations
- Same grids as before
- Similar to previous case, but rotated $45^\circ$. 

(a) Sample realization ($\log_{10} K$)  
(b) Reference Solution
Mean saturation error as a function of coarse grid size.
Mean and standard deviation of the saturation error for the coarse grid of size $8 \times 8$. 
Numerical Experiments
Diagonal Channels

One of the bad realizations for the MsMFEM and ALGU-NG:

(a) $\log_{10} K$
(b) $4 \times$ Reference

(c) MsMFEM
(d) ALGU-NG
Comparison
Anisotropic Medium / High Aspect Ratio

Spacially correlated log-normal permeability:

\[ \log_{10} K \]

\[ K_x / K_y = 10^4 \]

or

\[ l_x / l_y = 10^{-2} \]
Example: 3D (128x128x128), $\alpha = 1.2$ and $m = 3$
Direct solution more efficient, so why bother with multiscale?

- Full simulation: $\mathcal{O}(10^2)$ steps.
- Basis functions need not be recomputed

Also:
- Possible to solve very large problems
- Easy parallelization
A Few Words About Implementation

In our experience:

- **MsFVM and ALGU-NG:**
  - *Dual grid* — SPECIAL CASES (along external boundaries and internal structures)

- **MsMFEM:**
  - Coarse grid cell is union of fine grid cells —
    - Implementation straightforward given a fine grid method.
    - Method quite independent of coarse grid cell geometry.

(a) Shale Barriers  (b) MsMFEM coarse grid!
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Conclusions

- All three methods: High accuracy on typical data.
- MsMFEM: Advantage for uncorrelated data and media where the trends are aligned with the grid.
- MsFVM: Advantage when the grid is not aligned with the main flow direction (multi-point stencil).
- MsFVM: Trouble for anisotropic media / high aspect ratios
- MsMFEM and MsFVM have similar computational complexity, ALGU-NG is less efficient