A front-tracking method for hyperbolic three-phase models

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Abstract

We develop and apply a front-tracking method for the numerical simulation of three-phase flow in porous media. The proposed framework combines analytical solutions to the corresponding Riemann problem with an efficient front-tracking method to study Cauchy and initial-boundary value problems. The method has the ability to track individual waves and give very accurate (or even exact) resolution of discontinuities. This numerical procedure is then used in combination with a streamline method, for the simulation of three-dimensional, three-phase flow problems. We demonstrate the applicability of the method through several numerical examples, including a streamline simulation of a water-alternating-gas (WAG) injection process in a three-dimensional, heterogeneous, shallow-marine formation.

1. Introduction

In recent years, streamline methods have emerged as an attractive way to simulate flow in highly heterogeneous formations. The essence of these methods is to decouple three-dimensional transport equations into a set of one-dimensional problems along streamlines. Streamline methods, however, rely heavily on the efficient solution of the transport equations in 1D. Here, we propose the use of a front tracking method for the solution of one-dimensional three-phase flow problems along streamlines.

The term "front tracking" refers to a family of numerical methods that perform some kind of tracking of shocks and other evolving discontinuities. Most front-tracking schemes consist in a finite-difference scheme coupled with a recipe for detecting and tracking discontinuities. Our method is different in the sense that no finite differences are involved. Instead, the numerical solution is computed by treating all waves as discontinuities. Smooth rarefaction waves are approximated by small discontinuities that violate the entropy condition, whereas shocks and real other discontinuities are tracked exactly [1]. The most appealing features of the method is that it is able to resolve discontinuities *exactly*, has no grid-dependence, and is unconditionally stable. Depending upon the availability of a fast Riemann solver and the complexity of the wave interactions of the problem, the method can be very efficient compared with conventional finite volume and finite element methods. In reservoir simulation, front tracking is a key technology in obtaining the high numerical efficiency of the two-phase version of the streamline simulator FrontSim [2] by Schlumberger.

The use of front tracking for simulation of three-phase flow has been limited, in part because of lack of general analytical solutions to the three-phase Buckley-Leverett problem. However, see e.g., [3] for early results on triangular three-phase models. Here, we implement and extend a recently developed analytical solution to the Riemann problem [4], and use it as a building block in the front-tracking algorithm. However, evaluation of the full three-phase Riemann solution is expensive in the context of a front-tracking method, because typical applications require hundreds of millions of calls to the Riemann solver. In this paper, we propose a hierarchical data reduction algorithm that accomplishes two goals: (1) avoid blow up of the number of Riemann problems to be solved, and (2) obtain a much faster method by using an accurate (but not exact) Riemann solution.

In Section 2, we outline the mathematical model, and discuss the wave structure of the Riemann problem. In Section 3 we describe the front-tracking method, with particular reference to the way in which rarefaction waves are discretized. The data reduction algorithm is given in

Section 4. In Section 5 we present two numerical simulations. The first example is a one-dimensional problem modeling water-alternating-gas injection in an oil and gas reservoir. The second example is a three-dimensional three-phase flow simulation in a highly heterogeneous formation, where the front tracking algorithm is used in combination with a streamline method. These numerical simulations illustrate the potential of this approach for fast and accurate simulations in real three-dimensional, heterogeneous reservoirs. We summarize the main conclusions in Section 6.

2. Riemann solver for three-phase flow

Under certain assumptions, the mathematical model describing three-phase flow in porous media may be expressed in terms of a pressure equation, and a system of saturation equations. For one-dimensional flow, the system of saturation equations takes the form (after re-scaling of the space variable):

$$u_t + f(u)_x = 0, \tag{1}$$

where $u = (S_w, S_g)$ is the vector of water and gas saturations, and $f = (f_w, f_g)$ is the vector of fractional flow functions. The oil saturation is determined by the algebraic relation $S_o = 1 - S_w - S_g$. If the effects of miscibility, compressibility, capillarity and gravity are neglected, the fractional flow of phase i is simply:

$$f_i = \frac{\lambda_i}{\lambda_T}, \tag{2}$$

where λ_i is the relative mobility of phase i, and $\lambda_T = \lambda_w + \lambda_g + \lambda_o$ is the total mobility. The relative mobility is defined as:

$$\lambda_i = \frac{k_{ri}}{\mu_i}, \tag{3}$$

where k_{ri} and μ_i are the relative permeability and the dynamic viscosity of phase i, respectively. The relative permeabilities are normally understood as functions of the fluid saturations alone. It is well known that most relative permeability models used today give rise to elliptic regions, that is, open sets in the saturation space where the system (1) is locally elliptic rather than hyperbolic [5,6,7]. There is an ongoing debate on whether elliptic regions are physical, or simply an unintended consequence of the severe modeling assumptions made in development of three-phase flow models. In this paper, we adopt the view that elliptic regions are the result of an incomplete model, and we use relative permeability functions that render the system hyperbolic [7].

The Riemann problem is a particular case of the system (1), in which the initial condition is given by piecewise constant data, separated by a single discontinuity:

$$u_0(x) = \begin{cases} u_L & \text{if } x<0, \\ u_R & \text{if } x\geq 0. \end{cases} \tag{4}$$

Analytical solutions to the Riemann problem of three-phase flow have been studied extensively (see [8] for an overview). In a recent paper [4], a complete catalogue of solutions was identified, and efficient algorithms for the computation of the solution were given. The main assumptions used to limit the admissible wave structure are: (1) the system is strictly hyperbolic; and (2) both characteristic fields are nongenuinely nonlinear. Both conditions are natural extensions of the corresponding conditions in the two-phase flow case. Under those assumptions, the solution to the Riemann problem comprises two separated waves, connecting three constant states:

$$u_L \xrightarrow{W_1} u_M \xrightarrow{W_2} u_R. \tag{5}$$

Using a result that limits the admissible types of waves that may be present [9], it is concluded that only nine combinations of waves are possible: each of the two waves can only be a single rarefaction R, a single shock S, or a composite rarefaction-shock RS. Efficient algorithms for the calculation of all solution types have also been devised [4].

The algorithm starts by setting an initial guess u_M^{tr} of the intermediate state, and by assuming a trial solution of type R_1R_2, that is, a solution consisting in two rarefaction waves. The heart of the algorithm involves two actions:

1. Compute the intermediate state, given a *trial* wave structure $W_1^{\text{tr}}W_2^{\text{tr}}$ and an initial guess u_M^{tr}. This step is performed following the algorithms given in [4].
2. Ascertain what the wave structure of the solution would be *if* the intermediate state were the one just computed. The wave type is inferred separately for each individual wave ($i = 1, 2$).

It is not sufficient to check the admissibility of individual waves. If the solution involves shocks, one must also check that they form an increasing sequence of wave speeds, that is, $\sigma_1 < \sigma_2$. The algorithm terminates if the trial wave structure $W_1^{\text{tr}}W_2^{\text{tr}}$ is admissible. Otherwise, both the intermediate state and the wave structure are updated from the computed values. Because rarefaction curves and shock curves typically have similar paths on the saturation space, the intermediate state is usually not very sensitive to the solution type, and the procedure often converges after one iteration.

3. The front-tracking method

Front tracking is an algorithm for constructing exact or approximate solutions to hyperbolic conservation laws of the form

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x). \tag{6}$$

Assume that the initial function $u_0(x)$ is a piecewise constant function so that the Cauchy problem consists of a series of local Riemann problems. In the previous section we discussed how to solve the Riemann problem exactly to produce a similarity solution, which is commonly referred to as the Riemann fan. Each Riemann fan is local in time and space and consists of a set of constant states separated by simple waves. By connecting the local Riemann fans, one obtains a solution that is global in space. Since each simple wave has a finite speed of propagation, the global solution is well defined up to the time when the first waves from two neighboring Riemann fans interact. If the two interacting waves are discontinuities, the interaction defines a new local Riemann problem and the new global solution can be constructed by inserting the corresponding local Riemann fan, see Figure 1. If all simple waves admitted by the system are discontinuities, all local Riemann problems produce constant states separated by discontinuities. In this case our construction can be repeated to compute the *exact* solution of the Cauchy problem up to an arbitrary desired time level. If the system admits rarefactions, as is the case for the three-phase model, the above construction cannot be used directly to construct an exact solution. However, an *approximate* solution can be constructed if we approximate each Riemann fan by a step function so that the approximate Riemann fan consists of constant states separated by space-time rays of discontinuities. To this end, we discretize the smooth rarefaction waves by a series of (small) jump discontinuities and keep the shocks (and the linear discontinuities).

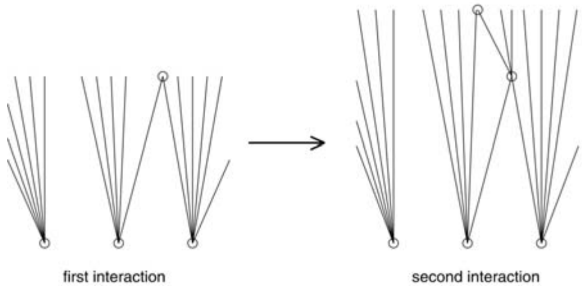

Fig. 1. Construction of a global solution by connecting local Riemann fans in the (x,t)-plane.

We now use the algorithm outlined above to construct a global approximate solution (in space and time) in the same way as one builds scaffolding. Start by resolving Riemann the initial problems and connect the local approximate Riemann fans. The result is a set of constant states separated by space-time rays of discontinuity, henceforth referred to as *fronts*. Then, track all fronts until the first two fronts collide, resolve the corresponding Riemann problem, insert the approximate Riemann fan, and so on. This is the front-tracking algorithm.

An important point that we have so far not discussed is how to approximate rarefaction waves. One possibility is to discretize each rarefaction wave uniformly in wave speed. However, since the integral curves are given as either $S_w = R(S_g)$ or $S_g = R(S_w)$, we can simply discretize the rarefactions by sampling uniformly along the integral curves in state space; that is, discretize each rarefaction wave by a set of constant states $\{u^i\}$ such that $|u^i - u^{i-1}| \approx \delta_u$, for some prescribed δ_u. Since the rarefaction waves are discretized in state space, the wave velocities for each discontinuity must be determined. There are several natural candidates like the characteristic speed of the left or the right state, or the average of the characteristic speeds, see e.g., [1]. We use the Rankine–Hugoniot wave speed given by the left and right state of each discontinuity, ensuring that each discontinuity in the approximate Riemann fan satisfies the equation in the weak sense, regardless of whether it is admissible or not.

4. Data reduction strategy

A potential pitfall of the front-tracking algorithm is that the number discontinuities in the solution may blow up in finite time for general systems. Generally, one must therefore use Glimm-type interaction estimates to do some kind of data reduction, see [1]. Then one can prove that the algorithm converges as the piecewise constant approximation of the initial data and the sampling of the Riemann fans are refined.

We are interested only in solutions with finite accuracy, and the algorithm terminates in a finite number of steps if we remove small waves (or Riemann problem) below some prescribed tolerance. Moreover, to speed up the algorithm it is feasible to give weak waves a simplified treatment in terms of an approximate Riemann solver. Inspired by [10], we propose a general four point strategy for solving the Riemann problem (u_L, u_R) approximately:

1. If $|u_L - u_R| \le \delta_1$, ignore the Riemann problem
2. If $\delta_1 < |u_L - u_R| \le \delta_2$, approximate the Riemann problem by a single discontinuity with shock speed equal the average of the Rankine–Hugoniot velocity of each component.
3. If $\delta_2 < |u_L - u_R| \le \delta_3$, approximate the Riemann problem by a two-shock solution $S_1 S_2$. If $\sigma_1 \not\le \sigma_2$, solve the full Riemann problem.
4. Otherwise solve the full Riemann problem.

5. Numerical simulations

In the following we will present two examples to demonstrate the behavior of the front-tracking algorithm. We have chosen a simple three-phase model given by the relative mobility functions

$$\begin{aligned}\lambda_w(S_w) &= (a_w S_w + (1-a_w)S_w^2)/\mu_w,\\ \lambda_g(S_g) &= (a_g S_g + (1-a_g)S_g^2)/\mu_g,\\ \lambda_o(S_w,S_g) &= (1-S_w-S_g)(1-S_w)(1-S_g)/\mu_o,\end{aligned} \tag{7}$$

where $a_w = 0$, $a_g = 0.1$, $\mu_w = 0.35$, $\mu_g = 0.012$, and $\mu_o = 0.8$. With this choice, the system is strictly hyperbolic in the entire saturation triangle, except at the vertex of 100% gas saturation, where the eigenvalues are equal [4].

5.1. One-dimensional WAG injection

We model a simplified, one-dimensional, water-alternating-gas (WAG) process, in which the injected conditions vary in time. We consider a linear reservoir with initial saturations of 20% gas and 80% oil. The process starts by injecting pure water and, at time $t = 0.2$, the injected state changes to 99% gas saturation. Subsequent cycles of water and gas injection are established, with each injection phase lasting for a period $\Delta t = 0.2$.
Because of the stepwise change in the left boundary condition, the solution is *not* a single Riemann fan. Figure 2 shows the fronts in the (x,t)-plane for a very coarse approximation $\delta_u = 0.05$, with and without the data reduction on waves. Notice how all the strong waves are preserved under the data reduction and that in both cases; in particular, the oil bank reaches the production well at $t \approx 0.564$. Moreover, one can clearly see that each interactions of two weak waves results in a S_1S_2 solution. The number of Riemann problems was 5563 and 1833, respectively, with 1605 being S_1S_2 and 234 full for the run with data reduction. The ratio of runtimes was $4.6:1$.

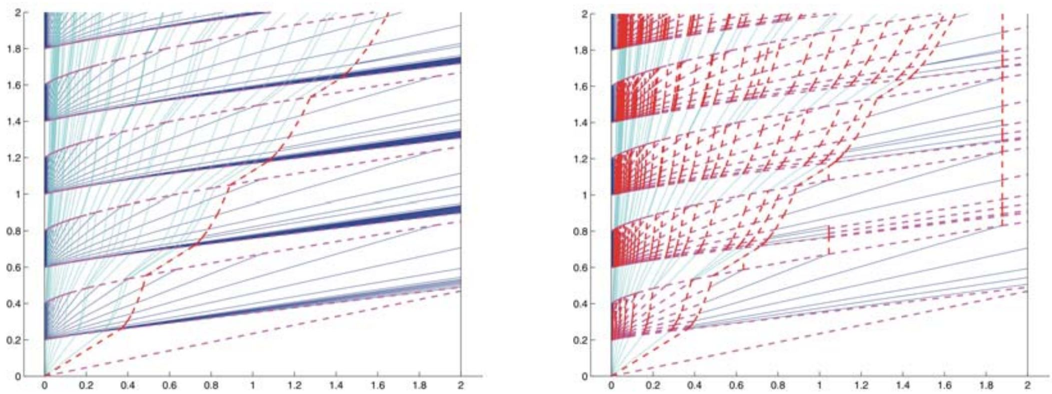

Fig. 2. Fronts in the (x,t)-plane for the one-dimensional WAG process, using $\delta_u = 0.05$. Shocks are shown as dashed lines, whereas rarefactions are solid lines. (Left) Full resolution of all waves with $\delta_1 = \delta_2 = \delta_3 = 10^{-5}$. (Right) Data reduction with $\delta_3 = 0.2$ and $\delta_1 = \delta_2 = 10^{-5}$.

In Figure 3 we plot the saturation profiles at times $t = 0$, 0.1, 0.2, 0.5, 1.5 and $t = 2$, calculated with the front-tracking method and the standard first-order upwind finite difference method with the cycle period reduced to $\Delta t = 0.1$. In the front-tracking solution, we used $\delta_u = 0.005$ for an accurate sampling of the rarefaction waves. The front-tracking method requires in this case the solution of about 1.6 million Riemann problems. The finite difference solution was computed with 100 grid cells, and a fully implicit time stepping procedure. We employed a constant time step $\Delta t = 0.005$, which corresponds to a Courant number $Co = \sigma_2 \Delta t/\Delta x \approx 1$. With this discretization, the computational effort of both methods is comparable. It is apparent that the front-tracking method gives a very accurate resolution of moving fronts, while the solution obtained with the finite difference upwind method is greatly affected by numerical diffusion.

Finally, in Figure 4 we plot the water saturation profiles obtained using $\delta_3 = 0.0$, 0.1, and 0.2 for two values of the rarefaction parameter. For $\delta_3 = 0.1$, the results are inseparable to plotting accuracy from those with full Riemann solution, whereas for $\delta_3 = 0.2$ there is a minor difference. Table 1 shows the corresponding run statistics and seen together with Figure 4 demonstrates the feasibility of the data reduction.

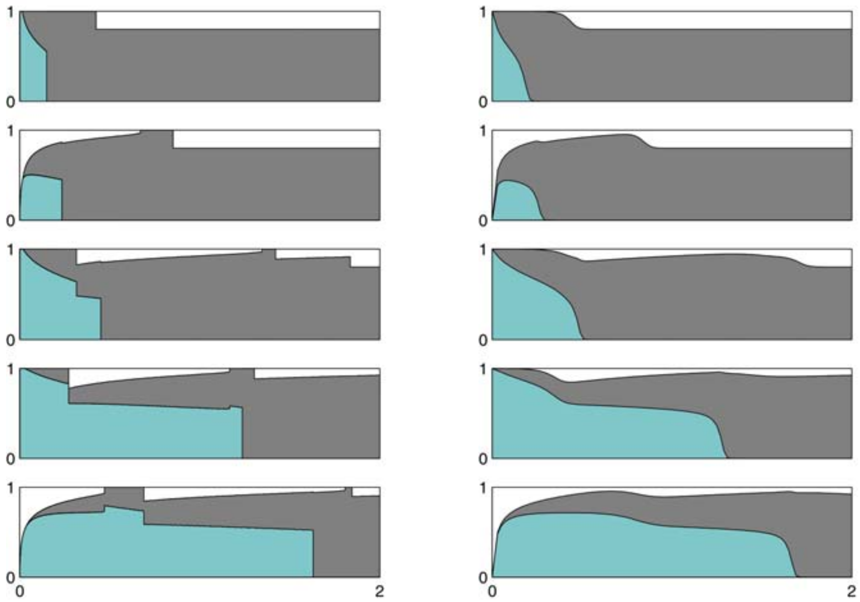

Fig. 3. Saturation profiles at $t = 0$, 0.1, 0.2, 0.5, 1.5 and $t = 2$, calculated by: front-tracking using $\delta_u = 0.005$ (left), and upwind finite differences with 100 grid cells (right).

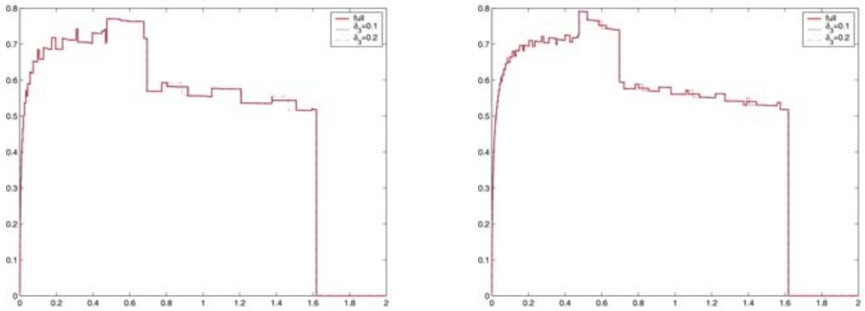

Fig. 4. Water saturation profiles with varying degree of data reduction: $\delta_u = 0.05$ (left), and 0.025 (right).

Table 1. Run statistics for Figure 4.

	$\delta_u = 0.05$			$\delta_u = 0.025$		
δ_3	0.0	0.1	0.2	0.0	0.1	0.2
Runtime	3.9	1.3	0.32	14.2	3.9	1.03
# W_1W_2	26587	1645	745	107315	3554	1615
# S_1S_2	0	26616	5320	0	97946	23242

5.2. Three-dimensional WAG injection

An interesting application of the front-tracking method for three-phase flow is in combination with streamlines to simulate multidimensional displacement scenarios. In a streamline simulation, the pressure and the transport equations are decoupled and solved sequentially. In the fluid transport, each streamline is treated as an isolated flow system and the saturation is advanced forward in time by solving a one-dimensional hyperbolic system, here by using the front-tracking method. At the end of the transport step, the water saturations are projected back onto the background grid, the fluid mobilities are updated, and the pressure recomputed. For a full three-dimensional simulation, the saturation step typically involves several thousand streamlines, resulting in a very large number of calls to the Riemann solver. To increase the speed significantly, we use the *adaptive* Riemann solver described in Section 4, in which the wave structure of strong Riemann problems is resolved exactly, whereas weak Riemann problems are approximated by a two-shock solution. In particular, we use $\delta_1 = \delta_2 = 10^{-5}$, $\delta_3 = 0.2$ and $\delta_u = 0.05$.

As an application example, we consider a synthetic, full three-dimensional reservoir model consisting of a five spot well configuration (one injection well at the center and four production wells at the corners) in a highly heterogeneous, shallow-marine Tarbert formation. The heterogeneity model is a subsample of the recent 10th SPE comparative solution project [11] on a $30 \times 110 \times 15$ grid. The field has large (but smooth) permeability variations: 6 orders of magnitude in the horizontal direction and 10 orders in the vertical direction see Figure 5. The porosity is strongly correlated to the permeability.

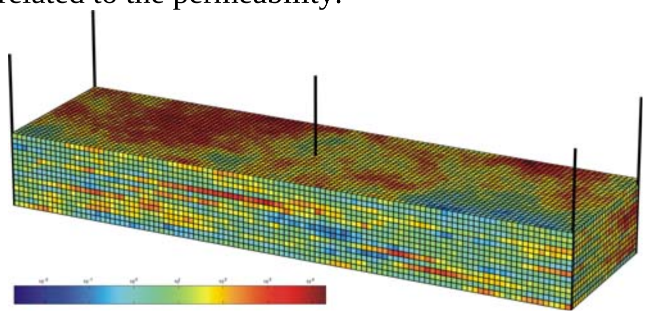

Fig. 5. Logarithm of horizontal permeability and well configuration for the Tarbert formation.

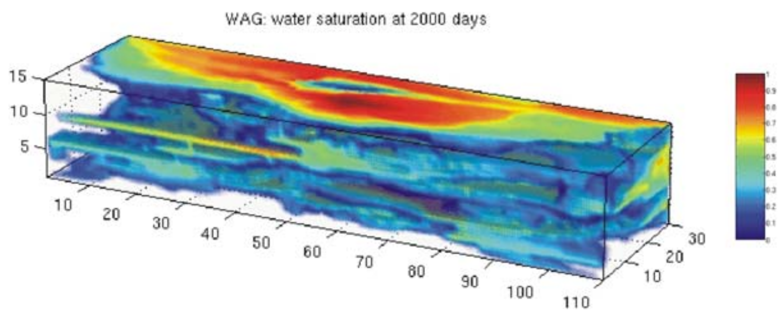

Fig. 6. Water saturation after 2000 days of production with a WAG injection scheme.

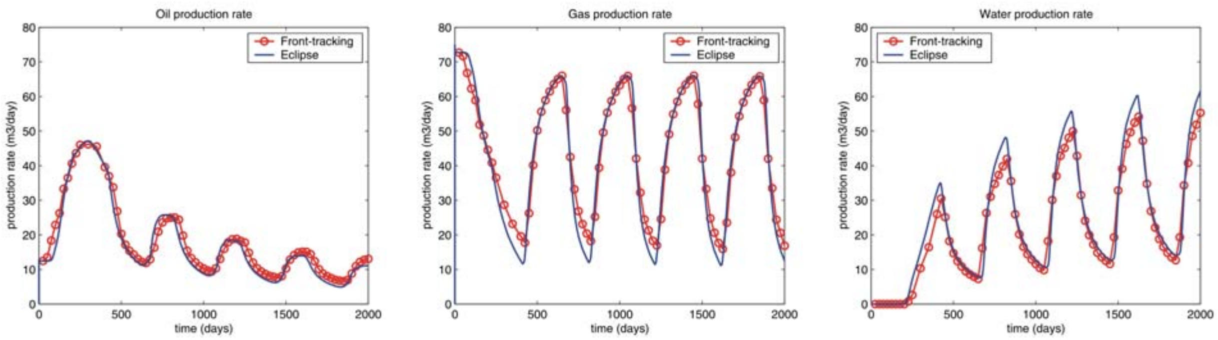

Fig. 7. Oil, gas, and water production rates computed by the front-tracking simulator and Eclipse.

For simplicity, we neglect gravity and assume incompressible flow. The three-phase model is as given in the previous section. The initial saturation is $(S_w, S_g) = (0.0, 0.2)$. We consider 2000 days of production by a WAG cycle, where the injected fluid composition is changed between $(1.0, 0.0)$ and $(0.05, 0.95)$ every 200 days, starting at day 400. Figure 6 shows the water saturation after 2000 days of production. In Figure 7, we plot the production rates of all three fluids, water, oil, and gas. We compare the results calculated by the front-tracking method with those produced by the black-oil reservoir simulator Eclipse 100. Given the completely different nature of both simulators, the agreement is remarkable. The WAG simulation involved 632 million calls to the Riemann solver, out of which 617 million were approximated by a two-shock solution, and ran about 6 times faster than the Eclipse simulation.

6. Conclusions

We have presented a front-tracking method for the numerical solution of three-phase porous media flow. The method yields very accurate (even exact) solutions to one-dimensional problems with general initial and boundary data. Two distinctive features of our method are (1) the use of an analytical solution of the three-phase Riemann problem, and (2) a hierarchical data reduction algorithm to speed up the Riemann solver. The combination of these two features makes the front-tracking technique very attractive as a computational method. We have illustrated the performance of the method with a one-dimensional simulation of WAG injection. The front-tracking method produces a very accurate solution, where strong shocks are resolved exactly (both in amplitude and speed). A solution of similar quality with finite differences would require an impractical grid resolution.

We have also shown the efficacy of the front-tracking algorithm in combination with streamline methods, for the simulation of oil recovery in real reservoirs. We simulated a WAG injection scheme in a complex, highly heterogeneous, three-dimensional reservoir model adapted from the 10th SPE comparison solution project. Recovery predictions computed with the streamline/front-tracking simulator were found to be in excellent agreement with Eclipse 100, but the run time was significantly less.

The integration of analytical Riemann solvers, the front-tracking method, and streamline tracing, offers the potential for fast and accurate prediction of three-phase flows in real reservoirs. This technology becomes particularly relevant for screening purposes and for risk assessment, which require the simulation of a large number of scenarios.

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