Phase field, level set method and PCLSM

Xue-Cheng Tai*

Xue-Cheng.Tai@mi.uib.no, http://www.mi.uib.no/%7Etai.

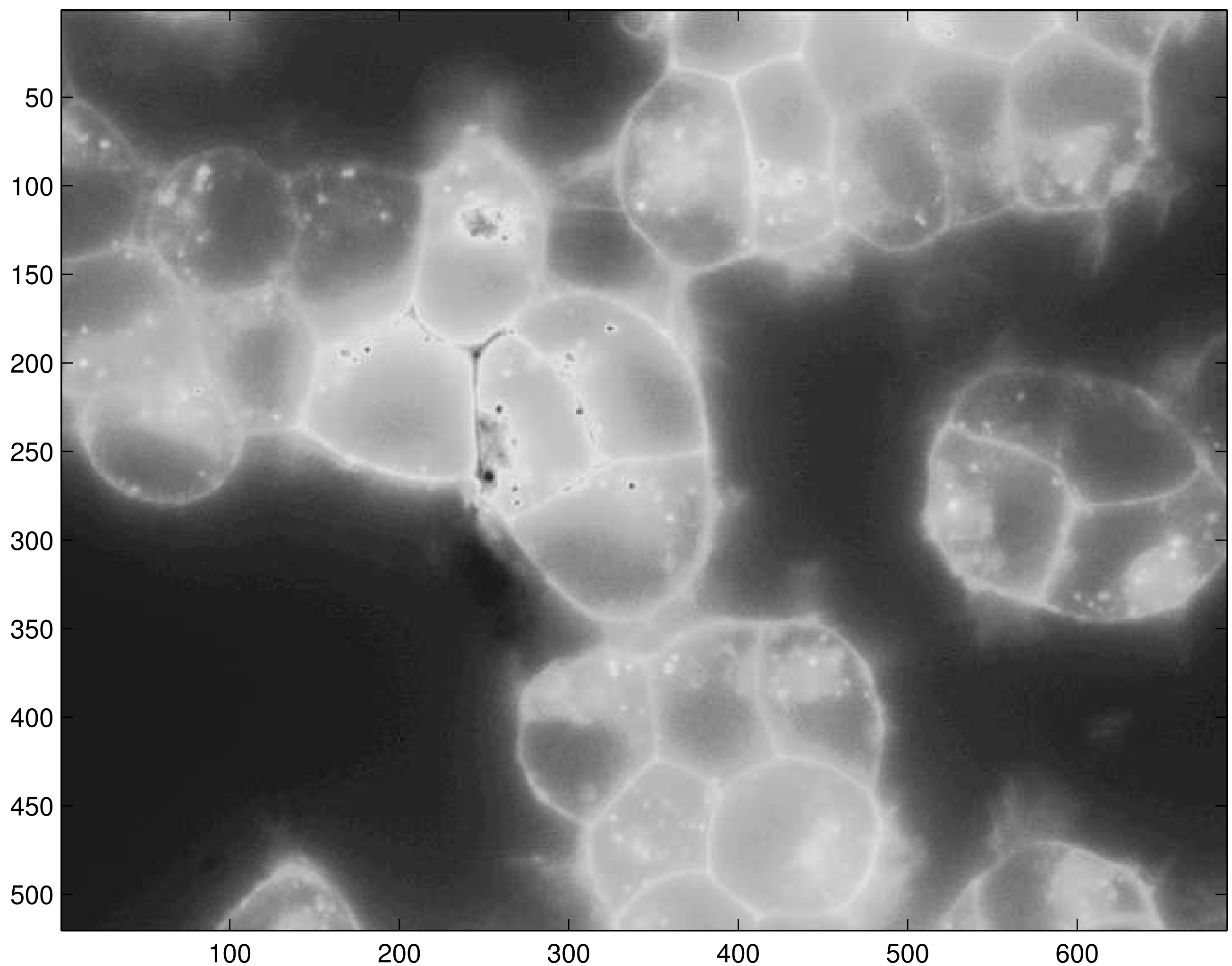
* Department of Mathematics,

University of Bergen,

Johannes Brunsgt. 12,

5007 Bergen, Norway,

Phase field models



Curve or surface to minimize energy

$$\min_{\Gamma} E(\Gamma).$$

Examples:

- Curvature energy

$$E = \int_{\Gamma} \frac{c}{2} \kappa^2 ds.$$

  κ is the curventure in 2D and the mean curvature in 3D, i.e. $\kappa = (\kappa_1 + \kappa_2)/2$ and κ_1 κ_2 are the principle curvatures.

- Hook's law:

$$E = \int_{\Gamma} (a + b(\kappa - c_0) + cG)^2 ds.$$

  G is the Gaussian curvature for 3D problems.

Curve or surface to minimize energy

$$\min_{\Gamma} E(\Gamma).$$

Examples:

- Bulk energy

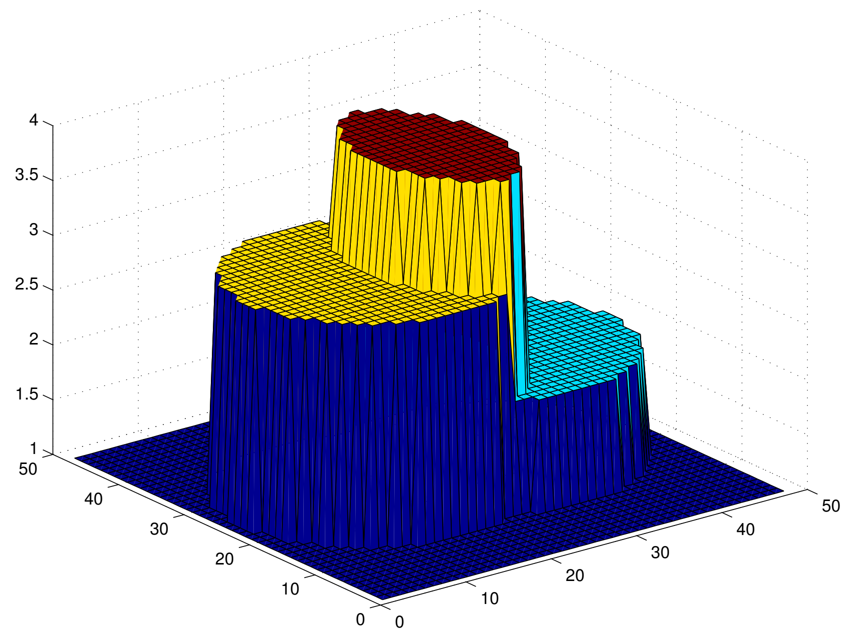
$$E = E_1 + E_2.$$

$$E_1 = \sum_{1 \le i,j \le n} f_{ij} length(\Gamma_{ij}).$$

$$E_2 = \sum_{1 \le i,j \le n} e_i area(\Omega_i).$$

Elliptic inverse problems

$$-\nabla \cdot (q(x)\nabla u) = f, \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$



Our interests: q is discontinuous and is piecewise smooth. We want to use some information from u to recover $q(x)$. The conduct media only contains four types of materials in this example.

Applications

Fig. 3 *The ACT 3 system with 32 electrodes encircling the chest of a subject. Here the goal is to make images that show the changes in volumes of air and blood that occur with breathing and the pulsatile circulation of the blood. Such images are referred to as ventilation and perfusion images, respectively. This is one possible positioning of electrodes for this purpose.*

images. However, EIT is low cost, noninvasive, and provides information about the electrical parameters of the body, which is information that cannot be obtained by these other methods. Specific clinical applications still need to be explored.

Appendix 1: Derivation of (2.1) from Maxwell's Equations. The fixed-frequency version of Maxwell's equations is

$$\nabla \wedge E = -i\omega\mu H, \tag{A.1}$$

Applications

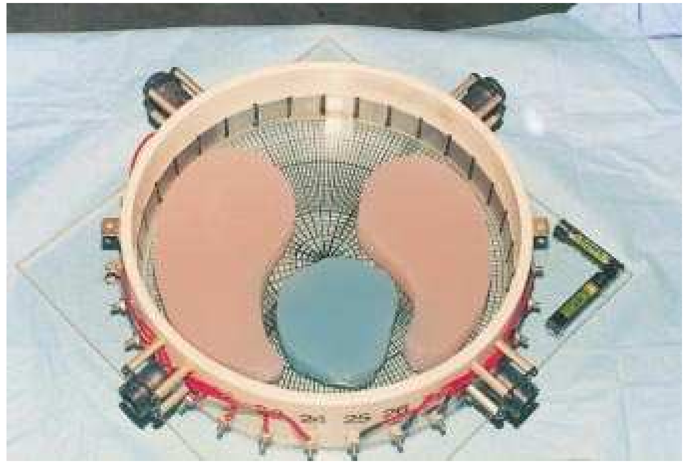


Fig. 1 *A test tank containing "lungs" and "heart" made of agar with varying amounts of added salt. This tank is filled with salt water, and used as a test body for the EIT system. Note the large electrodes around the inner circumference of the tank.*

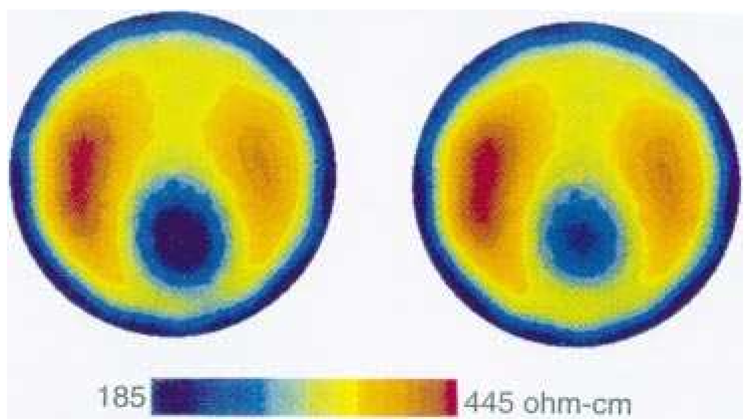


Fig. 2 *Images of the resistivity of two different test tanks like the one shown in Figure 1. The two different tanks had hearts of different sizes, meant to simulate different times during the heart's cycle.*

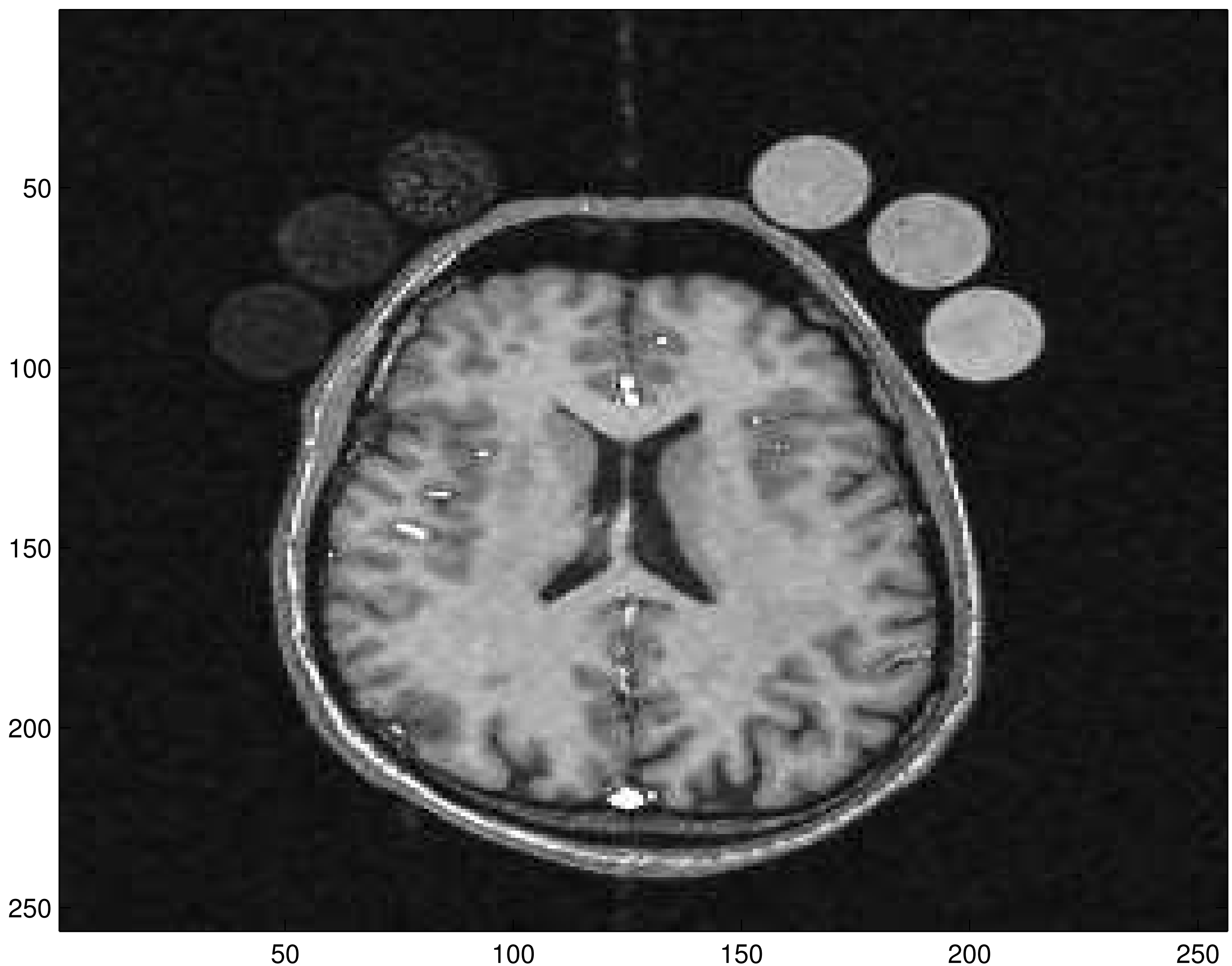
Finally, the ill-posedness of the problem also makes it unlikely that EIT images, even with many electrodes, will have resolution comparable to that of CT or MRI

Applications

Computer graphics: use computer to identify number of car plates.

Applications

Applications



Other Applications

- Reservoir simulations, find the zones
- Tumors or demaged detection in human heart and brain
- no-damage detections in industrial applications
- ...

Phase field method

The idea for phase field model is related to several well-known equations:

- Allen-Cahn equation:

$$u_t - \Delta u + \frac{1}{\epsilon^2} f(u) = 0, \text{ in } \Omega \times [0, T].$$

- Ginzburg-Landau equation.
- Cohn-Hillard equation
- Liquid Crystale
- ...

Phase field method

Two-phase case (stationary):

$$-\epsilon \Delta u + \frac{1}{\epsilon} u(u^2 - 1) = 0 \text{ in } \Omega \quad u = g \text{ on } \partial\Omega.$$

$$\min_{u=g} \int_\Omega \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{4\epsilon} (u^2 - 1)^2 dx.$$

Properties when $\epsilon \longrightarrow 0$

$$\min_{u=g} \int_\Omega \frac{\epsilon}{2}\|\nabla u\|^2 + \frac{1}{4\epsilon}(u^2-1)^2 dx.$$

It was proved that when $\epsilon \longrightarrow 0$, we have

- Values:

$$u \longrightarrow \pm 1.$$

- Length:

$$\epsilon \int_\Omega \|\nabla u\|^2 \longrightarrow 0.94\, length(\Gamma).$$

- Area:

$$\frac{1}{2}\int_\Omega (u+1) \longrightarrow area(\Omega_1).$$

$$\frac{1}{2}\int_\Omega (1-u) \longrightarrow area(\Omega_2).$$

- Curvature:

$$\epsilon \int_\Omega |\Delta u|^2 dx \longrightarrow \alpha \int_\Gamma \kappa^2 ds.$$

Properties when $\epsilon \to 0$

$$-\epsilon \Delta u + \frac{1}{\epsilon} u(u^2 - 1) = 0 \text{ in } \Omega \quad u = g \text{ on } \partial\Omega.$$

$$\min_{u=g} \int_\Omega \frac{\epsilon}{2} \|\nabla u\|^2 + \frac{1}{4\epsilon}(u^2 - 1)^2 dx.$$

$$u \to u^* = \pm 1.$$

Thus for $g = \pm 1$, the limit u^* is the solution of

$$\min_{u=g} Length(\Gamma).$$

Other energy functionals

$$-\epsilon\Delta u + \epsilon\Delta^2 u + \frac{1}{\epsilon}u(u^2-1) = 0 \text{ in } \Omega \quad u = g \text{ on } \partial\Omega.$$

$$\min_{u=g}\int_\Omega \frac{\epsilon}{2}|\nabla u|^2 + \frac{\epsilon^2}{2}|\Delta u|^2) + \frac{1}{4\epsilon}(u^2-1)^2 dx.$$

$$u \to u^* = \pm 1.$$

Thus for $g = \pm 1$, the limit u^* is the solution of

$$\min_{u=g}\int_\Gamma ds + \int_\Gamma \kappa^2 ds.$$

Time evolution

$$v_t - \epsilon \Delta v + \frac{1}{\epsilon} v(v^2 - 1) = 0 \text{ in } \Omega \quad v = g \text{ on } \partial\Omega.$$

$$v_t - \epsilon \Delta v + \epsilon^2 \Delta^2 v + \frac{1}{\epsilon} v(v^2 - 1) = 0 \text{ in } \Omega \quad v = g \text{ on } \partial\Omega.$$

For fixed ϵ, we have

$$v(x,t) \to u(x) \text{ as } t \to \infty.$$

This offers some fast methods for solving the stationary equation for fixed ϵ.

Phase field for more than two phases

Use a vector function for the phase values:

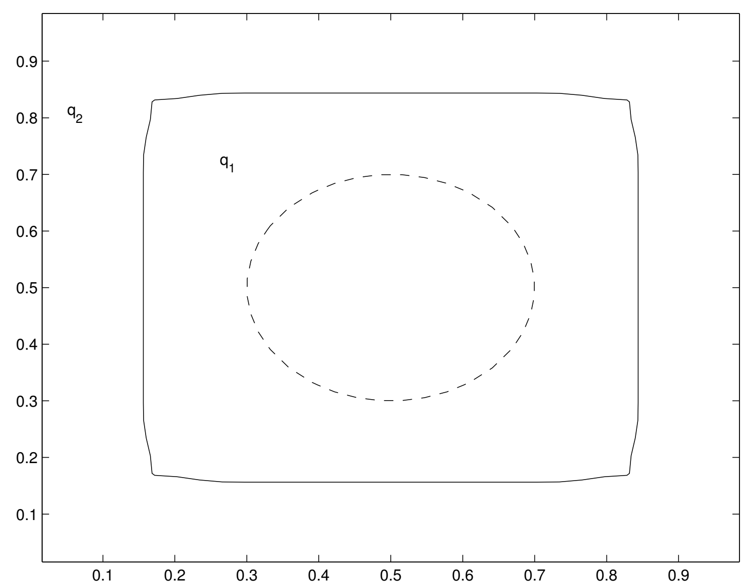
$$\mathbf{u} = (u_1, u_2, \cdot, u_m).$$

Levelset Methods.

Optimal shape design approach

Assume that the values of the constants are known and we only need to identify the discontinuities:

$$\Gamma : x = x(t), y = y(t), t \in [a, b].$$



Output-least-squares:

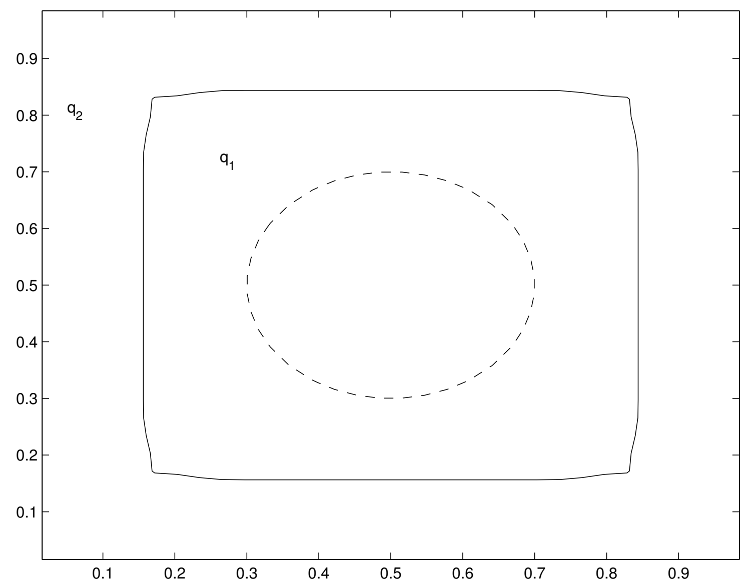
$$\min F(\Gamma), \qquad F(\Gamma) = \|u(\Gamma) - u_d\|_G + \beta L(\Gamma)$$

Optimal shape design approach

$$\vec{x}(t) = (x(y), y(t)).$$

$$\vec{x}(t) := \vec{x}(t) + \delta(t)\vec{n}(t).$$

$$a = t_0 < t_1 < \cdots < t_n = b.$$



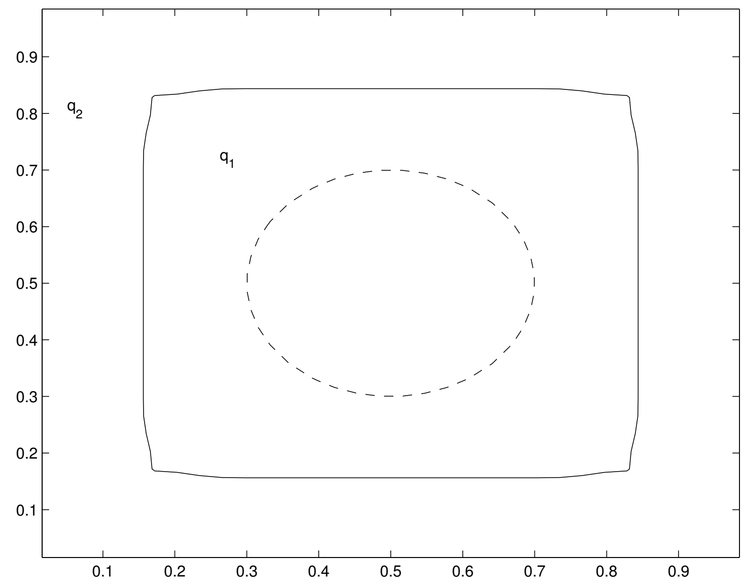
Find the value of $\delta(t)$ that gives the optimal values.

Weak point of the shape derivative approach

- Diffcult to handle complicated geometries.
- Diffulct if a curve disappears.
- Diffulct if a curve splits.
- Diffulct if curves merge with each other.
- The points could get clustered.

The level set idea (Osher and Sethian)

Start external animation



$$\Gamma \equiv \{x | \ \phi(x) = 0\}.$$

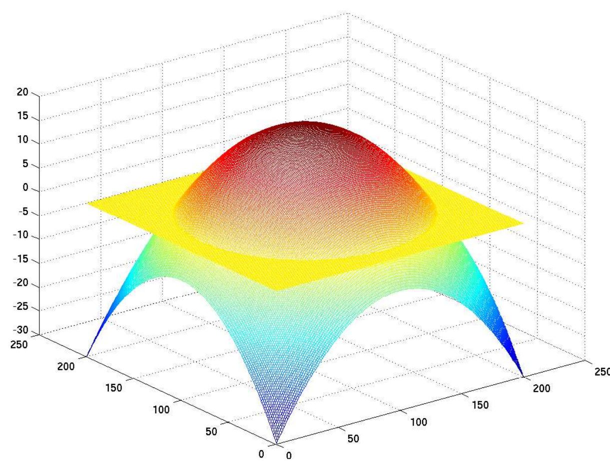
$$\phi(x) = \begin{cases} \text{distance}(x, \Gamma), & x \in \text{interior of } \Gamma \\ -\text{distance}(x, \Gamma), & x \in \text{exterior of } \Gamma, \end{cases}$$

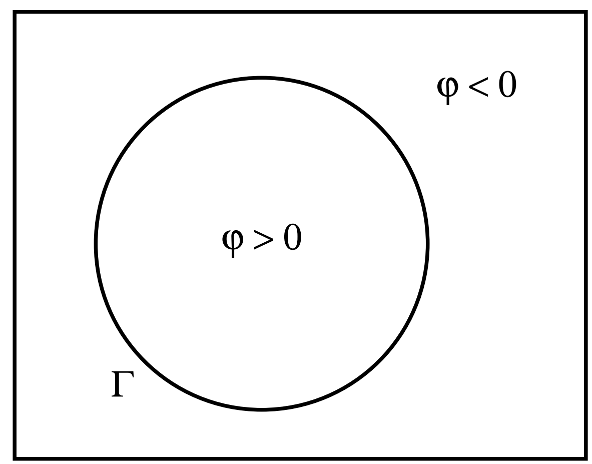
Representing p.w. constant functions by level set – two region case

Heaviside function:

$$H(\phi) = \begin{cases} 1, & \phi > 0 \\ 0, & \phi \leq 0, \end{cases}$$

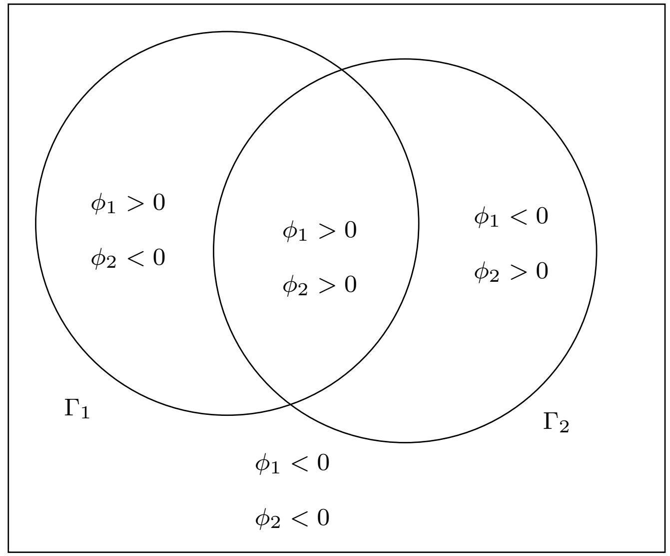
$$q = q_1 H(\phi) + q_2 \left(1 - H(\phi)\right). \quad (1)$$





More than two regions–multiple level-sets

(Chan and Vese, 2000)



$$
\begin{aligned}
\Omega_{++} &= \{x \in \Omega, \quad \phi_1 > 0, \quad \phi_2 > 0\} \\
\Omega_{+-} &= \{x \in \Omega, \quad \phi_1 > 0, \quad \phi_2 < 0\} \\
\Omega_{-+} &= \{x \in \Omega, \quad \phi_1 < 0, \quad \phi_2 > 0\} \\
\Omega_{--} &= \{x \in \Omega, \quad \phi_1 < 0, \quad \phi_2 < 0\} .
\end{aligned}
$$

$$
\begin{aligned}
q &= q_1 H(\phi_1)H(\phi_2) + q_2 H(\phi_1)(1 - H(\phi_2)) + \\
&\quad + q_3(1 - H(\phi_1))H(\phi_2) + q_4(1 - H(\phi_1))(1 - H(\phi_2)).
\end{aligned}
$$

The general case

$$
\begin{aligned}
q \quad = \quad & q_1 H(\phi_1) H(\phi_2) \cdots H(\phi_n) + \\
& + q_2 (1 - H(\phi_1)) H(\phi_2) \cdots H(\phi_n) + \\
& \quad \vdots \\
& + q_{2^n} (1 - H(\phi_1)) (1 - H(\phi_2)) \cdots (1 - H(\phi_n)) .
\end{aligned}
$$

- For n regions, only needs $\log_2(n)$ level-set functions.
- No need to know the number of regions–some of the regions are allowed to be empty automatically.
- We do not need to know ϕ_j exactly. We just need to know where it is positive and where it is negative.
- We will try to identify the level set functions ϕ_j and also the constant values q_i.
- Easy to extend to cases that q is piecewise polynomial or piecewise smooth (Chan and Vese).

The essential idea

$$\{\Omega_i\}_{i=1}^{m} \Longleftrightarrow \{\Gamma_i\}_{i=1}^{n} \Longleftrightarrow \{\phi_i\}_{i=1}^{n} \Longleftrightarrow \{H(\phi_i)\}_{i=1}^{n}$$

For general shape optimization problems (Chan Tai, JCP03, vol.193)

- Consider

$$\min_{q \in K} F(q).$$

- Gradient method needs $\frac{\partial F}{\partial q}$.
- Represent q by level-set functions and the constant values

$$q = q(q_i, \phi_i).$$

- In order find the constant q_i values and the level-set functions ϕ_i, we need

$$\frac{\partial F}{\partial \phi} \text{ and } \frac{\partial F}{\partial q_i}.$$

- Use the chain rule:

$$\frac{\partial F}{\partial \phi} = \frac{\partial F}{\partial q}\frac{\partial q}{\partial \phi}. \qquad \frac{\partial F}{\partial q_i} = \int_\Omega \frac{\partial F}{\partial q}\frac{\partial q}{\partial q_i}.$$

Level Sets Methods for Elliptic Inverse Problems

Tony F. Chan and Xue–Cheng Tai

Xue-Cheng.Tai@mi.uib.no, http://www.mi.uib.no/%7Etai.

UCLA, Department of Mathematics, California

Mathematics Department, University of Bergen, Norway.

(Chan Tai, JCP03, vol.193)

An inverse problem

$$-\nabla \cdot (q(x)\nabla u) = f, \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

We shall use some observations of u to recover $q(x)$.

$$\min F(q), \qquad F(q) = \|u(q) - u_d\|_G + \beta R(q)$$

or

$$\min_{-\nabla\cdot(q(x)\nabla u)=f} F(q, u), \qquad F(q, u) = \|u - u_d\|_G + \beta R(q)$$

β is a small regularization parameter to be chosen properly.

The problem is identifiable if $u \in H_0^{1+\epsilon}(\Omega)$ and $\nabla u \neq 0$ a.e. in Ω. (There are many references for this).

The Augment Lagrangian for the inverse problem

Assume that we have an observation u_d everywhere for the solution u, try to solve

$$\min_{-\nabla\cdot(q\nabla u)=f} \frac{1}{2}\int_\Omega |u(q)-u_d|^2 dx + \beta R(q) dx.$$

Define equation error by

$$-\Delta e = -\nabla \cdot (q\nabla u) - f.$$

Use augmented Lagrangian method

$$L(q,u,\lambda) = \frac{1}{2}\|u-u_d\|^2 + \frac{c}{2}\|\nabla e\|^2 + (\nabla\lambda, \nabla e) + \beta R(q) dx.$$

Tests have also been done without using augmented Lagrangian method.

For general shape optimization problems

- Consider

$$\min_{q\in K} F(q).$$

- Gradient method needs $\frac{\partial F}{\partial q}$.
- Represent q by level-set functions and the constant values

$$q = q(q_i, \phi_i).$$

- In order find the constant q_i values and the level-set functions ϕ_i, we need

$$\frac{\partial F}{\partial \phi} \text{ and } \frac{\partial F}{\partial q_i}.$$

- Use the chain rule:

$$\frac{\partial F}{\partial \phi} = \frac{\partial F}{\partial q}\frac{\partial q}{\partial \phi}. \qquad \frac{\partial F}{\partial q_i} = \int_\Omega \frac{\partial F}{\partial q}\frac{\partial q}{\partial q_i}.$$

Computing the gradients

In case of two regions:

$$q = q_1 H(\phi) + q_2(1 - H(\phi)).$$

$$\frac{\partial q}{\partial \phi} = (q_1 - q_2)\delta(\phi).$$

$$\frac{\partial q}{\partial q_1} = H(\phi) \quad \frac{\partial q}{\partial q_2} = 1 - H(\phi).$$

$$\frac{\partial R}{\partial \phi} = -\nabla \cdot \left(\frac{\nabla q}{|\nabla q|}\right)(q_1 - q_2)\delta(\phi)$$

A two-level-set example

Gradient with respect to ϕ_j:

$$
\begin{aligned}
\frac{\partial L}{\partial \phi_1} &= \delta(\phi_1)\Big(\big[(q_1 - q_3)H(\phi_2) + (q_2 - q_4)(1 - H(\phi_2))\big]\big[\nabla u \cdot \nabla(ce - \lambda)\big] \\
&\qquad - \beta \nabla \cdot \frac{\nabla \phi_1}{|\nabla \phi_1|}\Big) \\
\frac{\partial L}{\partial \phi_2} &= \delta(\phi_2)\Big(\big[(q_1 - q_2)H(\phi_1) + (q_3 - q_4)(1 - H(\phi_1))\big]\big[\nabla u \cdot \nabla(ce - \lambda)\big] \\
&\qquad - \beta \nabla \cdot \frac{\nabla \phi_2}{|\nabla \phi_2|}\Big).
\end{aligned} \tag{2}
$$

The Uzawa algorithm

Choose initial q_i^0, ϕ_j^0, u^0 and λ^0, c^0. Set $k = 0$.

- Solve the following minimization problem:

$$\left(q_i^{k+1}, \tilde{\phi}_j^{k+1}, u^{k+1}\right) = \arg \min_{\substack{q_i \in R \\ \phi_j \in W^{1,1}(\Omega) \\ u \in H_0^1(\Omega)}} L\left(q_i, \phi_j, u, \lambda^k\right). \tag{3}$$

- Re-initialize the level-set functions: $\Gamma_j^k = \{\tilde{\phi}_j^k = 0\}$.

$$\phi_j^{k+1} = \text{sign } distance(x, \Gamma_j^{k+1})$$

.

- Update λ:

$$\lambda^{k+1} = \lambda^k - c\, e\left(q_i^{k+1}, \phi_j^{k+1}, u^{k+1}\right).$$

The used algorithm

- **(Update q_i's)** Let $p^k = \left\{ -\frac{\partial L_r(q_i^k, \phi_j^k, u^k, \lambda^k)}{\partial q_i} \right\}_{i=1}^{2^n}$. Find α^k such that

$$\alpha^k = \arg \min_{q_i^k + \alpha p_i^k \in [a_i, b_i], i=1,\cdots,2^n} L_r(q_i^k + \alpha p_i^k, \phi_j^k, u^k, \lambda^k). \tag{4}$$

  Set $\{q_i^{k+1}\}_{i=1}^{2^n} = \{q_i^k\}_{i=1}^{2^n} + \alpha^k p^k$.

- **(Update ϕ_j's)** For $j = 1, 2, \cdots n$, define $\psi_j^k = -\frac{\partial L_r(q_i^{k+1}, \phi_j^k, u^k, \lambda^k)}{\partial \phi_j}$ and find σ_j^k such that

$$\sigma_j^k = \arg \min_{\sigma_j \in R} L_r(q_i^{k+1}, \phi_j^k + \sigma_j \psi_j^k, u^k, \lambda^k). \tag{5}$$

  Set $\tilde{\phi}_j^k = \phi_j^k + \sigma_j^k \psi_j^k$.

- **(Update u)** Calculate $v^k = -\frac{\partial L_r(q_i^{k+1}, \phi_j^{k+1}, u^k, \lambda^k)}{\partial u}$. Find ξ_k such that

$$\xi^k = \arg \min_{\xi \in R} L_r(q_i^{k+1}, \phi_j^{k+1}, u^k + \xi v^k, \lambda^k). \tag{6}$$

  Set $u^{k+1} = u^k + \xi^k v^k$.

- Go to the next iteration for k.

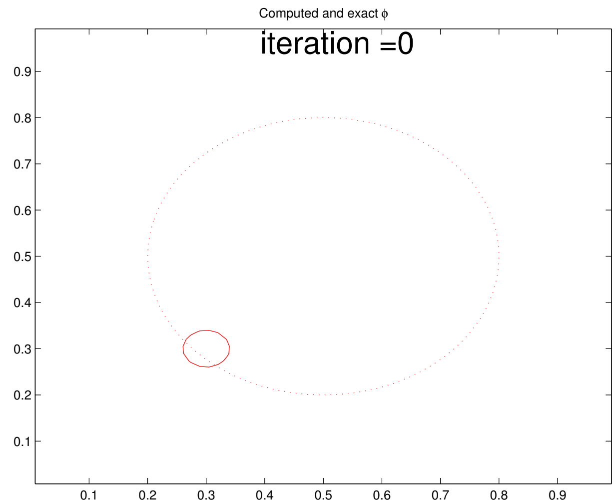
Use smoothed H and δ functions

$$H_\epsilon(\phi) = \frac{1}{\pi}\tan^{-1}\frac{\phi}{\epsilon} + \frac{1}{2}, \quad (7)$$

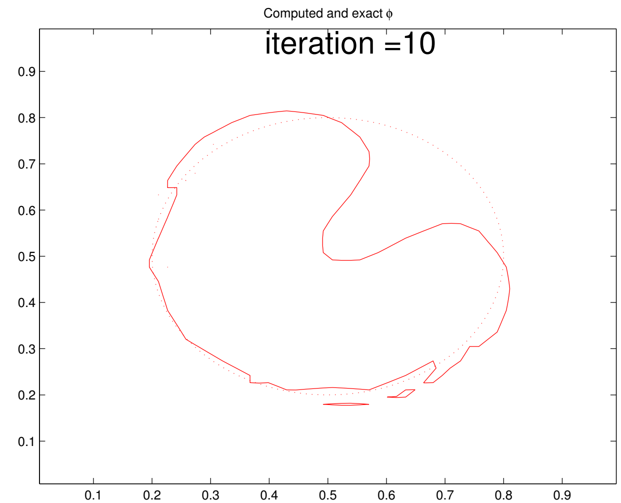
$$\delta_\epsilon(\phi) = \frac{\epsilon}{\pi\left(\phi^2 + \epsilon^2\right)}. \quad (8)$$

$$\begin{aligned}\frac{\partial L}{\partial \phi_1} &= \delta(\phi_1)\Big(\big[(q_1 - q_3)H(\phi_2) + (q_2 - q_4)(1 - H(\phi_2))\big]\left[\nabla u \cdot \nabla(ce - \lambda)\right] \\ &\quad -\beta\nabla\cdot\tfrac{\nabla\phi_1}{|\nabla\phi_1|}\Big) \\ \frac{\partial L}{\partial \phi_2} &= \delta(\phi_2)\Big(\big[(q_1 - q_2)H(\phi_1) + (q_3 - q_4)(1 - H(\phi_1))\big]\left[\nabla u \cdot \nabla(ce - \lambda)\right] \\ &\quad -\beta\nabla\cdot\tfrac{\nabla\phi_2}{|\nabla\phi_2|}\Big).\end{aligned} \quad (9)$$

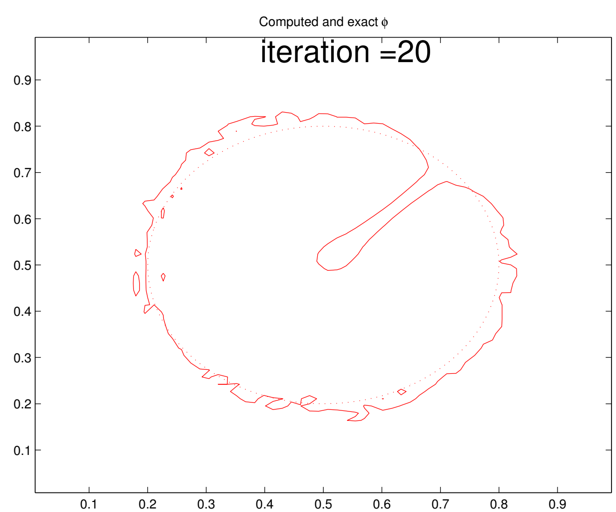
Experiments–A simple example, 20% *noise.*



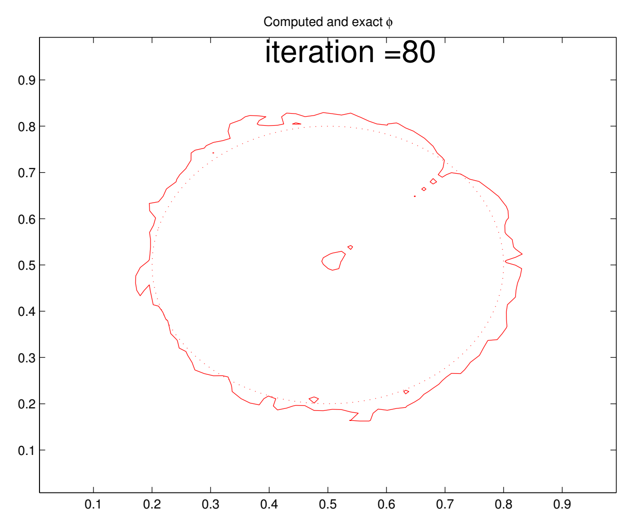
(a) Initial



(b) 10 iterations

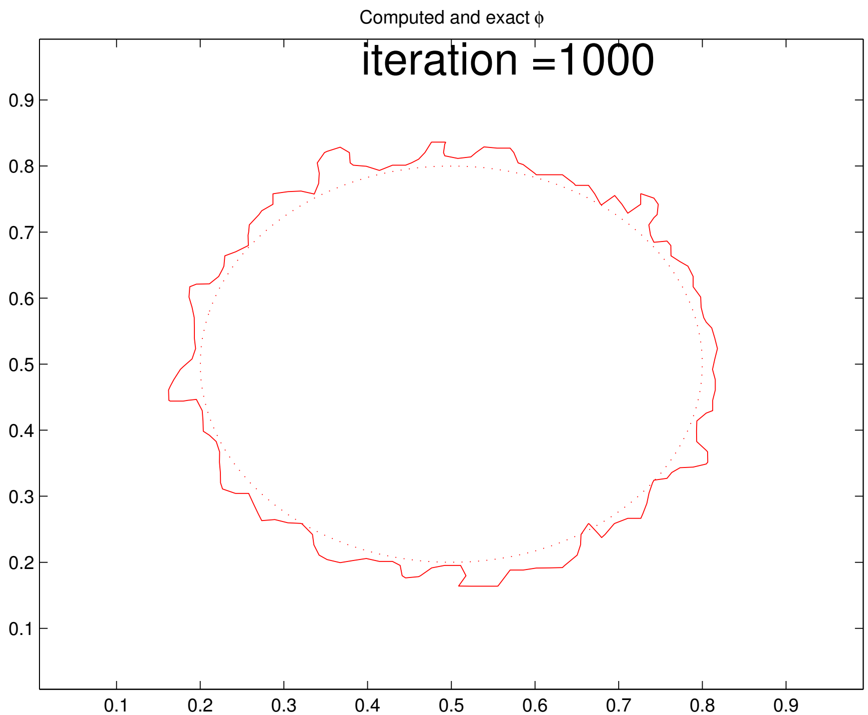


(c) 20 iterations

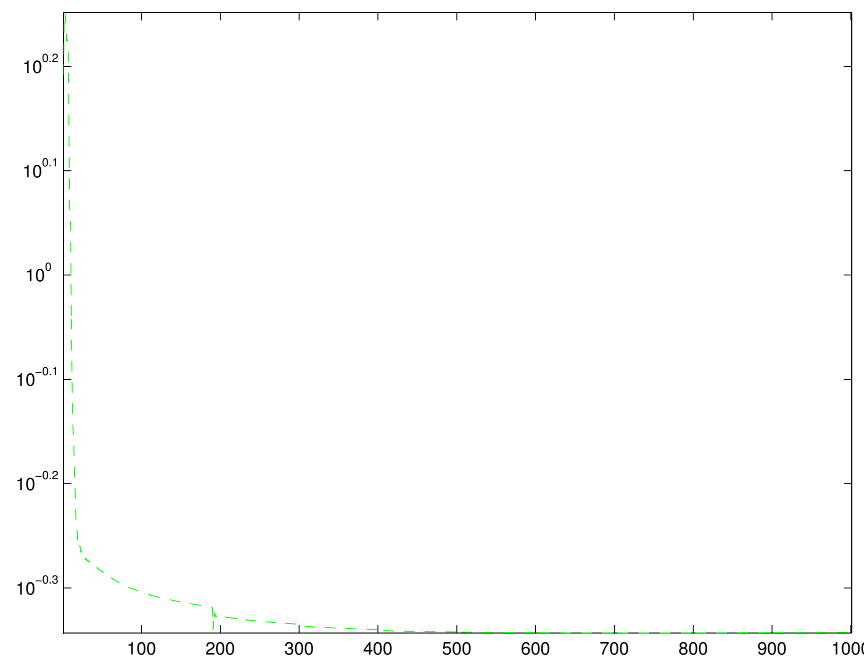


(d) 80 iterations

Experiments–A simple example, 20% noise.

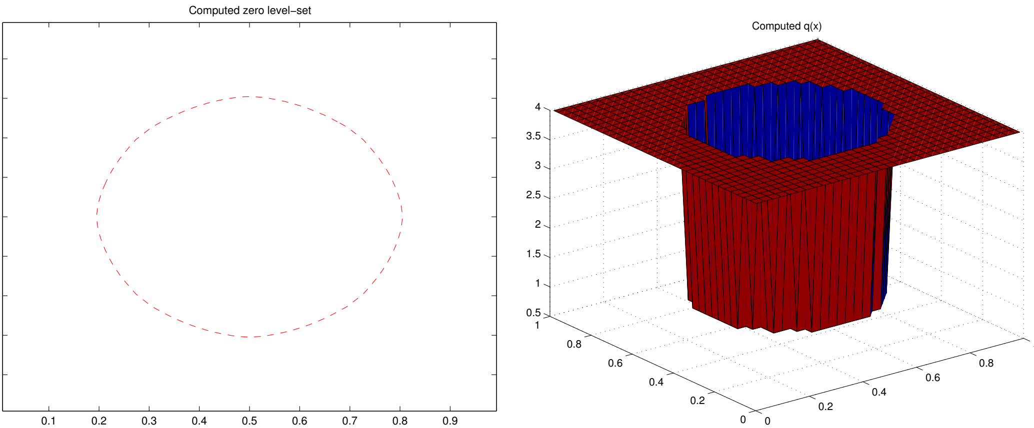


(e) 1000 iterations

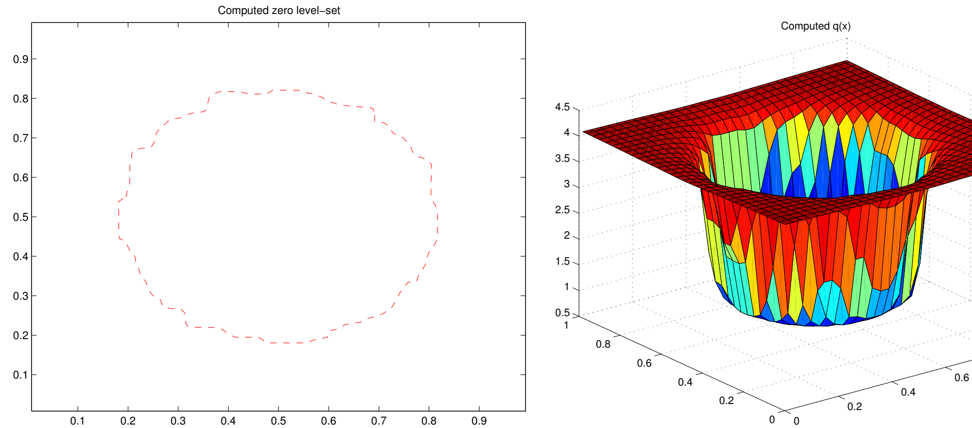


(f) $\|q - q^k\|_{L^2(\Omega)}$

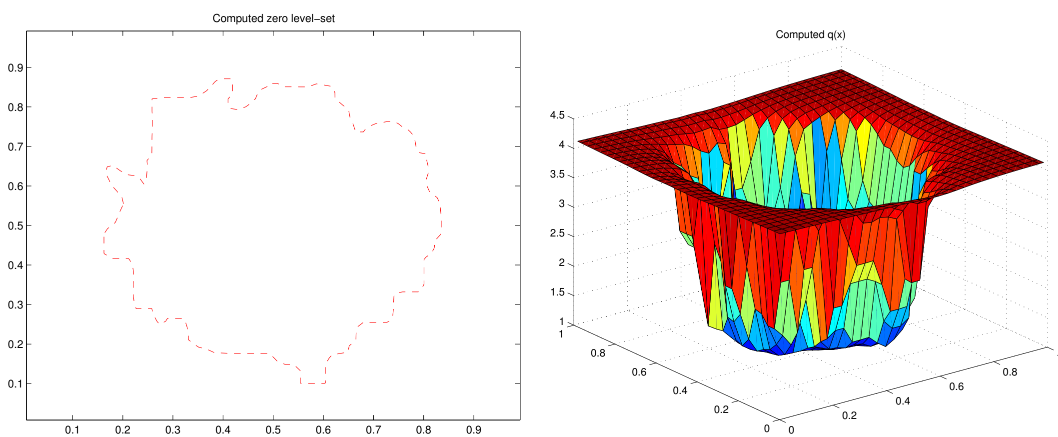
Experiments– A simple example, different noise levels, $0, 5\%, 20\%$



(g) With $\sigma = 0, r = 10^{-4}, \beta = 10^{-9}$

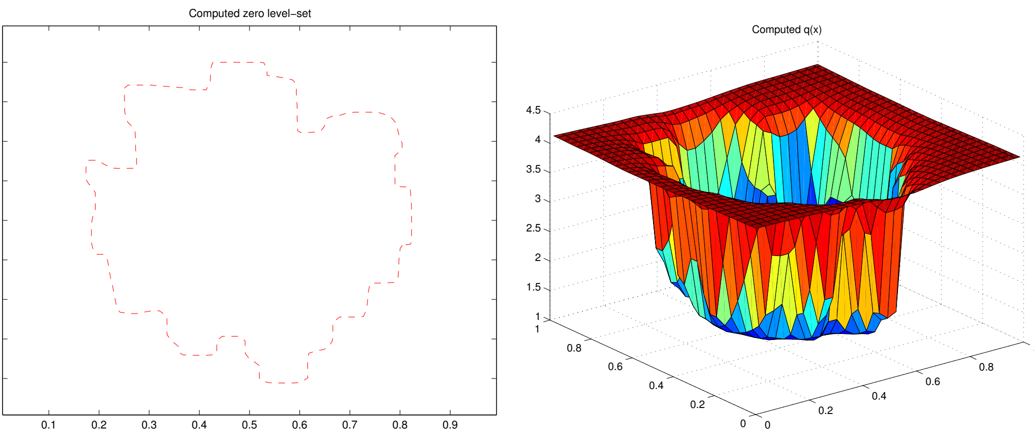


(h) With $\sigma = 5\%, r = 10^{-3}, \beta = 10^{-5}$

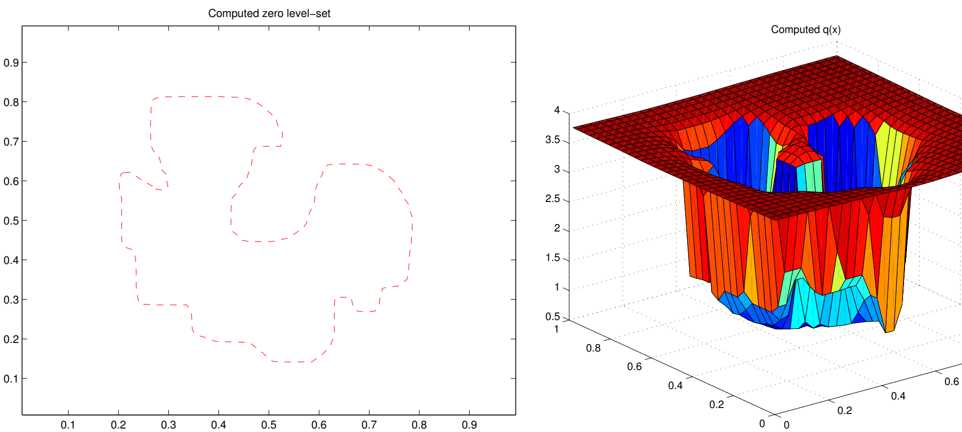


(i) With $\sigma = 20\%, r = 10^{-3}, \beta = 10^{-5}$

Experiments– A simple example, different noise levels $40\%, 60\%$

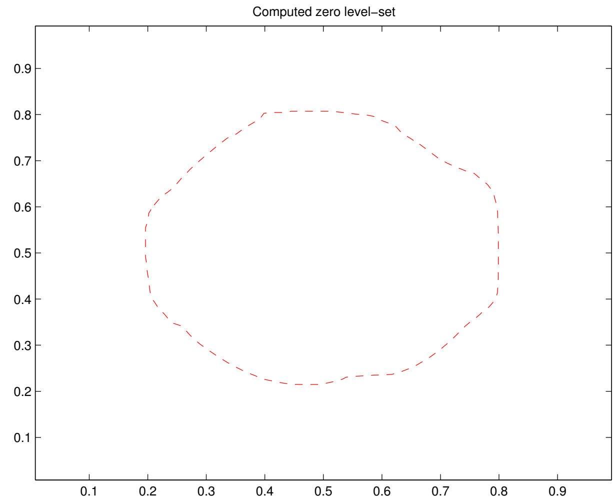


(a) With $\sigma = 40\%, r = 10^{-3}, \beta = 5 \times 10^{-5}$

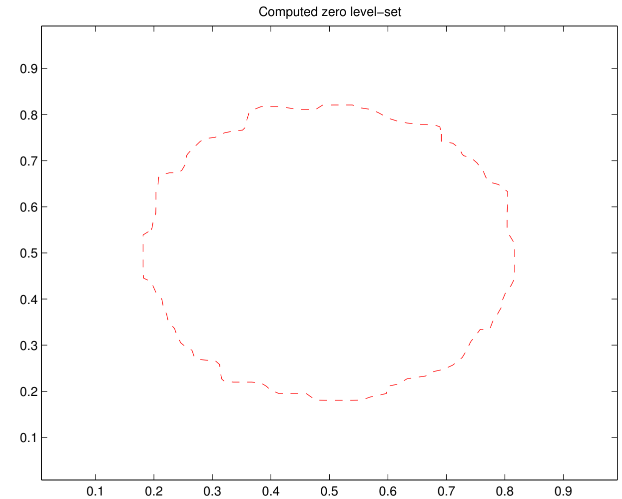


(b) With $\sigma = 60\%, r = 10^{-3}, \beta = 1$ 10^{-4}

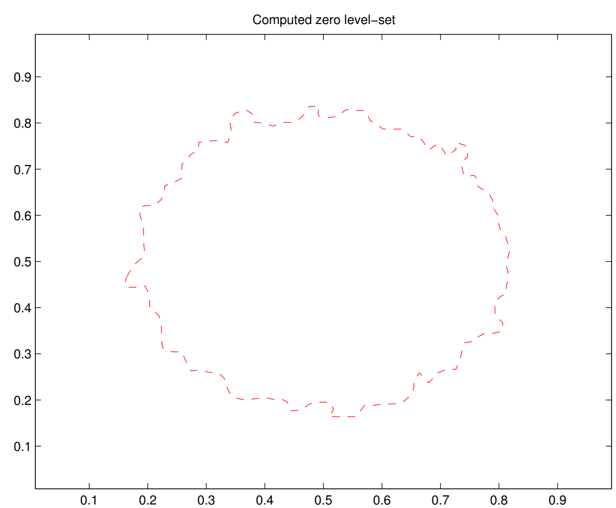
Experiments–A simple example, different β values.



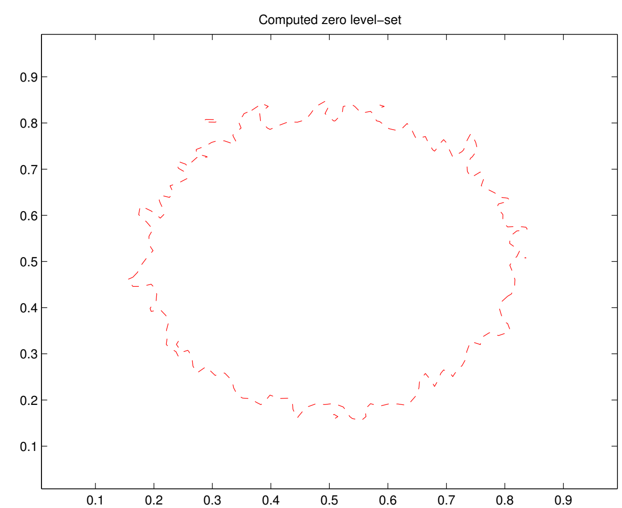
(c) $\beta = 10^{-3}$



(d) $\beta = 10^{-5}$

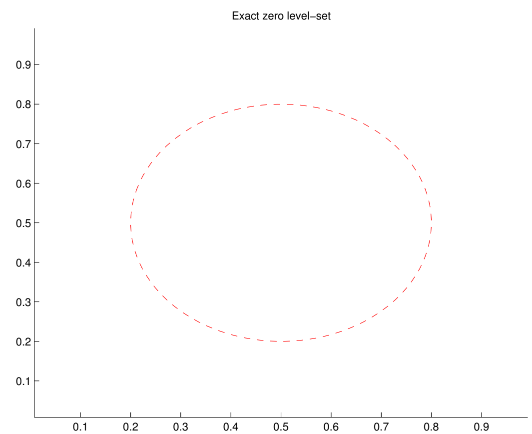


(e) $\beta = 10^{-6}$

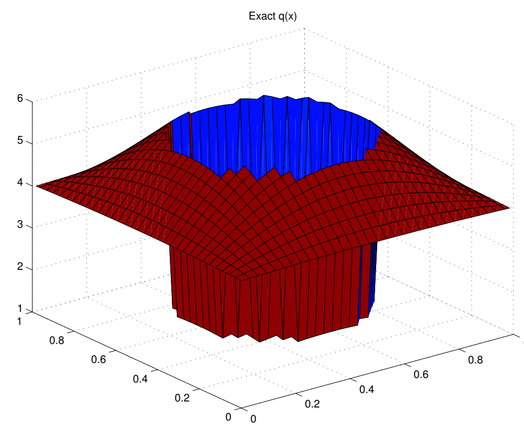


(f) $\beta = 10^{-9}$

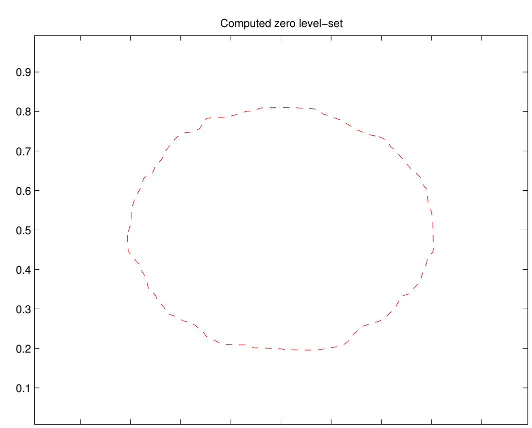
Experiments–piecewise smooth q, 5% noise.

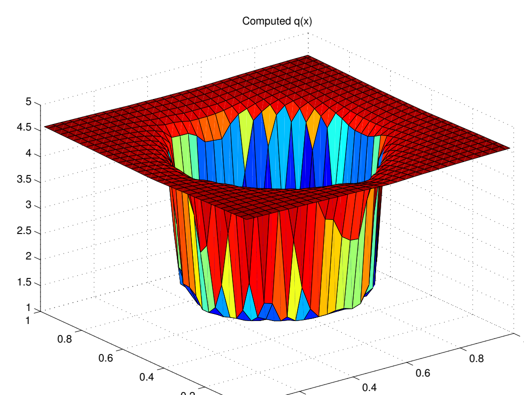


(a) Exact zero level curve

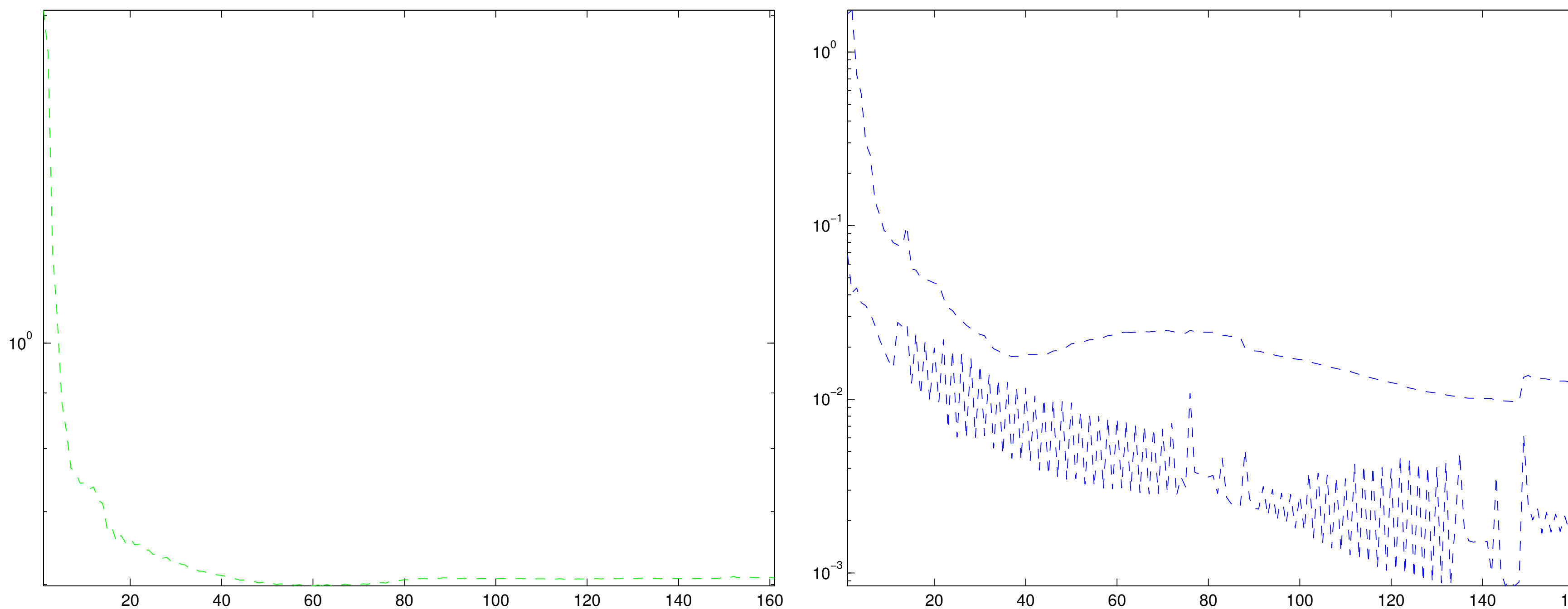


(b) Exact $q(x)$





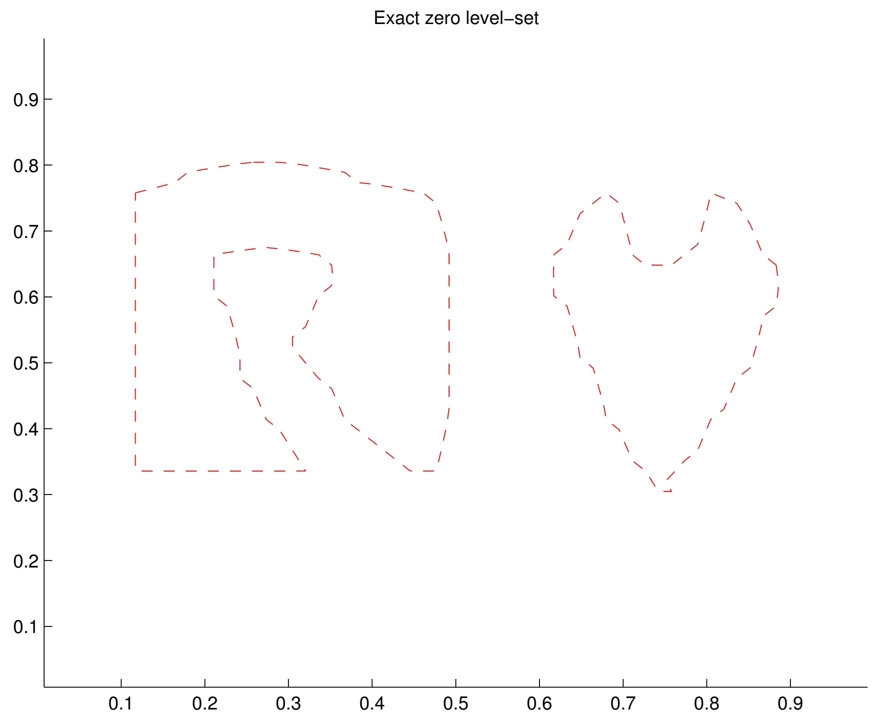
Experiments–piecewise smooth q, 5% noise.



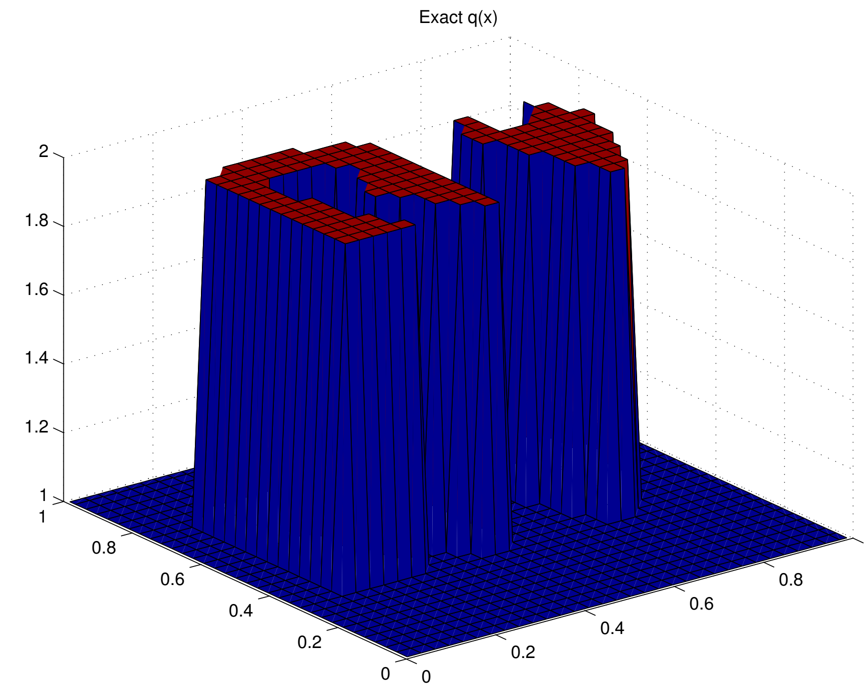
(a) $\|q - q^k\|_{L(\Omega)}$ (b) $\|e^k\|_{L^2(\Omega)}$ and $\|r^k\|_{L^2(\Omega)}$

Figure 4:

Experiments– Another example, one level set.



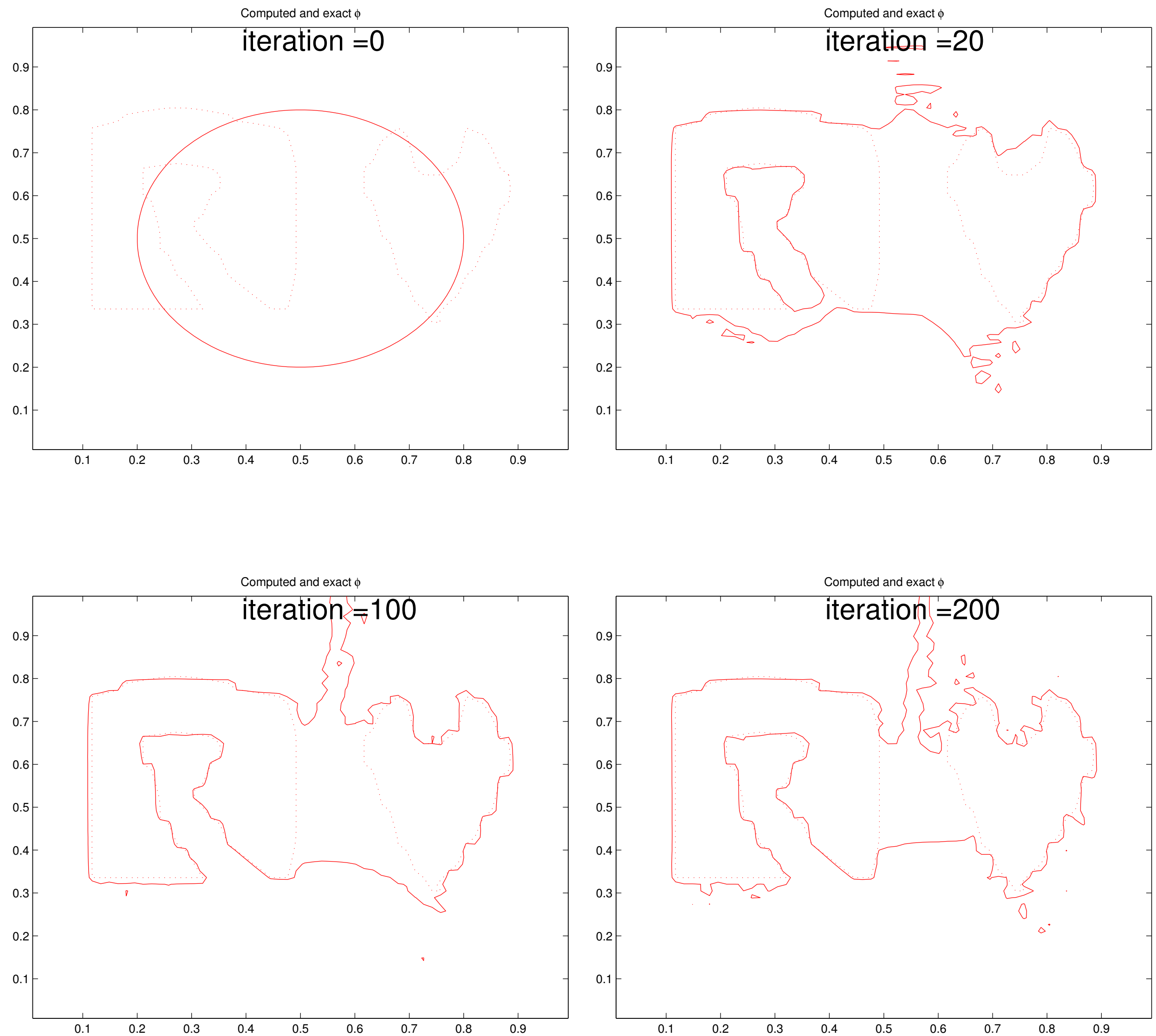
(a) Exact zero level sets



(b) Exact $q(x)$

Figure 5:

Experiments– Another example, one level set. Start with two-regions and it splits into two regions.



Experiments– Another example, one level set. Start with two-regions and it splits into two regions.

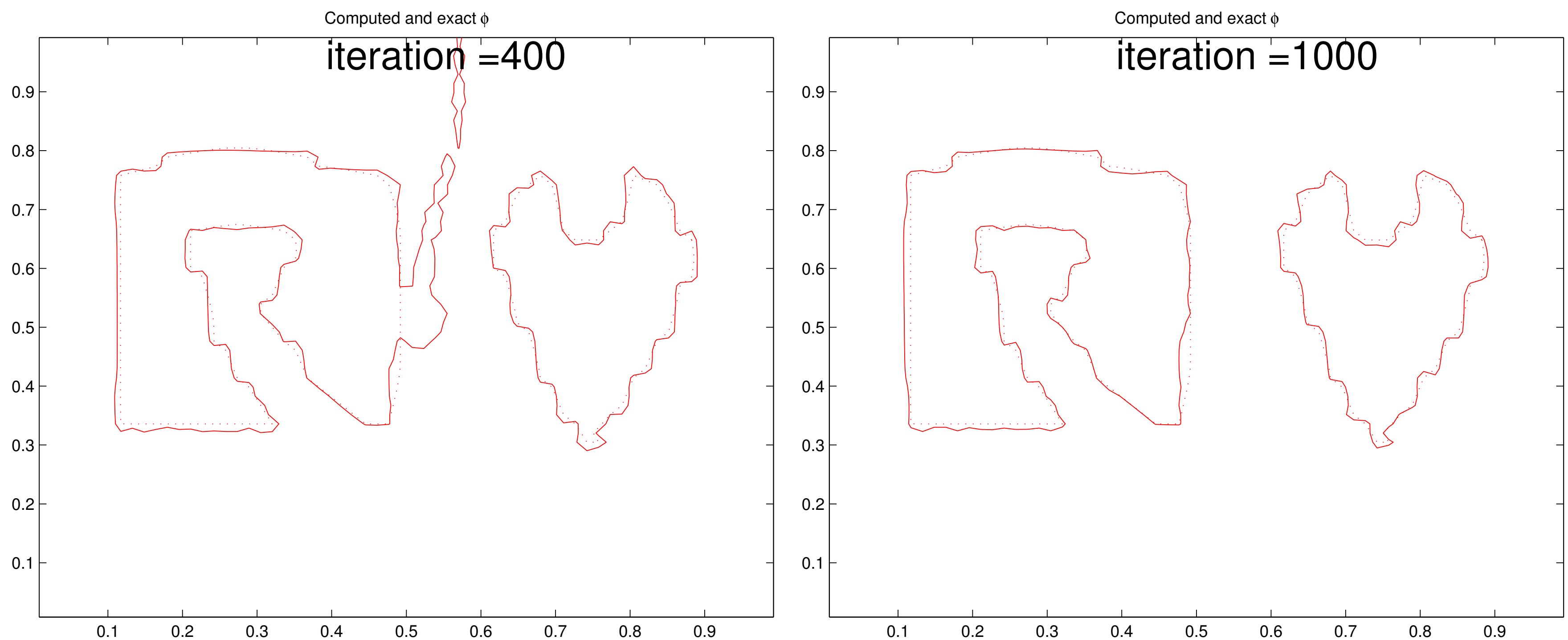
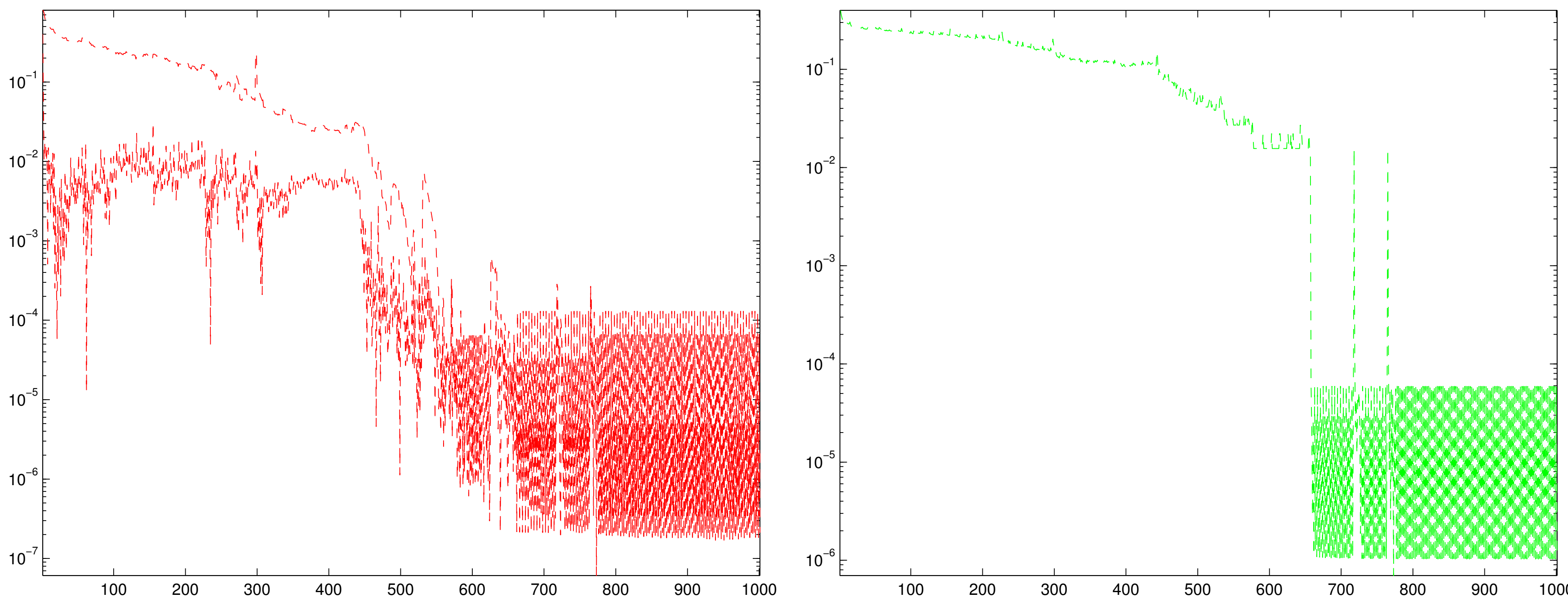


Figure 7:

Experiments– Another example, one level set convergence.

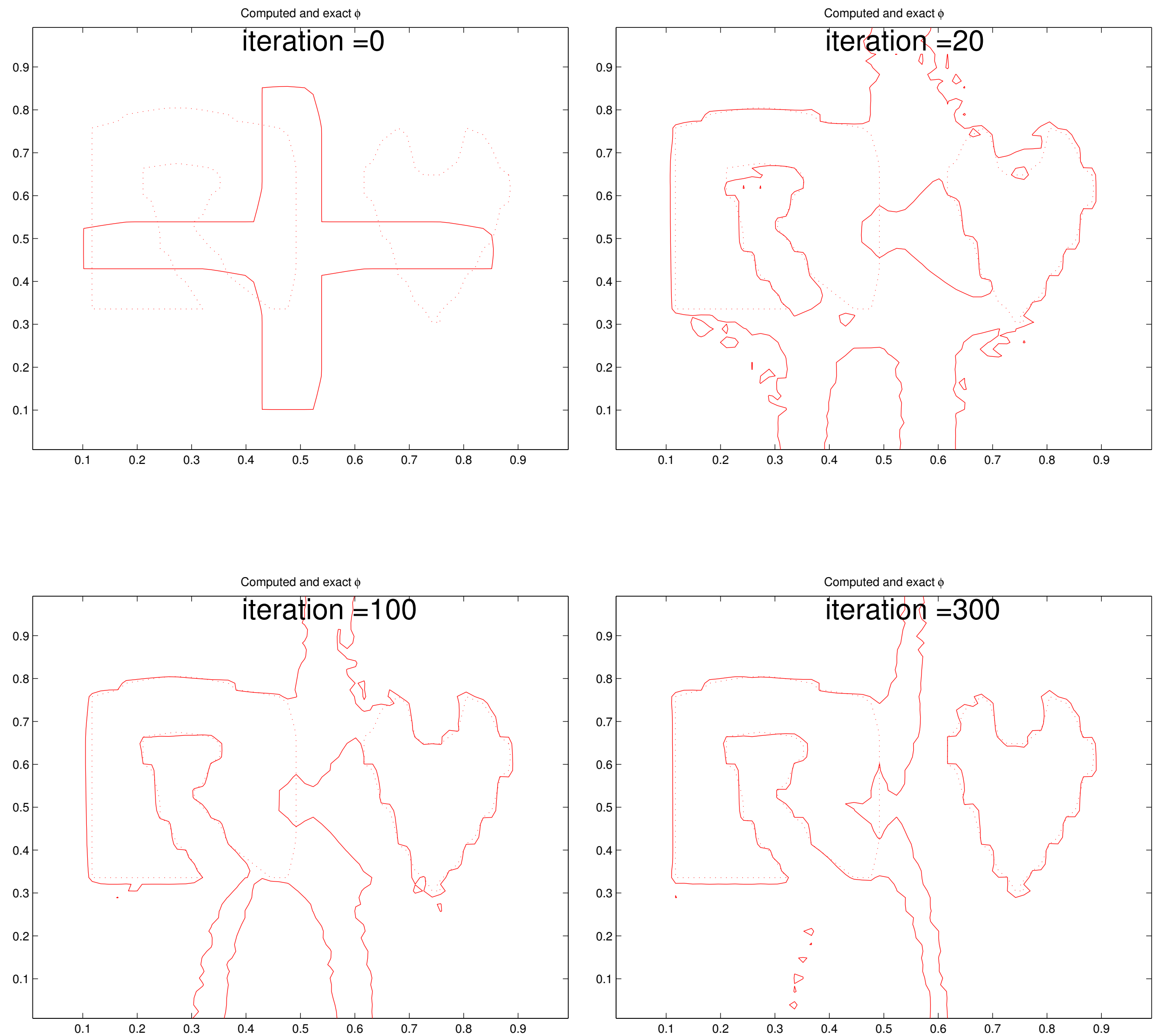


(a) Error $|q_i - q_i^k|$

(b) Error $\|q - q^k\|_{L^2(\Omega)}$

Figure 8:

Experiments– Another example, one level set. Another initial guess



Experiments– Another example, one level set. Another initial guess.

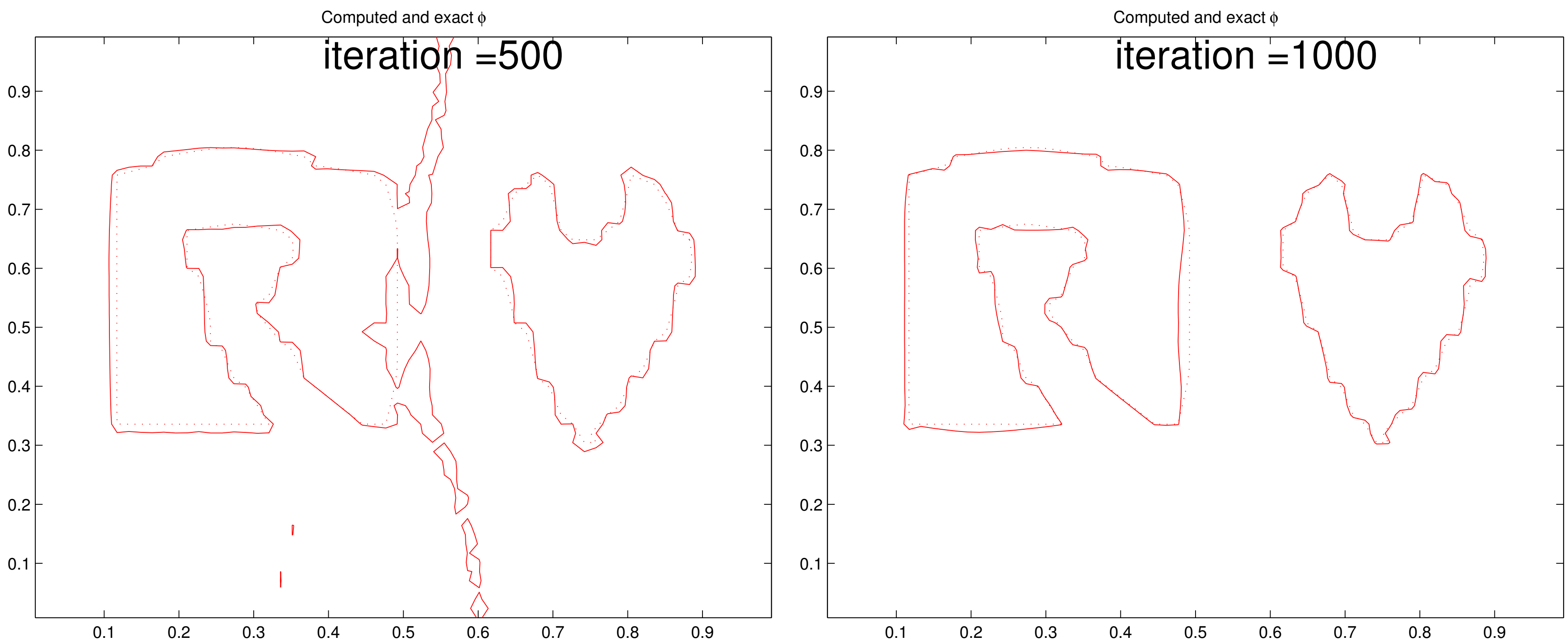
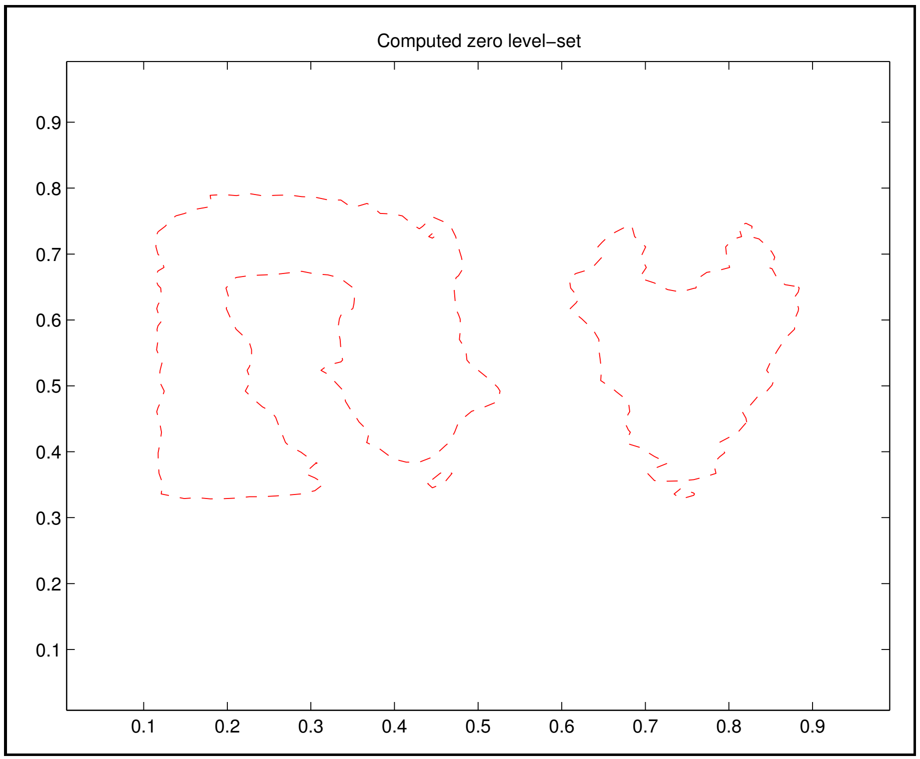
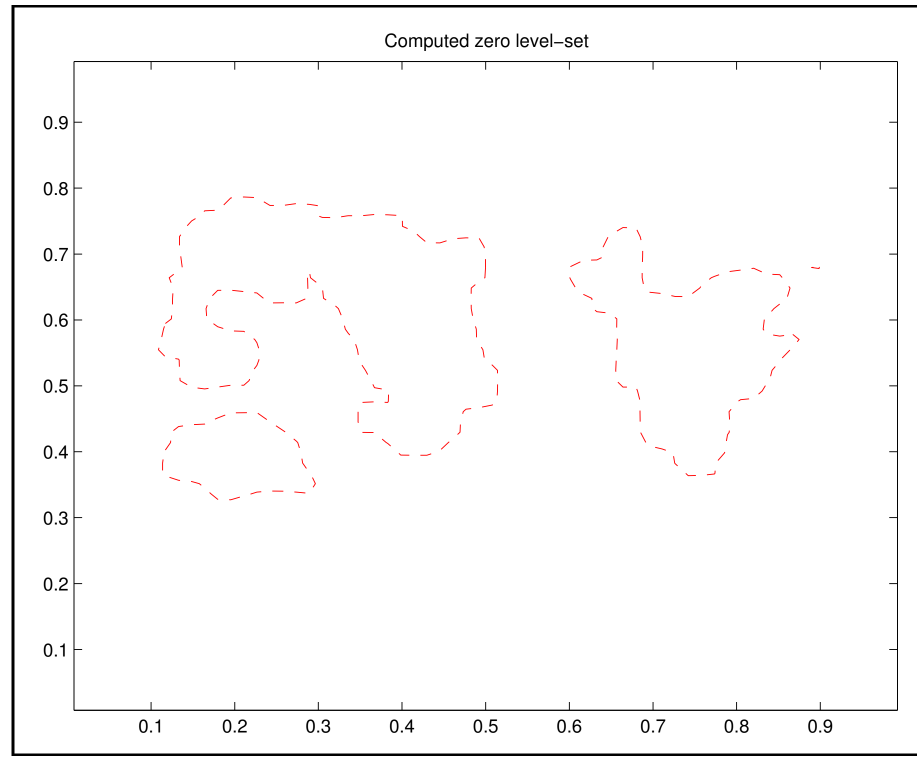


Figure 10:

Experiments–One level set. Different noise levels.



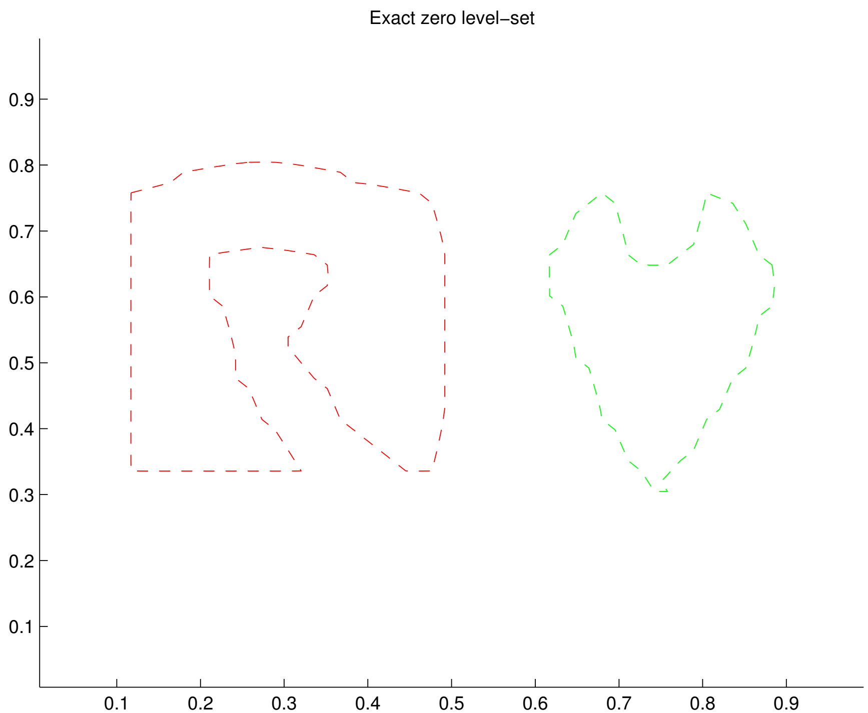
(a) With $\sigma = 1\%, r = 5 \times 10^{-4}, \beta = 5 \times 10^{-7}$



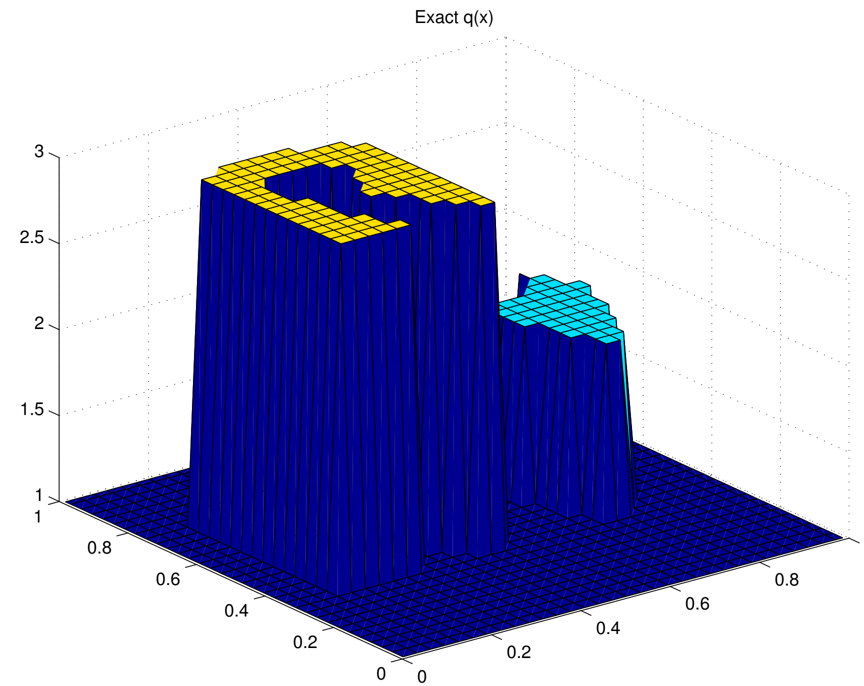
(b) With $\sigma = 5\%, r = 5 \times 10^{-4}, \beta = 5 \times 10^{-5}$

Figure 11:

Two-level sets – three regions.



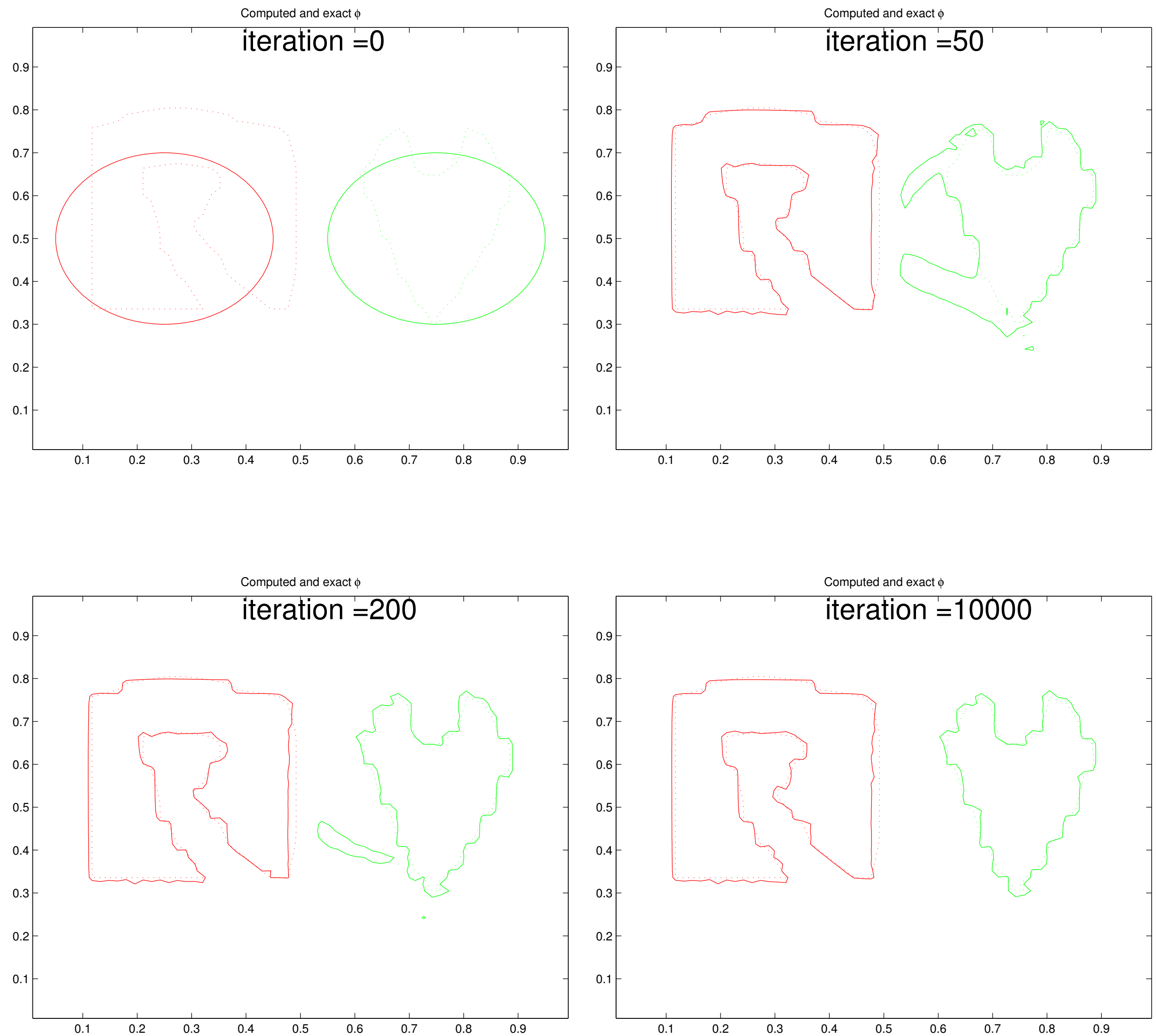
(a) Exact zero level sets



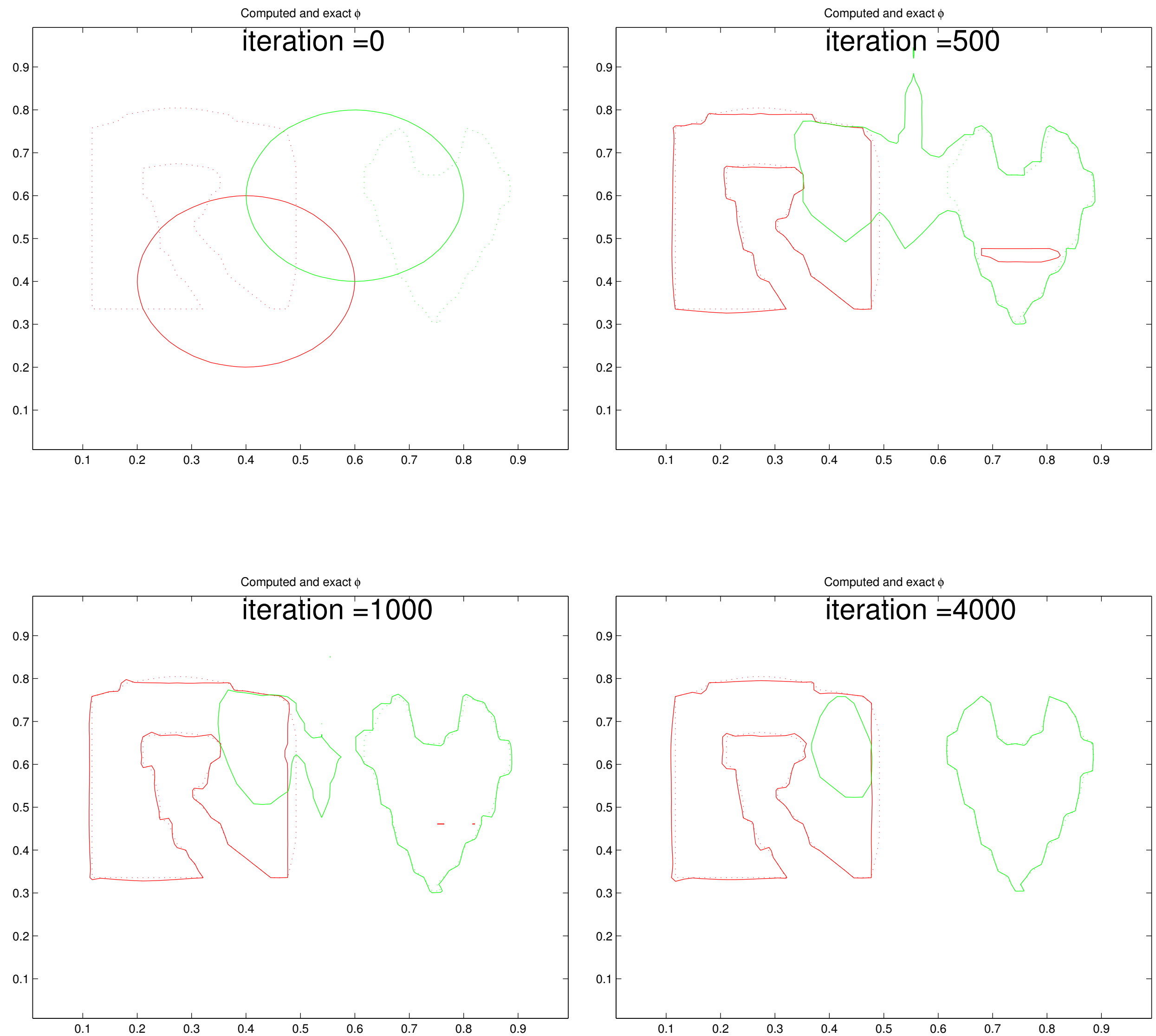
(b) Exact $q(x)$

Figure 12:

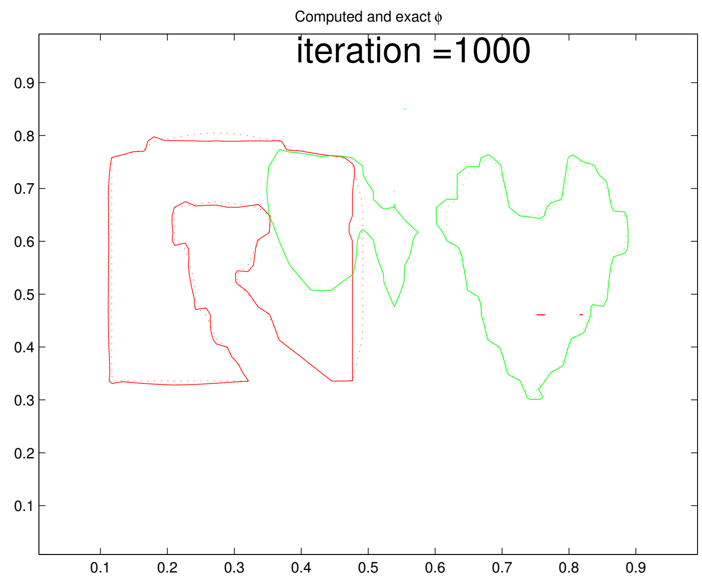
Two-level sets–three regions.



Two-level sets – three regions. Start with 4 regions.

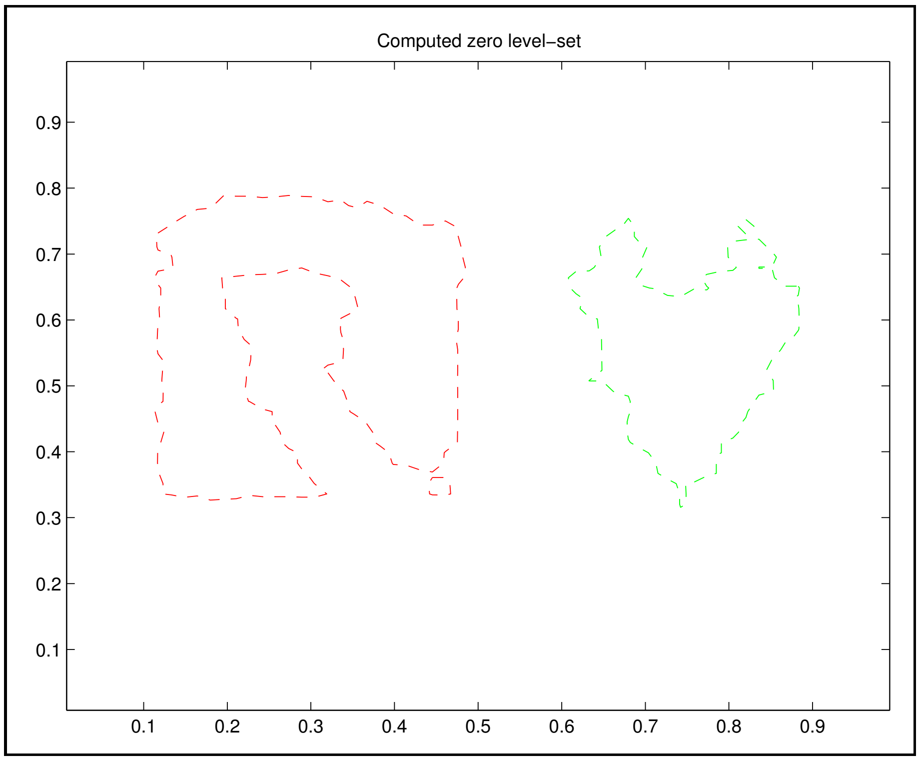


Two-level sets – three regions. Start with 4 regions.

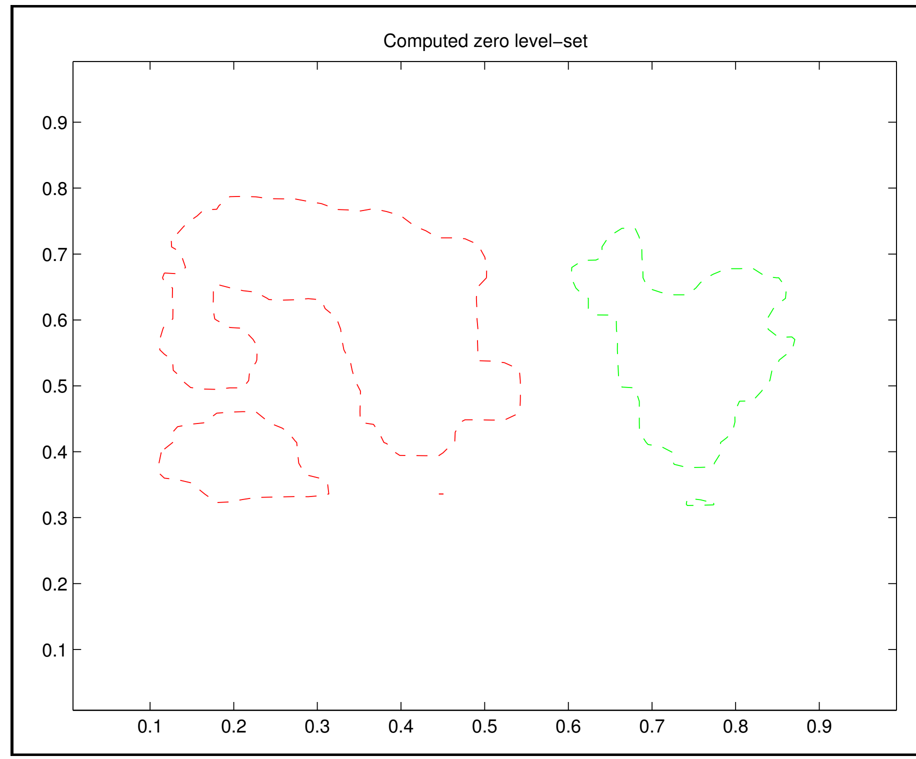


- Get four regions if the constant for the extra regions approaches to the constant for the regions surrounds it.
- Get three regions if the extra regions disappear before the constant for the extra regions approaches to the constant for the regions surrounds it.

Two-level sets – three regions. With different noise level.



(a) With $\sigma = 1\%, r = 5 \times 10^{-4}, \beta = 5 \times 10^{-6}$



(b) With $\sigma = 5\%, r = 5 \times 10^{-4}, \beta = 5 \times 10^{-5}$

Figure 15:

A level set method for electrical impedance tomography

Eric T. Chung and Tony F. Chan

Department of Mathematics

UCLA

and

Xue-Cheng Tai

Department of Mathematics

University of Bergen.

JCP 2005 (to appear).

Xue-Cheng.Tai@mi.uib.no, http://www.mi.uib.no/%7Etai.

Electrical impedance tomography

Problem: given $f(x)$ and $m(x)$, determine $q(x)$

$$\begin{aligned} -\nabla \cdot (q \nabla u) &= 0 \quad \text{in} \quad \Omega \\ q \frac{\partial u}{\partial n} &= f \quad \text{on} \quad \partial\Omega \\ u &= m \quad \text{on} \quad \partial\Omega \end{aligned}$$

where f is current density, m is measured potential, q is conductivity
Our work:
1. A numerical scheme based on level set method for piecewise constant $q(x)$

2. Numerical results showing that the scheme can be able to recover a sharp interface

and can tolerate higher level of noise in the data

Least squares and TV regularization

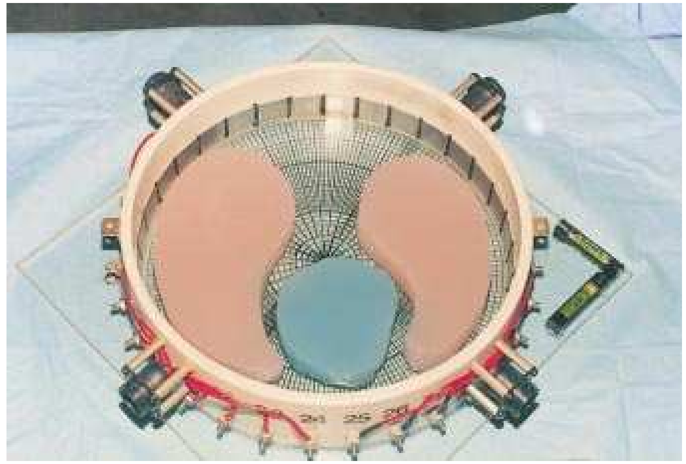


Fig. 1 *A test tank containing "lungs" and "heart" made of agar with varying amounts of added salt. This tank is filled with salt water, and used as a test body for the EIT system. Note the large electrodes around the inner circumference of the tank.*

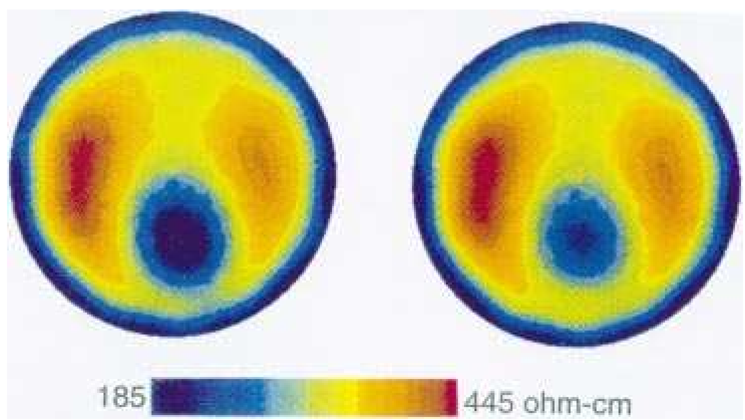


Fig. 2 *Images of the resistivity of two different test tanks like the one shown in Figure 1. The two different tanks had hearts of different sizes, meant to simulate different times during the heart's cycle.*

Finally, the ill-posedness of the problem also makes it unlikely that EIT images, even with many electrodes, will have resolution comparable to that of CT or MRI

Least squares and TV regularization

Assume q_1 and q_2 are known
Let N be number of measurements
Determine Γ by minimizing

$$F(\phi) = \frac{1}{2}\sum_{i=1}^{N}\int_{\partial\Omega} |u_i(s) - m_i(s)|^2 \, ds + \beta \int_{\Omega} |\nabla q| \, dx$$

using the gradient descent method:

$$\phi^{k+1} = \phi^k - \alpha_k \frac{dF}{d\phi}(\phi^k)$$

where $\alpha_k > 0$ is chosen such that $F(\phi^{k+1})$ is minimized

Computation of gradient

The gradient can be computed by

$$\frac{dF}{d\phi} = \frac{dF}{dq}\frac{dq}{d\phi} = \frac{dF}{dq}(q_1 - q_2)\delta(\phi)$$

and

$$\frac{dF}{dq} = -\sum_{i=1}^{N} \nabla u_i \cdot \nabla z_i - \beta \nabla \cdot (\frac{\nabla q}{|\nabla q|})$$

where z_i satisfies

$$\begin{aligned} -\nabla \cdot (q \nabla z_i) &= 0 \quad \text{in} \quad \Omega \\ q\frac{\partial z_i}{\partial n} &= u_i - m_i \quad \text{on} \quad \partial\Omega \end{aligned}$$

One test result

Parameters: $N = 60$, noise =0.05%, $\beta = 10^{-7}$.

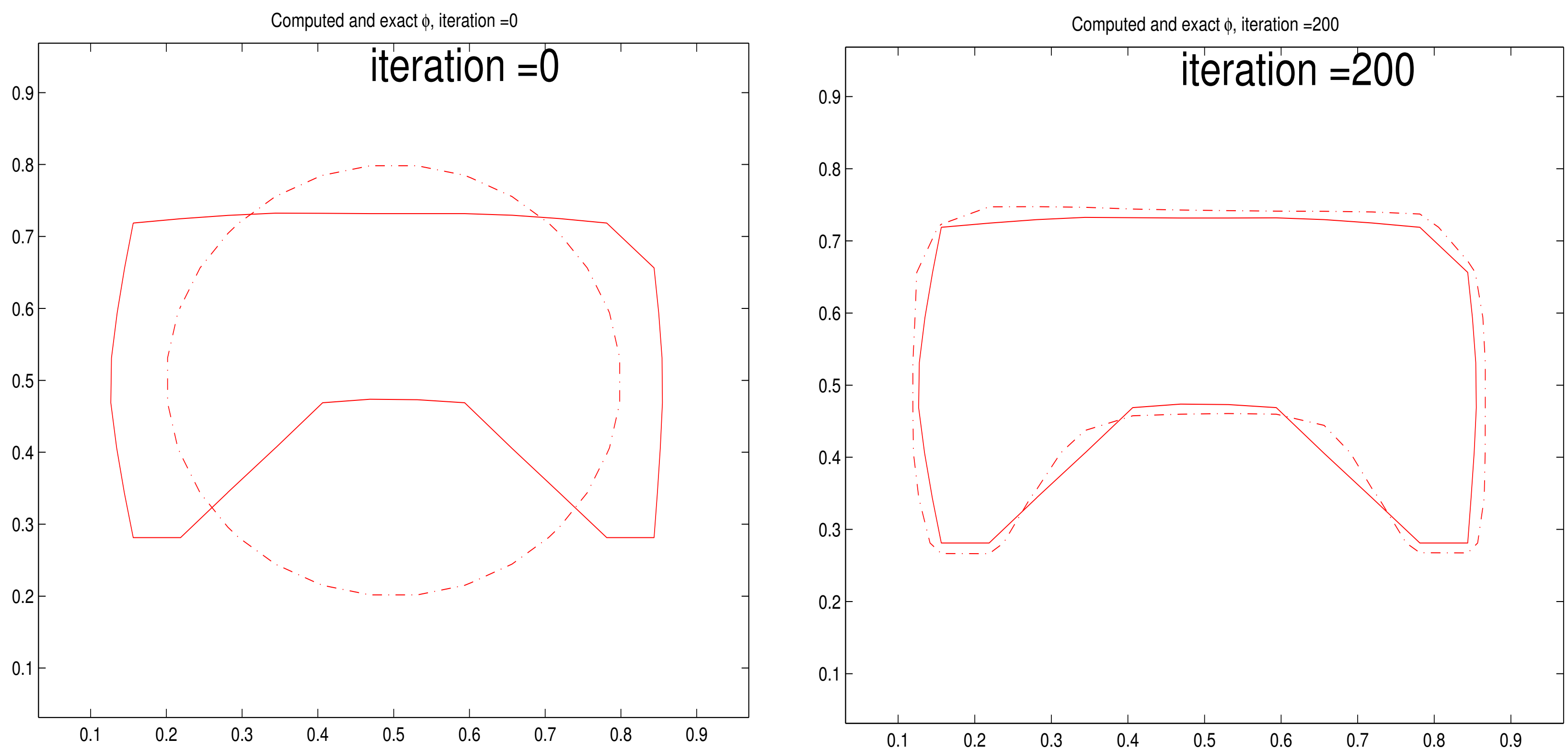


Figure 1: Left picture shows the initial guess. Right picture shows the numerical result after 200 iterations.

One test result

Parameters: $N = 60$, noise $=0.1\%$, $\beta = 10^{-6}$.

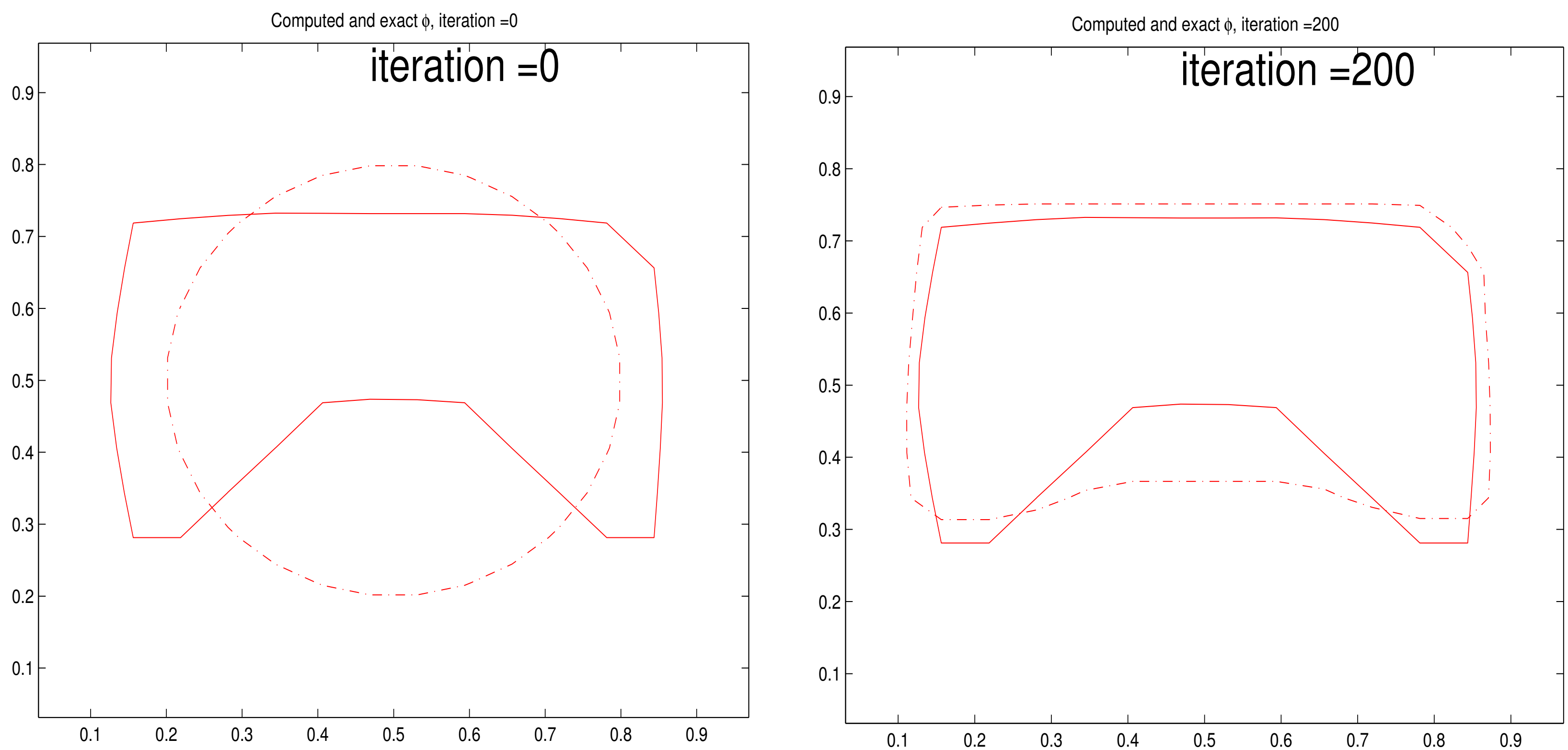


Figure 2: Left picture shows the initial guess. Right picture shows the numerical result after 200 iterations.

Another examples

Parameters: N=60, noise = 0.1%, $\beta = 10^{-6}$

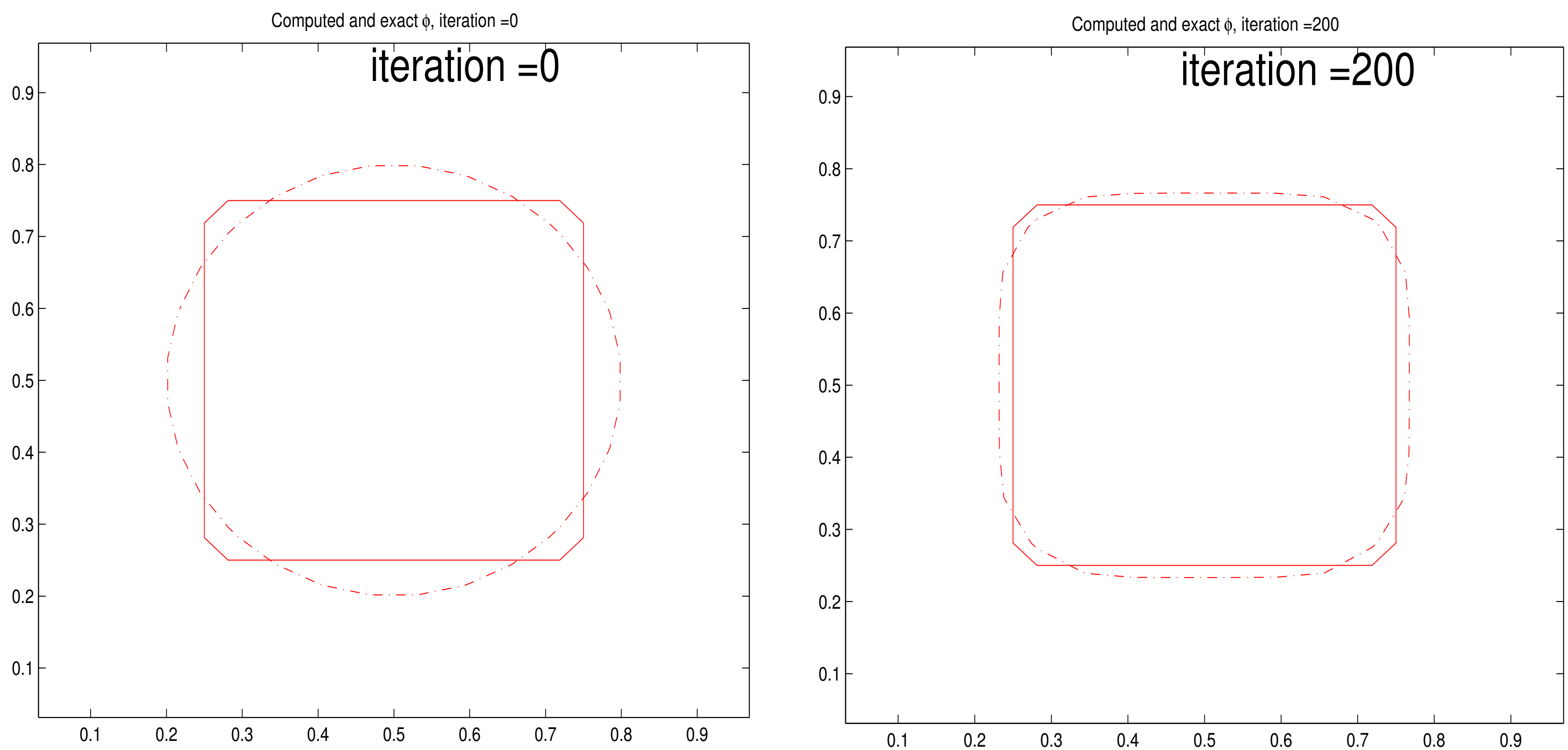
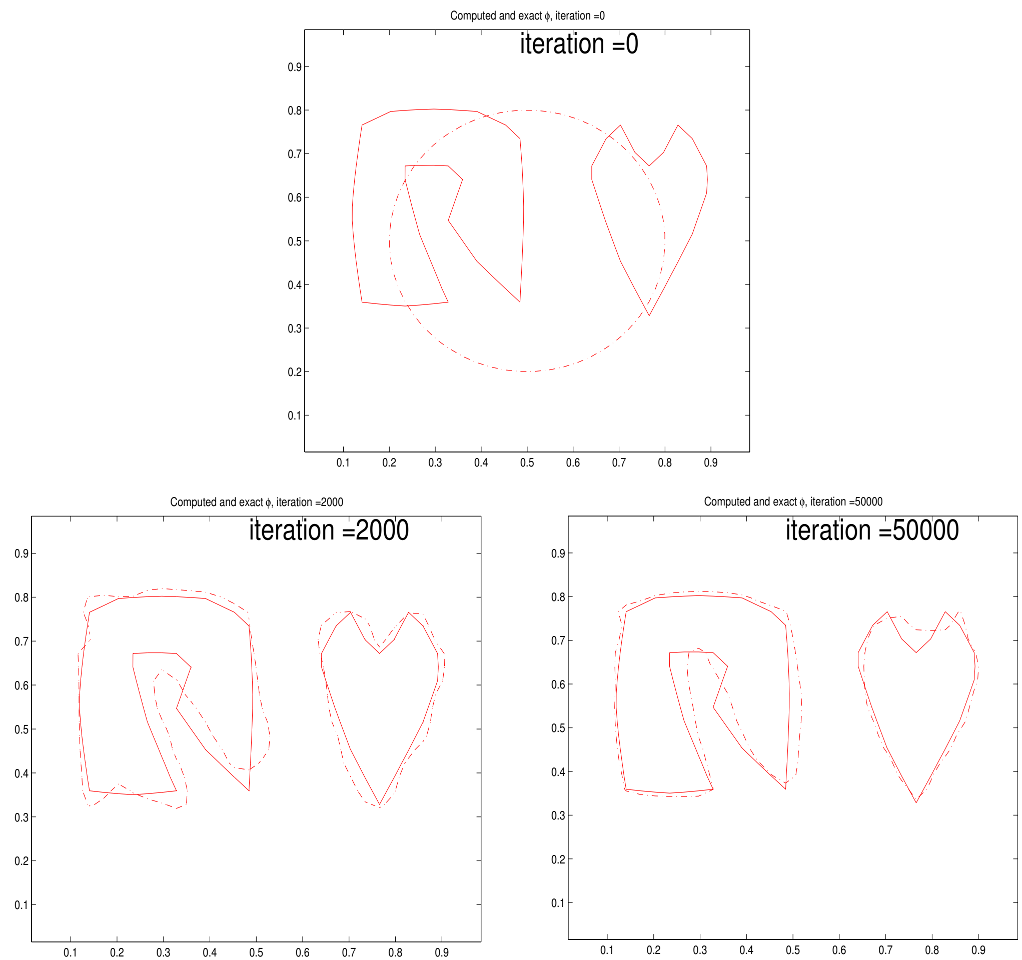


Figure 3: Left picture shows the initial guess. Right picture shows the numerical result after 200 iterations.

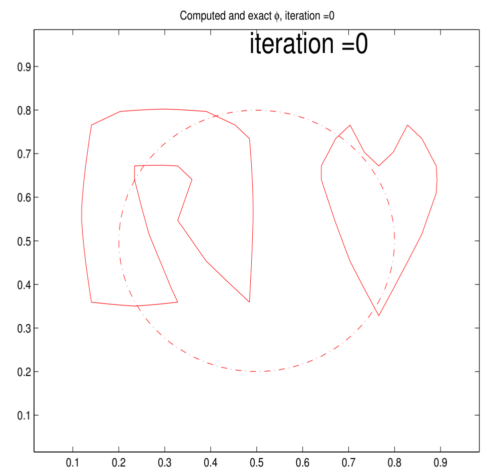
More complicated geometry

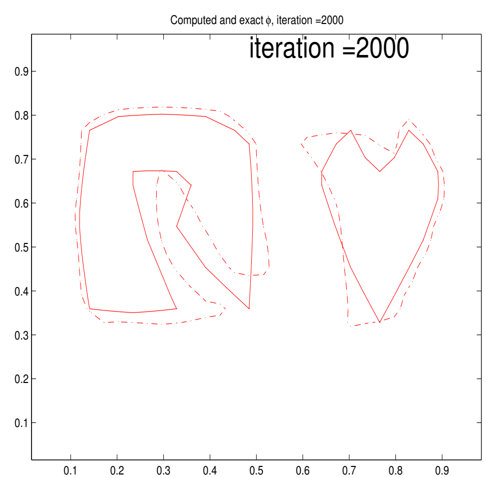
Parameters: $N = 60, h = 1/32, \beta = 10^{-12}$, no noise.

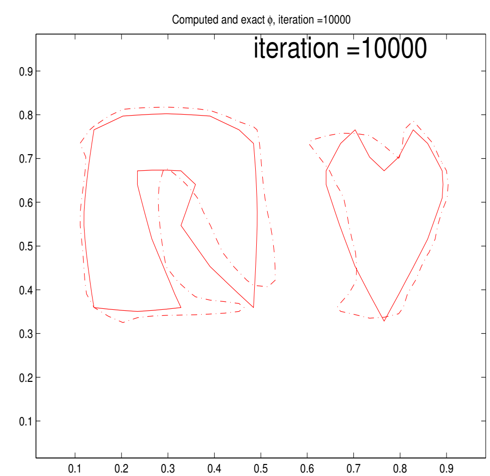


More complicated geometry and more nosie

Parameters: $N = 60, h = 1/32, \beta = 10^{-12}$, 0.1% noise.

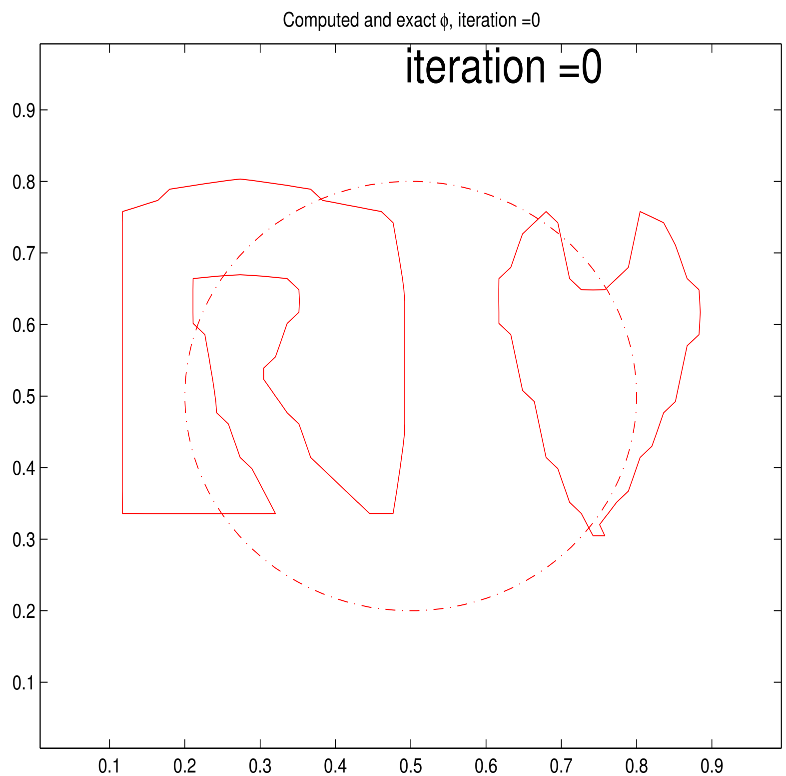


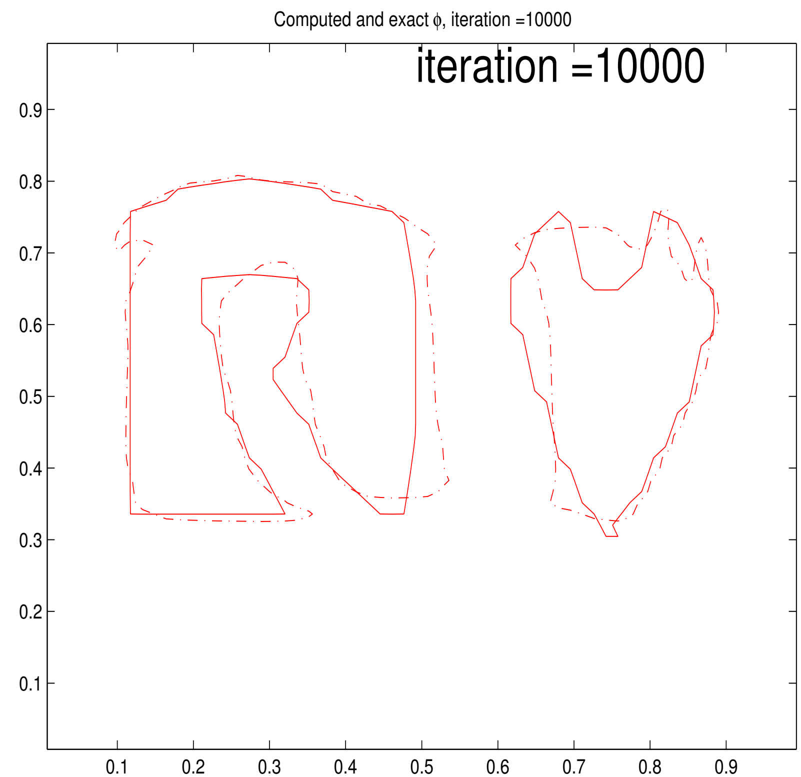




Finer mesh, better results

Parameters: $h = 1/64$, noise=same%





Identify both constants and the discontinuity

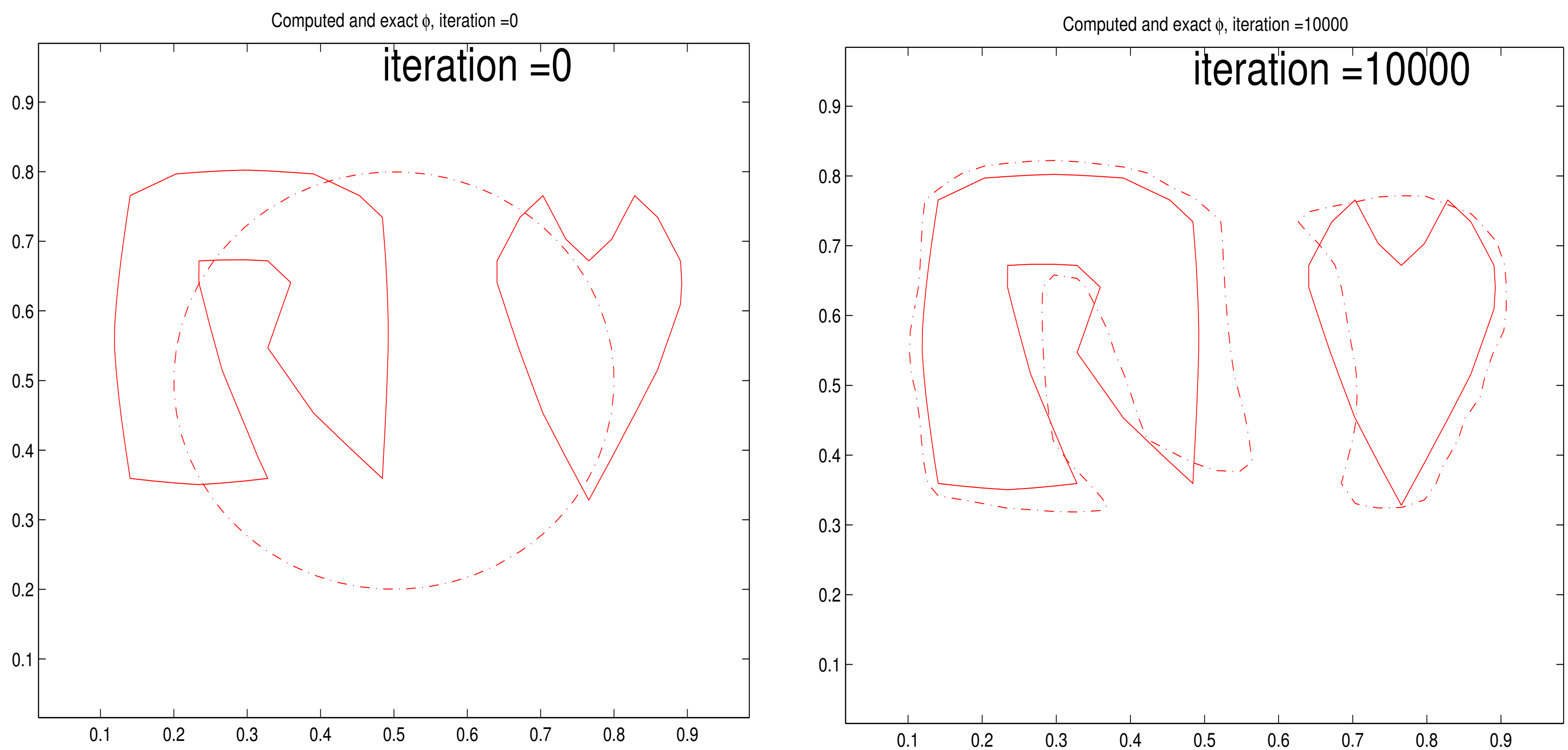


Figure 7: Result with 0.01% noise in the data. Left: initial guess. Right: numerical result after 10000 iterations with computed $q = 2.273$

Number of observations

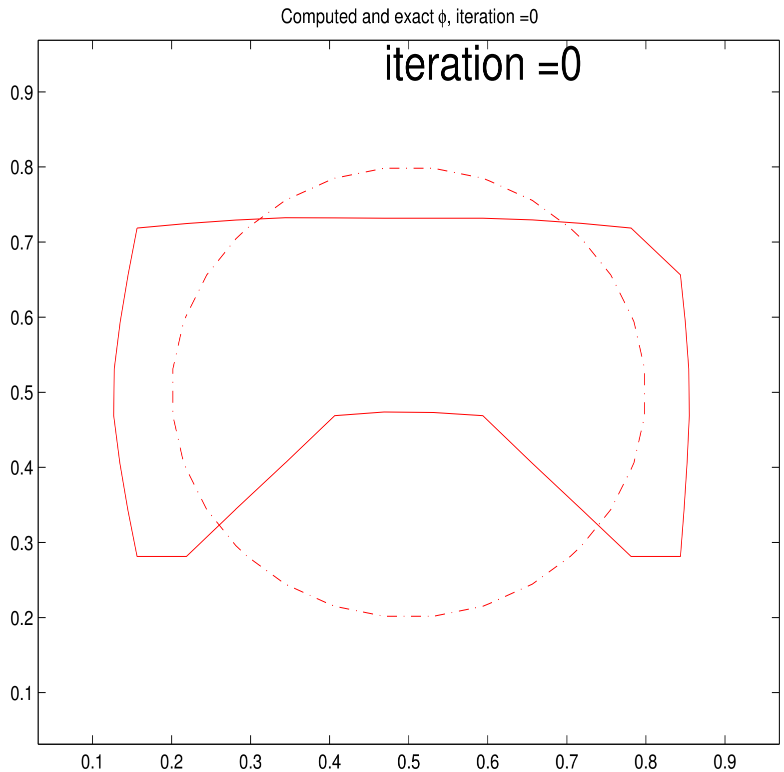


Figure 8: A circle is chosen to be an initial guess.

Number of observations

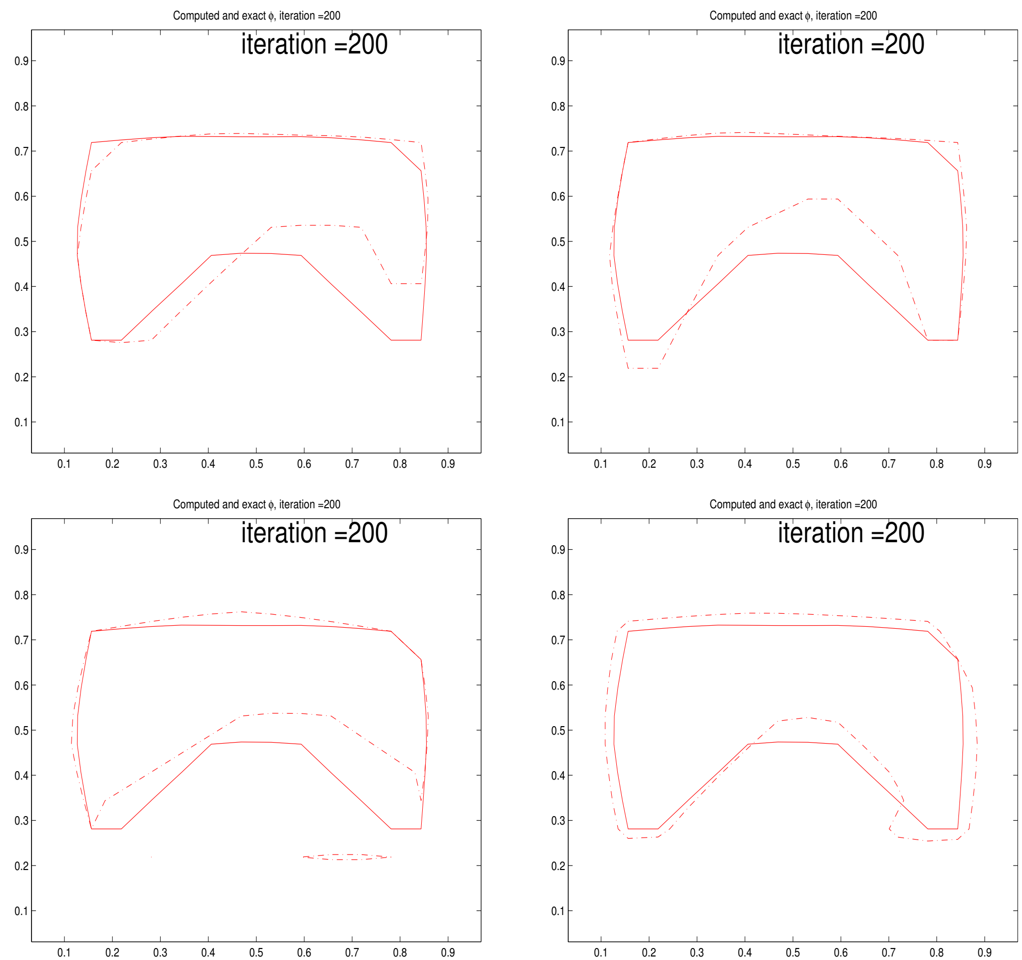


Figure 9: The upper left, upper right, lower left and lower right figures show results with 4, 12, 28 and 60 measurements.

Influence of noise level

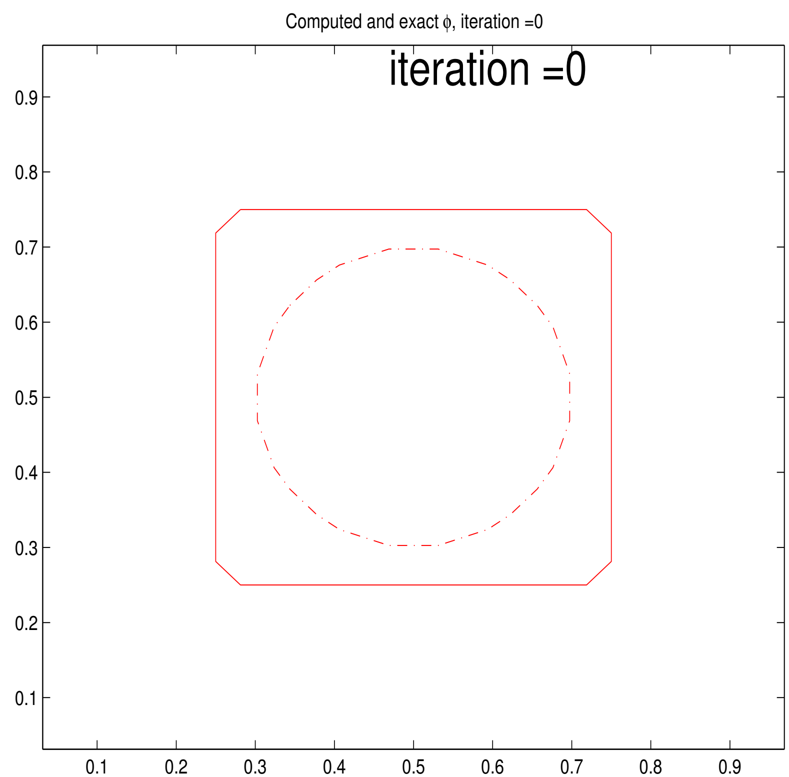


Figure 10: A circle is chosen to be an initial guess.

Influence of noise level

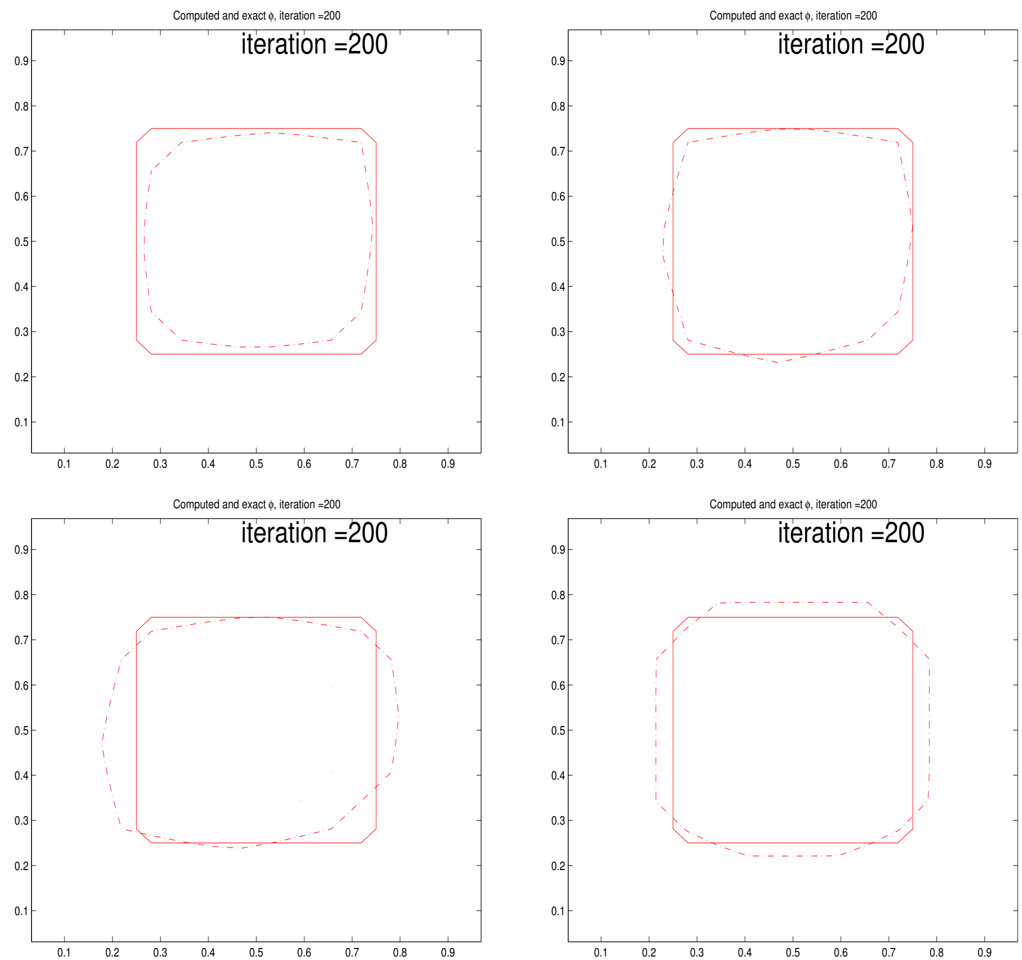


Figure 11: The upper left, upper right, lower left and lower right figures show results with 1%, 2%, 3% and 4% noise and regularization parameter 10^{-13}, 10^{-10}, 2×10^{-6} and 5×10^{-5} respectively.

Influence of noise level

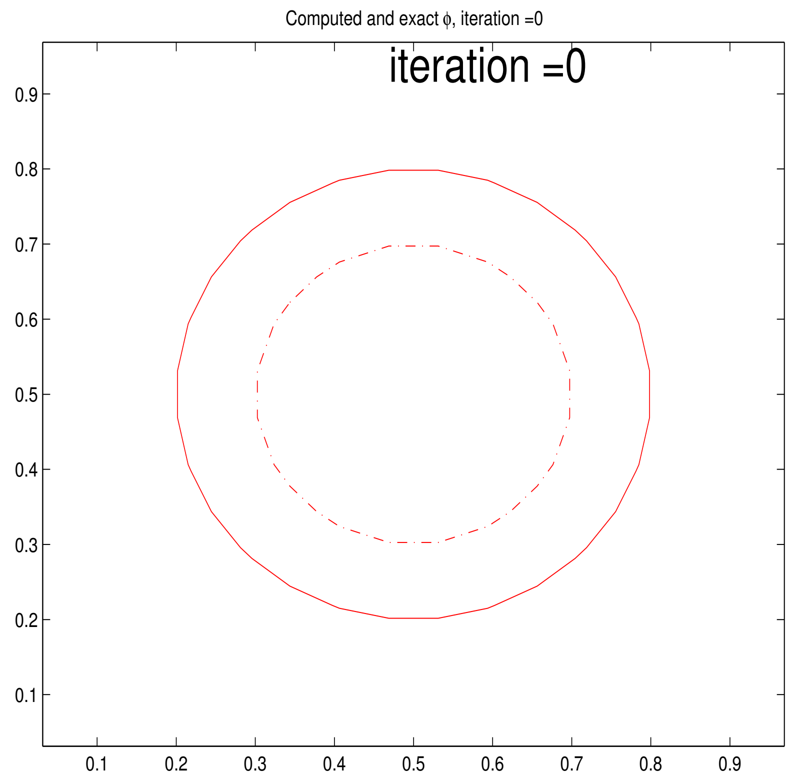


Figure 12: A circle is chosen to be an initial guess.

Influence of noise level

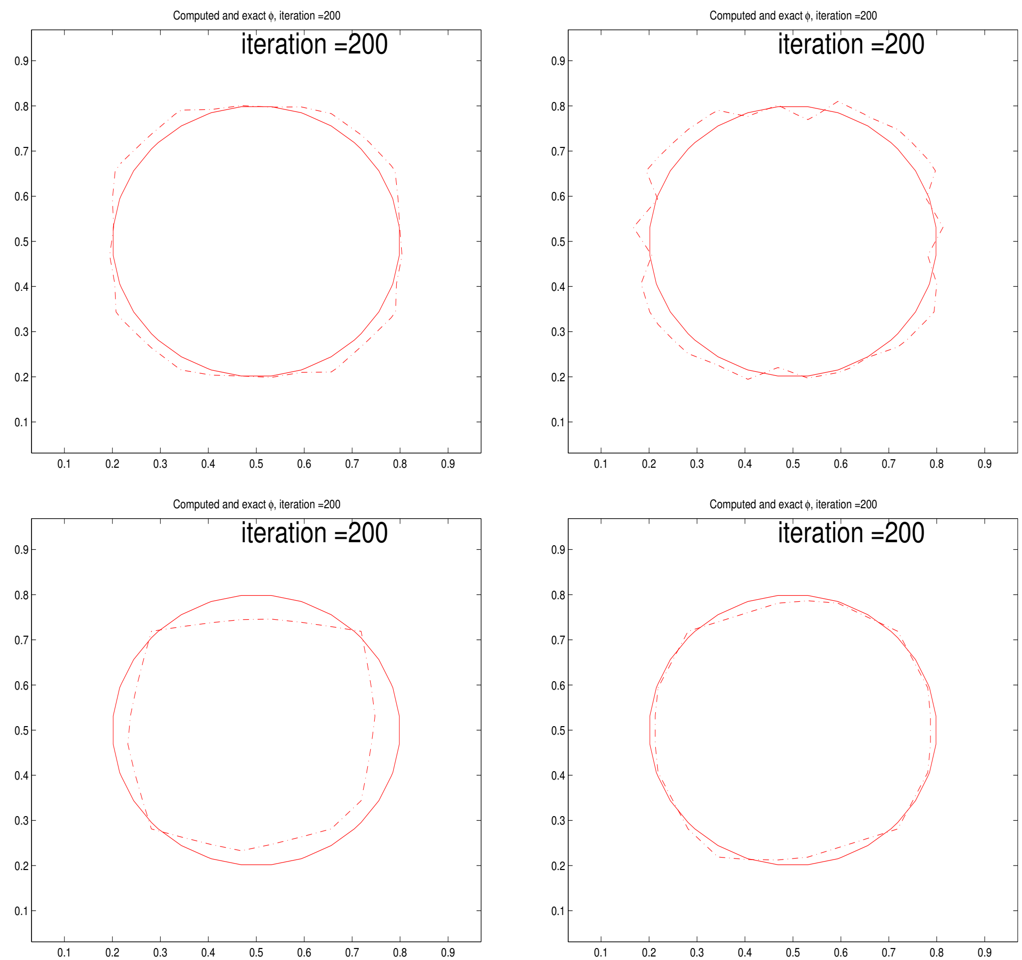


Figure 13: The upper left, upper right, lower left and lower right figures show results with the regularization parameter chosen to be 10^{-4}, 10^{-5}, 10^{-6} and 10^{-7} and noise level 7%, 2%, 1% and 0.5% respectively.

PET using TV-norm regularization and level-set method

M. Lysaker[1], X.-C. Tai[1],T. Chan[2] and H. Huang[3]

Xue-Cheng.Tai@mi.uib.no, http://www.mi.uib.no/%7Etai.

Department of Mathematics, University of Bergen[1]

Department of Mathematics, UCLA[2]

Department of Molecular and Medical Pharmacology, UCLA[3]

PET equation

$$L(\lambda) = \min_{\lambda} \sum_{b}^{B} \lambda_b - \sum_{t}^{T} n_t \log{(P\lambda)_t} + R(\lambda), \tag{10}$$

where $R(\lambda)$ is a regularization term introduced to improve image quality

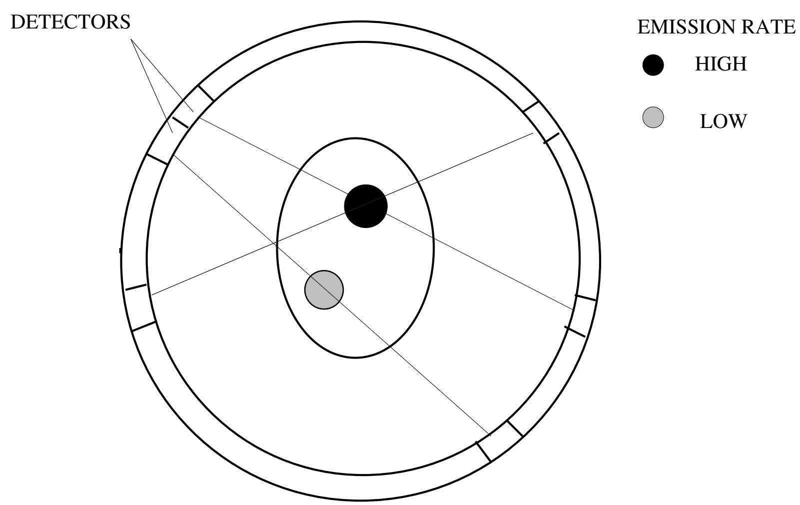


Figure 16: Gamma-rays escape the brain and an external detection is possible.

Level Set Method to Restore PET Images

If the PET image only takes 4 different intensity values, λ_k, k=1:4, we define:

$$\Omega_k := \{(x,y) \in \Omega | \lambda(x,y) = \lambda_k\},$$

where $\Omega = \bigcup_{k=1}^{4} \Omega_k$ is the image domain. The unknown $\lambda(x,y)$ now takes the form:

$$\lambda = \lambda_1 H(\phi_1)H(\phi_2) + \lambda_2 H(\phi_1)[1-H(\phi_2)] + \lambda_3[1-H(\phi_1)]H(\phi_2) + \lambda_4[1-H(\phi_1)][1-H(\phi_2)]$$

where the Heaviside function is given by:

$$H(\phi_j) = \begin{cases} 1 & \text{if } \phi_j \geq 0 \quad j = 1,2 \\ 0 & \text{if } \phi_j < 0 \quad j = 1,2. \end{cases}$$

With 2 level set the objective function to minimize is:

$$\begin{aligned} L(\phi_1,\phi_2) = {} & \lambda_1 \int_\Omega H(\phi_1)H(\phi_2)\,dx + \lambda_2 \int_\Omega H(\phi_1)[1-H(\phi_2)]\,dx \\ & + \lambda_3 \int_\Omega [1-H(\phi_1)]H(\phi_2)\,dx + \lambda_4 \int_\Omega [1-H(\phi_1)][1-H(\phi_2)]\,dx \\ & - \sum_t^T n_t \log \int_\Omega Q_t[\lambda]\,dx + \beta \sum_{j=1}^{2} \int_\Omega |\nabla H(\phi_j)|\,dx. \end{aligned} \tag{11}$$

PET equation

After some calculation we find that

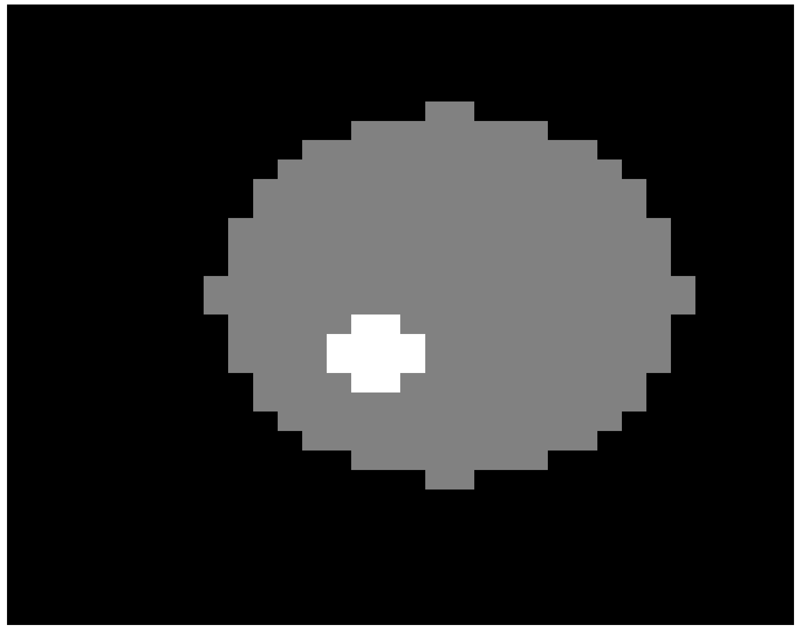
$$\begin{aligned}\frac{\partial L}{\partial \phi_1} &= \Big((\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)H(\phi_2) + \lambda_2 - \lambda_4\Big)\delta(\phi_1) \\ &- \sum_t^T n_t \frac{Q_t\Big[(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)H(\phi_2) + \lambda_2 - \lambda_4\Big]\delta(\phi_1)}{\int_\Omega Q_t[\lambda]\, dx} \\ &- \beta\nabla \cdot \frac{\nabla\phi_1}{|\nabla\phi_1|}\delta(\phi_1)\end{aligned} \tag{12}$$

and in a similar way we discover

$$\begin{aligned}\frac{\partial L}{\partial \phi_2} &= \Big((\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)H(\phi_1) + \lambda_3 - \lambda_4\Big)\delta(\phi_2) \\ &- \sum_t^T n_t \frac{Q_t\Big[(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)H(\phi_1) + \lambda_3 - \lambda_4\Big]\delta(\phi_2)}{\int_\Omega Q_t[\lambda]\, dx} \\ &- \beta\nabla \cdot \frac{\nabla\phi_2}{|\nabla\phi_2|}\delta(\phi_2).\end{aligned} \tag{13}$$

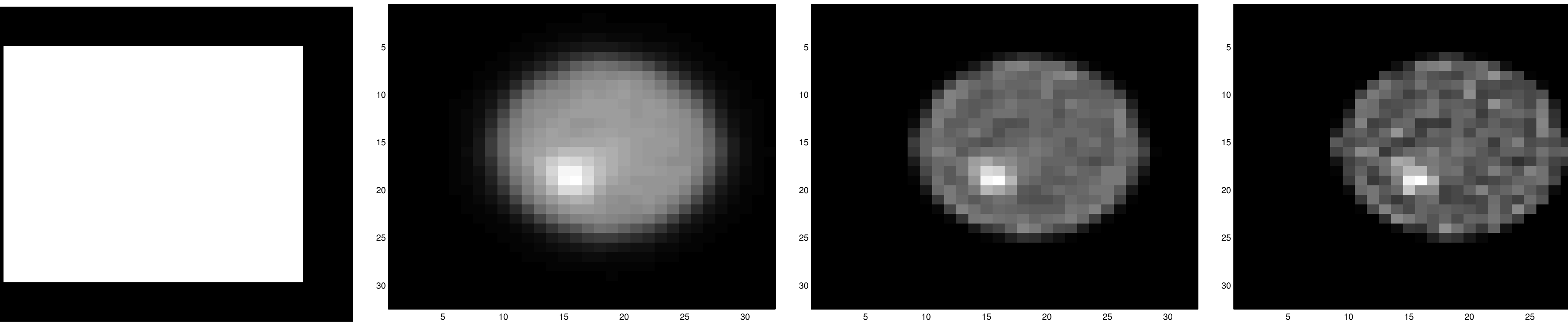
Experimental Result

In our first example we try to reconstruct a 32x32 image of two circles, one inside the other as shown below



Total 1536 (32 position and 48 angular views) observations was given to us, all exposed with noise. In this specific test we used intensity values: λ_4 (dark), λ_2 (gray) and $\lambda_1 = \lambda_3$ (light), all perturbated by $\pm$ 10 %.

Experimental Result

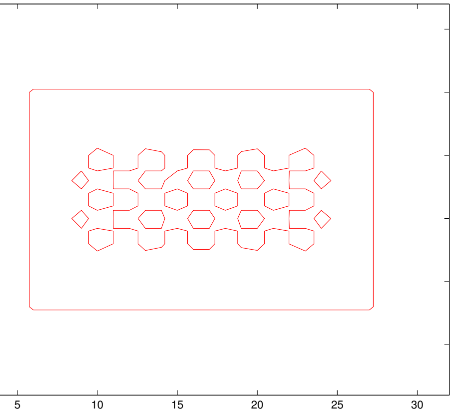


(a) Initial image (b) 5 iterations (c) 30 iterations (d) 100 iterations

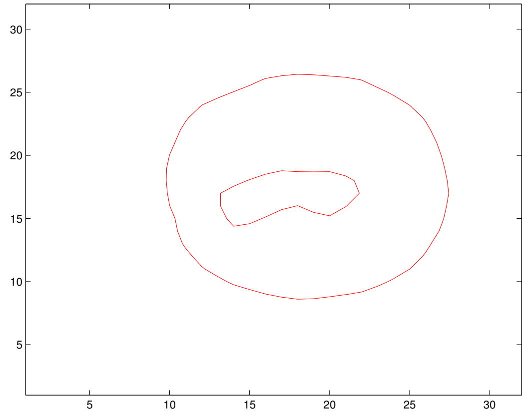
Figure 17: Evolution of a two circles with the OSL algorithm.

Experimental Result

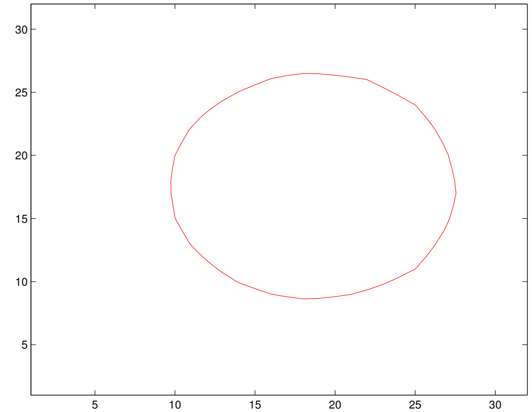
ă



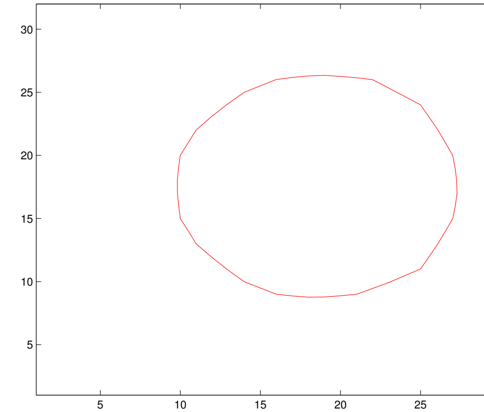
(a) Initial ϕ_1



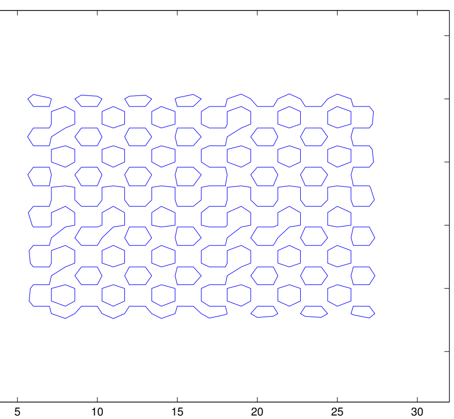
(b) 50 iterations



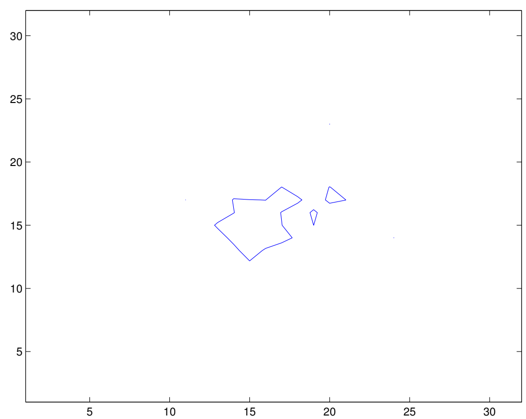
(c) 150 iterations



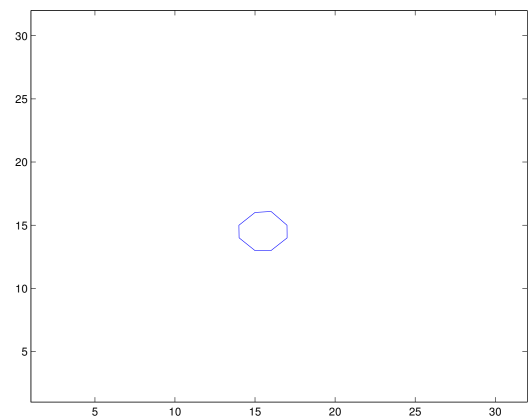
(d) 650 iterations



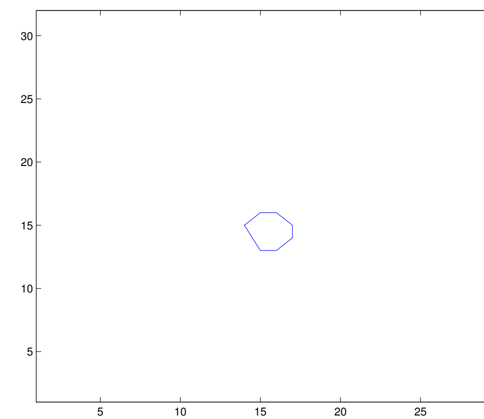
(e) Initial ϕ_2



(f) 50 iterations



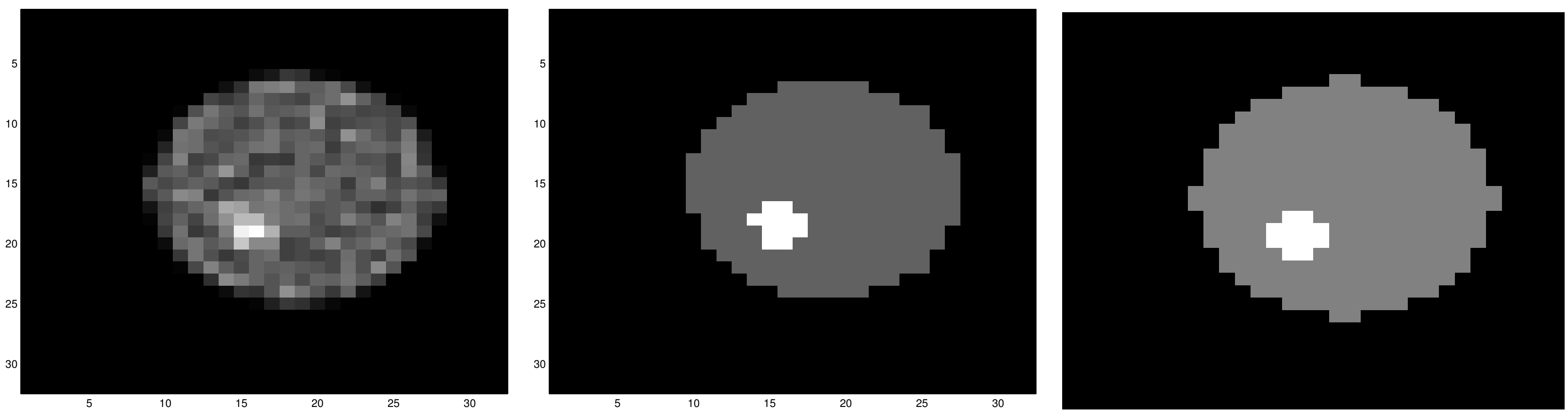
(g) 150 iterations



(h) 650 iterations

Experimental Result

- Sharp edges is implicit given for the reconstructed image when Level Set algorithm is used
- Two major drawbacks with OSL-algorithm is the lack of termination criteria and the introduction of noisy as the number of iteration increase.

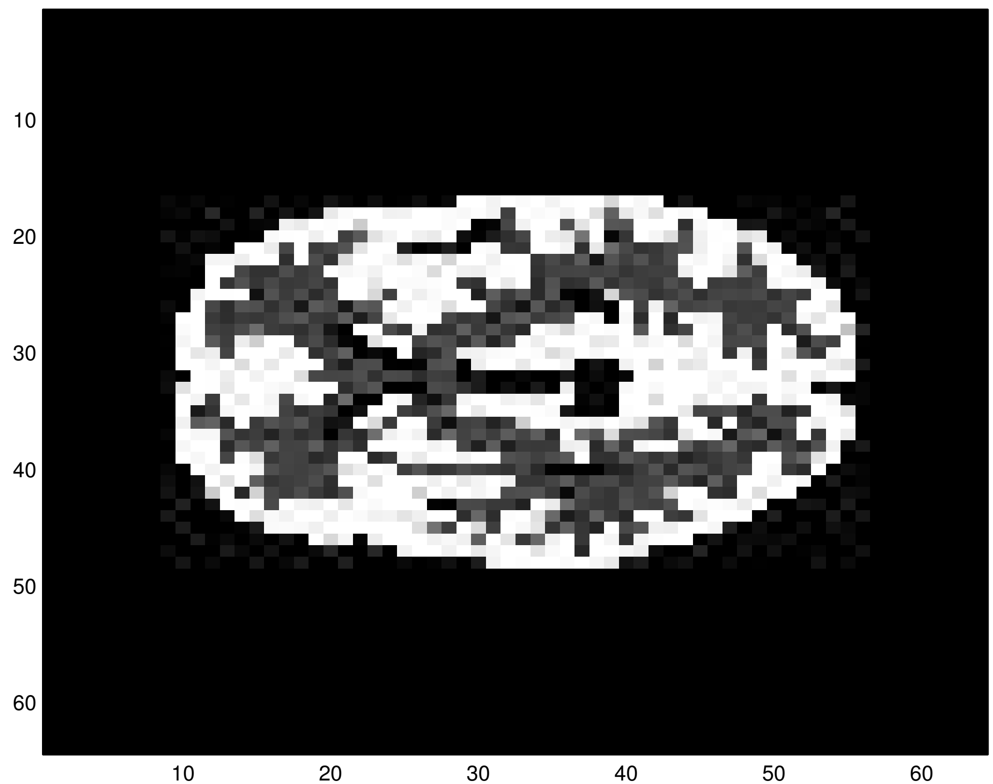


(i) OSL algorithm (j) LSEM algorithm (k) True image

Figure 18: Image of two circles constructed with different algorithms.

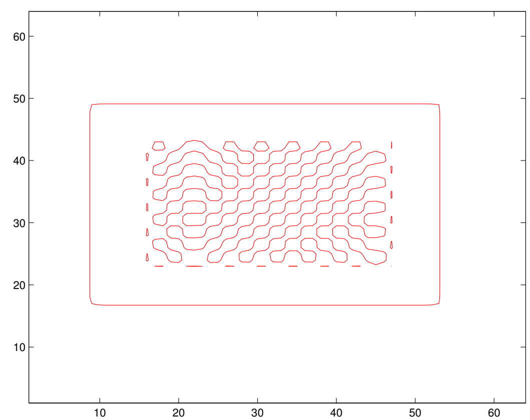
Experimental Result

In our last example we try to reconstruct a 64x64 image of the brain.

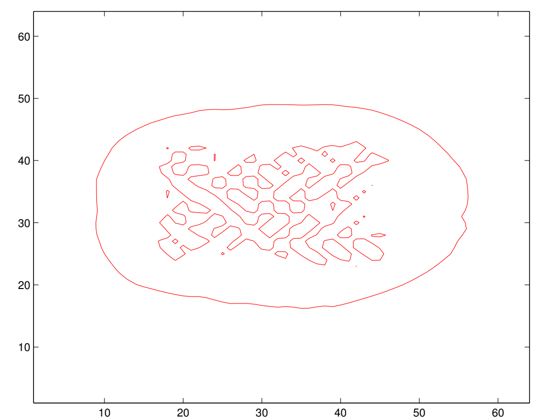


We used the true intensity values: λ_4 (background), λ_2 (gray matter) and $\lambda_1 = \lambda_3$ (white matter) and 6144 (64 position and 96 angular views) observations as input data.

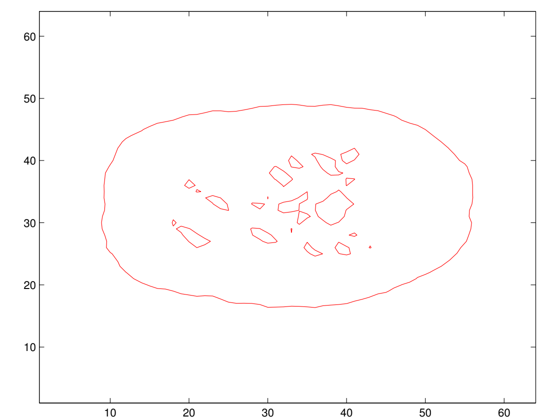
Experimental Result



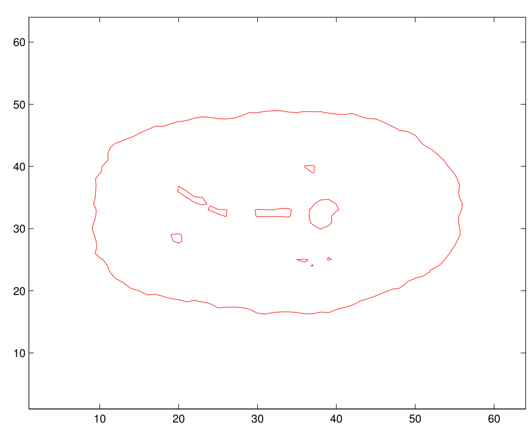
(a) Initial ϕ_1



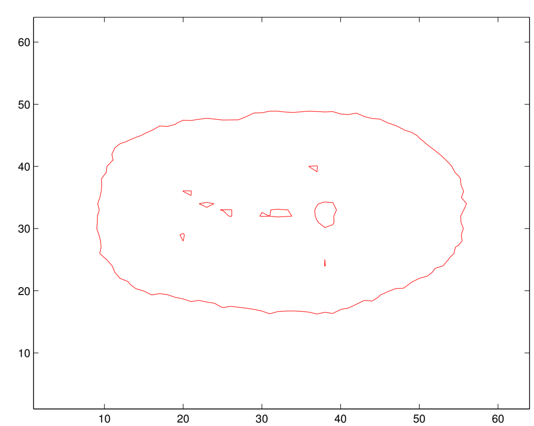
(b) 50 iterations



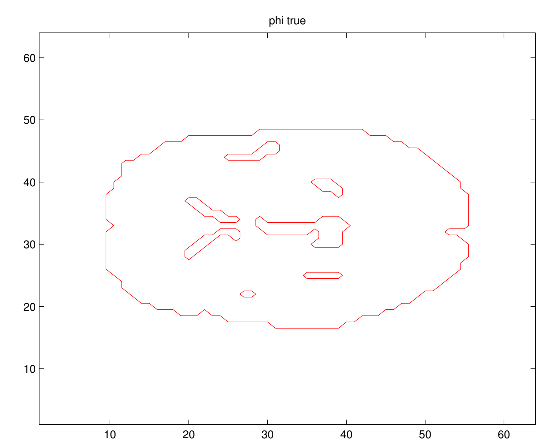
(c) 250 iterations



(d) 1000 iterations

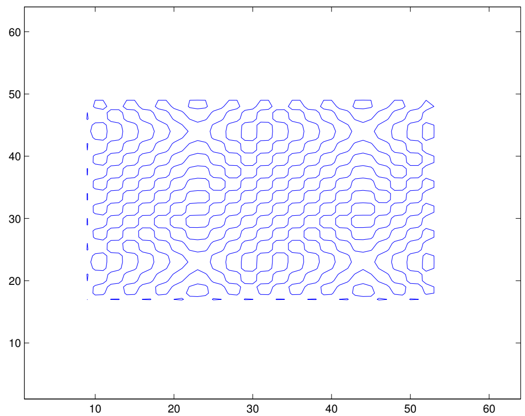


(e) 2800 iterations

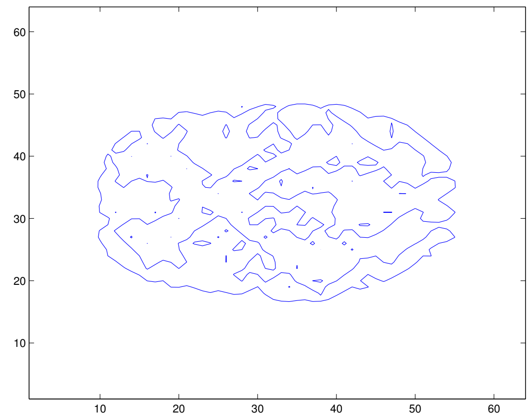


(f) True ϕ_1

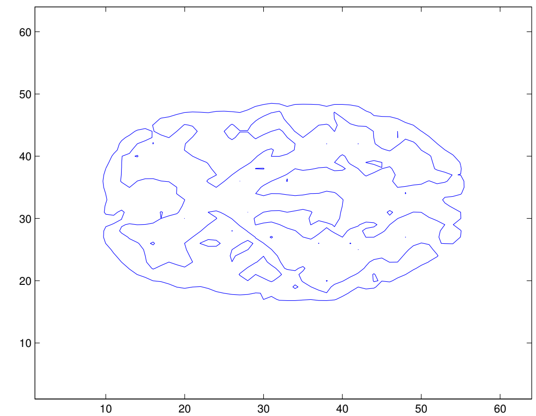
Experimental Result



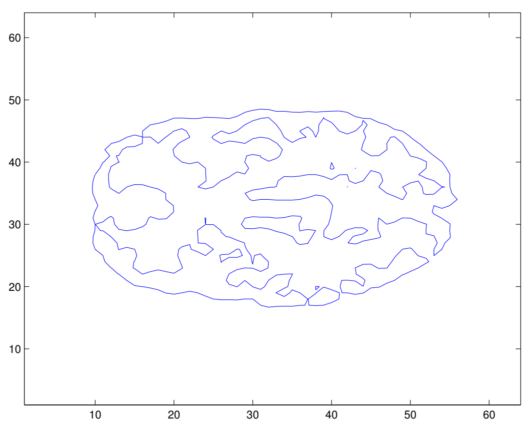
(a) Initial ϕ_2



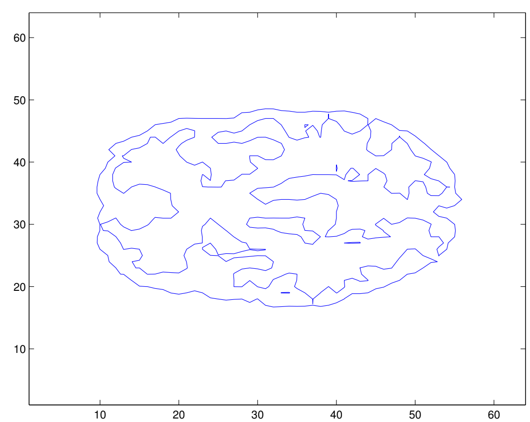
(b) 50 iterations



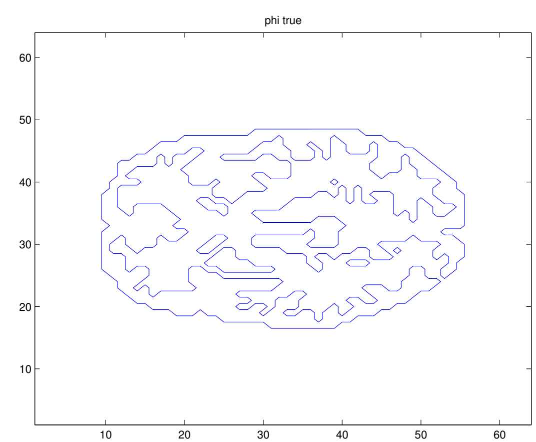
(c) 250 iterations



(d) 1000 iterations

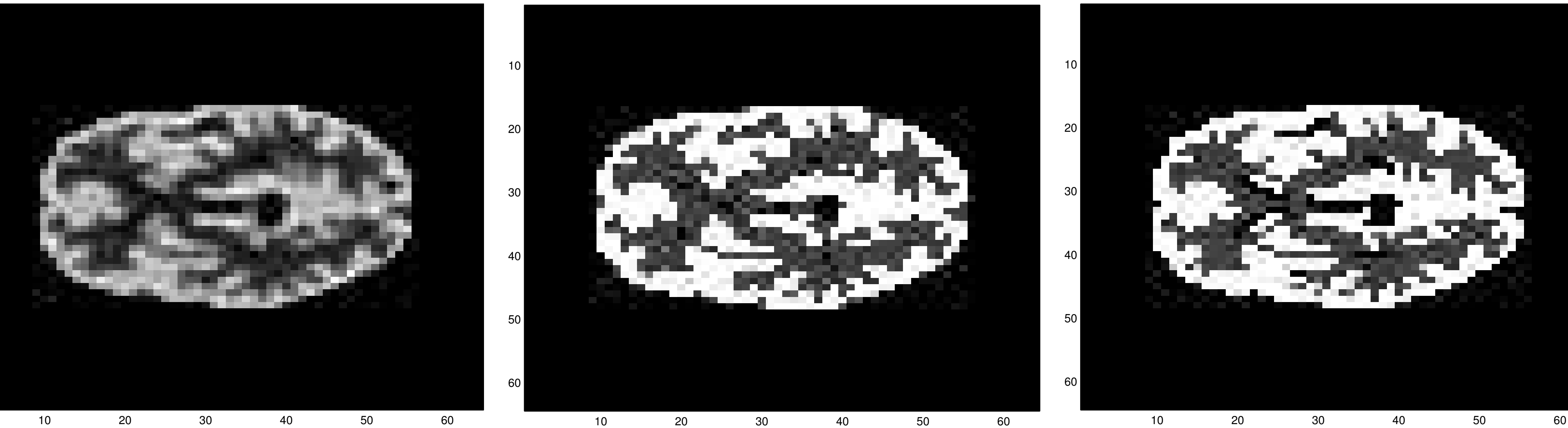


(e) 2800 iterations



(f) True ϕ_2

Experimental Result



(a) OSL algorithm (b) LSEM algorithm (c) True image

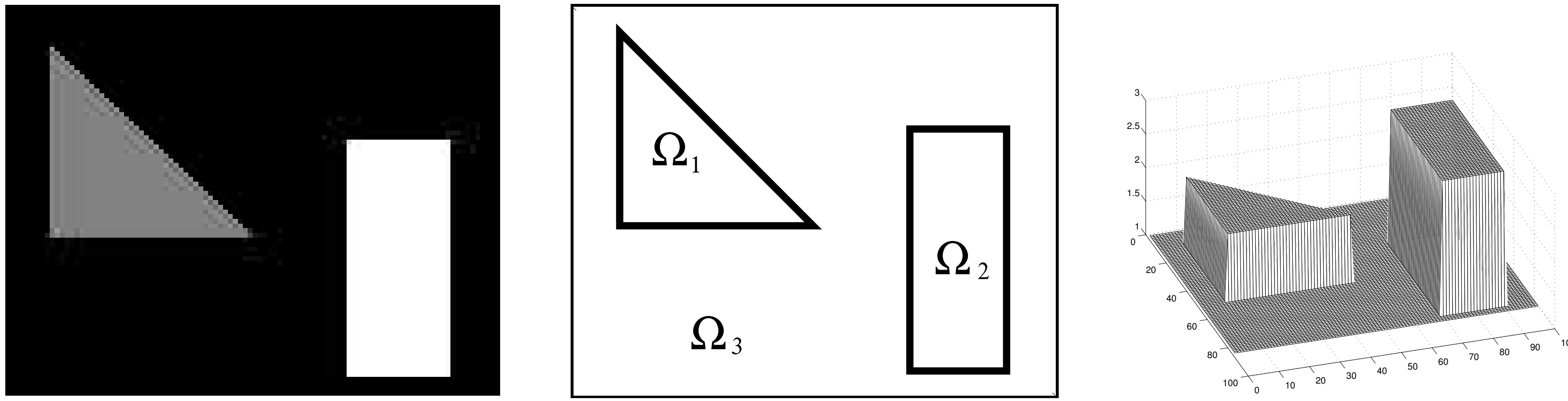
Figure 21: Image reconstruction

A third approach: Label techniques

Some new label techniques

- Johan Lie, Marius Lysaker and Xue-Cheng Tai:
  A Variant of the Level Set Method and Applications to Image Segmentation,
  September 2003. UCLA, Applied Mathematics, CAM-report-03-50.
- Johan Lie, Marius Lysaker and Xue-Cheng Tai:
  A Binary Level Set Method and Applications to Image Segmentation,
  CAM-report-04-31, May 2004.

Segmentation: Piecewise constant representation



(a) u with 3 object. (b) $\Omega = \cup_{i=1}^{3} \Omega_i$. (c) $\phi = 1 \vee 2 \vee 3$.

Figure 22:

Overlap or vacuum between the phases

$$K(\phi(x)) = (\phi - 1)(\phi - 2)\cdots(\phi - n) = \prod_{i=1}^{n}(\phi - i), \tag{14}$$

If $\phi(x) = i$ in Ω_i, then:

$$K(\phi(x)) = 0. \tag{15}$$

The basis functions

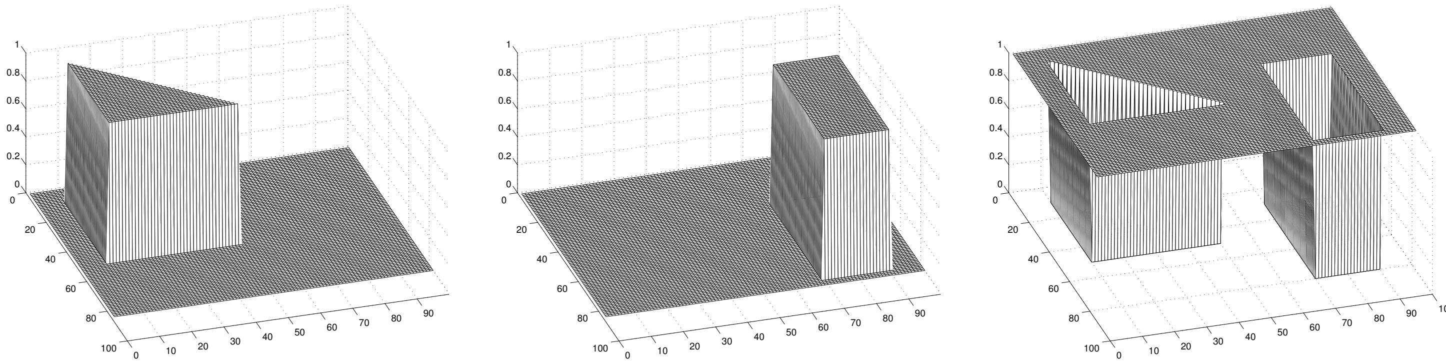
$$\psi_i(x) = \frac{1}{\alpha_i} \prod_{\substack{j=1 \\ j \neq i}}^{n} (\phi(x) - j) \quad \text{and} \quad \alpha_i = \prod_{\substack{k=1 \\ k \neq i}}^{n} (i - k). \tag{16}$$

It is clear that a function u given by:

$$u(x) = \sum_{i=1}^{n} c_i \psi_i(x) \tag{17}$$

is a piecewise constant function and $u = c_i$ in Ω_i.

Segmentation: Piecewise constant representation



(a) $\psi_1 = 0$ or 1.　(b) $\psi_2 = 0$ or 1.　(c) $\psi_3 = 0$ or 1.

Figure 23:

Area and arc length

The basis functions can further be utilized to express arc length of $\partial\Omega_i$ and the area Ω_i of each phase, i.e.

$$|\partial\Omega_i| = \int_\Omega |\nabla\psi_i| \, dx \text{ and } |\Omega_i| = \int_\Omega \psi_i \, dx. \tag{18}$$

Some conjectures

The basis functions can further be utilized to express the curvatue and normal vectors for the interface, i.e.

$$\vec{\mathbf{n}} \approx \frac{\nabla_h \phi}{|\nabla_h \phi|}, \qquad \kappa \approx \nabla \cdot \frac{\nabla_h \phi}{|\nabla_h \phi|} \text{ when mesh size goes to zero.} \tag{19}$$

Application to segmentation

Based on the above observations, we propose to solve the following constrained minimization problem for segmenting an image u_0:

$$\min_{\substack{\mathbf{c},\ \phi \\ K(\phi)=0}} \left\{ F(\mathbf{c}, \phi) = \int_\Omega |u - u_0|^2 dx + \beta \sum_{i=1}^{n} \int_\Omega |\nabla \psi_i| dx \right\}. \tag{20}$$

We see that large approximation errors will be penalized by the fidelity term $\int_\Omega |u - u_0|^2$. The penalization term is just the sum of the length of the sub-domain boundaries and it is treats all phases equally.

$$u = u(\mathbf{c}, \phi).$$

Application to other problems

$$\min_{\substack{\mathbf{c},\ \phi \\ K(\phi)=0}} \left\{ F(\mathbf{c}, \phi) = \int_{\Omega} |I(u) - u_0|^2 dx + \beta \sum_{i=1}^{n} \int_{\Omega} |\nabla \psi_i| dx \right\}. \tag{21}$$

$$u = u(\mathbf{c}, \phi).$$

Two-phase problems

$$\psi_1 = 2 - \phi, \quad \psi_2 = \phi - 1.$$

$$u = c_1\psi_1 + c_2\psi_2.$$

- No need to impose the constraint
- Minimization functional is strongly convex and smooth

$$\min_{\mathbf{c},\,\phi} \left\{ F(\mathbf{c}, \phi) = \int_\Omega |u - u_0|^2 dx + \beta \sum_{i=1}^{n} \int_\Omega |\nabla \psi_i| dx \right\}. \tag{22}$$

Three-phase problems

$$\psi_1 = (3-\phi)(2-\phi)/2, \quad \psi_2 = (\phi-1)(3-\phi) \quad \psi_3 = (\phi-1)(\phi-2)/2.$$

$$u = c_1\psi_1 + c_2\psi_2 + c_3\psi_3.$$

- No need to impose the constraint, but we often impose it.
- Minimization functional is locally convex and smooth.

$$\min_{\substack{\mathbf{c},\ \phi \\ K(\phi)=0}} \left\{ F(\mathbf{c}, \phi) = \int_\Omega |u-u_0|^2 dx + \beta \sum_{i=1}^{n} \int_\Omega |\nabla \psi_i| dx \right\}. \quad (23)$$

Augmented Lagrangian Method

We shall use the augmented Lagrangian method to solve the constrained minimization problem (21).

$$L(\mathbf{c}, \phi, \lambda) = F(\mathbf{c}, \phi) + \int_\Omega \lambda K(\phi)\, dx + \frac{r}{2}\int_\Omega |K(\phi)|^2 dx, \tag{24}$$

where $\lambda \in L^2(\Omega)$ is the multiplier and $r > 0$ is a penalty parameter which needs to be chosen properly. To find a minimizer for (21), we need to find the saddle points for L. We use the following Uzawa type algorithm to find a saddle point for $L(\mathbf{c}, \phi, \lambda)$:

Algorithm

Algorithm 1 Choose initial values for ϕ^0 and λ^0. For $k = 1, 2, \cdots$

1. Find $\mathbf{c}^k$ from

$$L(\mathbf{c}^k, \phi^{k-1}, \lambda^{k-1}) = \min_{\mathbf{c}} L(\mathbf{c}, \phi^{k-1}, \lambda^{k-1}). \tag{25}$$

2. Use (17) to update $u = \sum_{i=1}^{n} c_i^k \psi_i(\phi^{k-1})$.

3. Find ϕ^k from

$$L(\mathbf{c}^k, \phi^k, \lambda^{k-1}) = \min_{\phi} L(\mathbf{c}^k, \phi, \lambda^{k-1}). \tag{26}$$

4. Use (17) to update $u = \sum_{i=1}^{n} c_i^k \psi_i(\phi^k)$.

5. Update the Lagrange-multiplier by

$$\lambda^k = \lambda^{k-1} + rK(\phi^k). \tag{27}$$

Algorithm

$$\frac{\partial L}{\partial \phi} = (u - u_0)\frac{\partial u}{\partial \phi} - \beta \sum_{i=1}^{n} \nabla \cdot \left(\frac{\nabla \psi_i}{|\nabla \psi_i|} \right) \frac{\partial \psi_i}{\partial \phi} + \lambda \frac{\partial K}{\partial \phi} + rK \frac{\partial K}{\partial \phi}. \tag{28}$$

Use Gradient method to solve it:

$$\phi^{new} = \phi^{old} - \Delta t \frac{\partial L}{\partial \phi}(\mathbf{c}^k, \phi^{old}, \lambda^{k-1}). \tag{29}$$

$$\frac{\partial L}{\partial c_i} = \int_\Omega \frac{\partial L}{\partial u}\frac{\partial u}{\partial c_i} = \int_\Omega (u - u_0)\psi_i \; dx, \qquad \text{for } \; i = 1, 2, \cdots n. \tag{30}$$

Solve a system of equations $A\mathbf{c}^k = b$:

$$\sum_{j=1}^{n} \int_\Omega (\psi_j \psi_i) c_i^k \; dx = \int_\Omega u_0 \psi_i \; dx, \qquad \text{for } \; i = 1, 2, \cdots n. \tag{31}$$

Strength and Weakness

- Storage capacity
- Avoid reinitialization
- Avoid approximations for the Heaviside and the Delta function
- Not tracing curves
- advantage with inside "curves"
- Difficult to find a suitable way to represent the unit normal and the mean curvature for the curves
- Higher order polynomials are used as basis functions to represent u
- Coefficient matrix in is ill-conditioned

Strength and Weakness

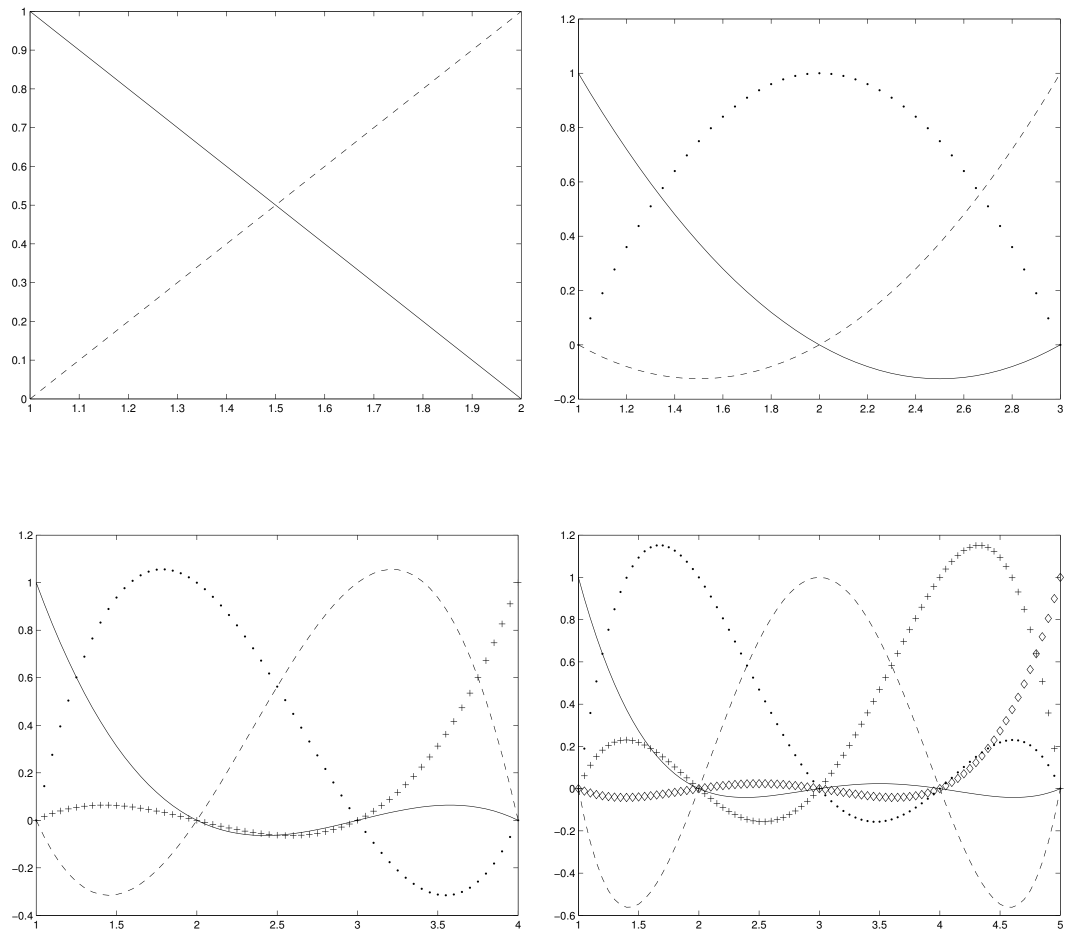
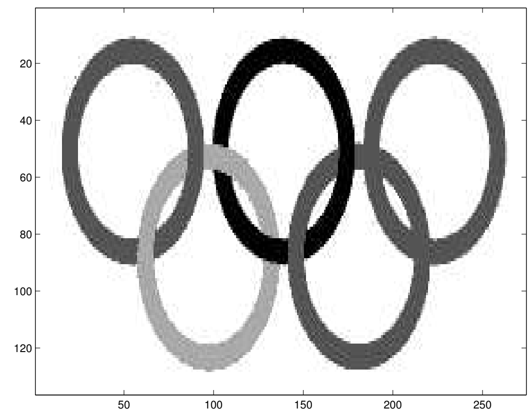
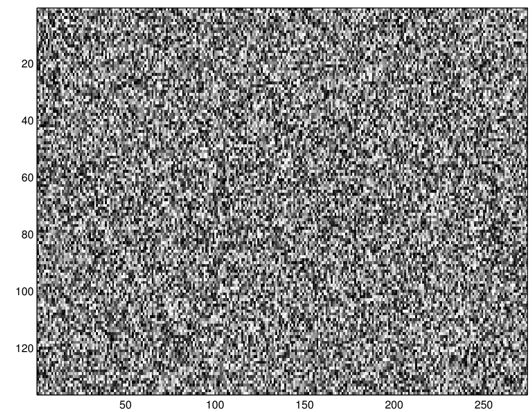


Figure 24: Different basis functions with $n =$

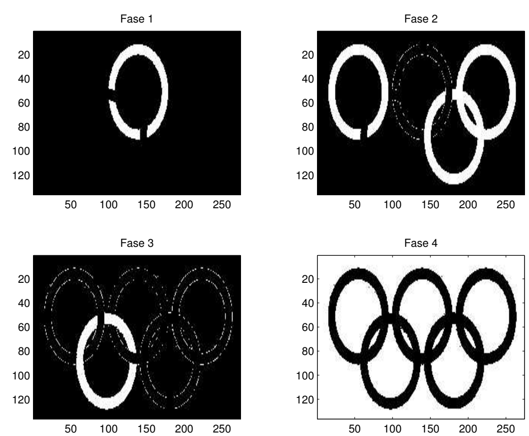
Segmentation: global method



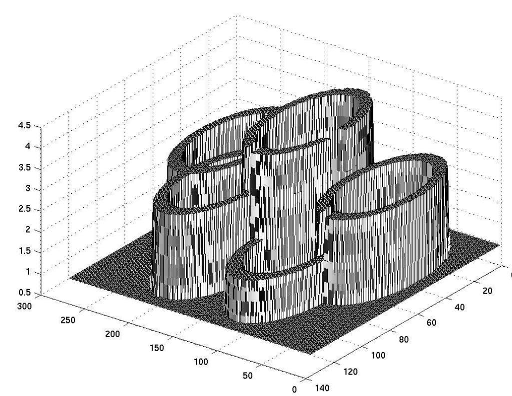
(a) Observed image u_0



(b) Initial level set function



(c) Different phases using



(d) At converges ϕ approach 4 con-

Segmentation

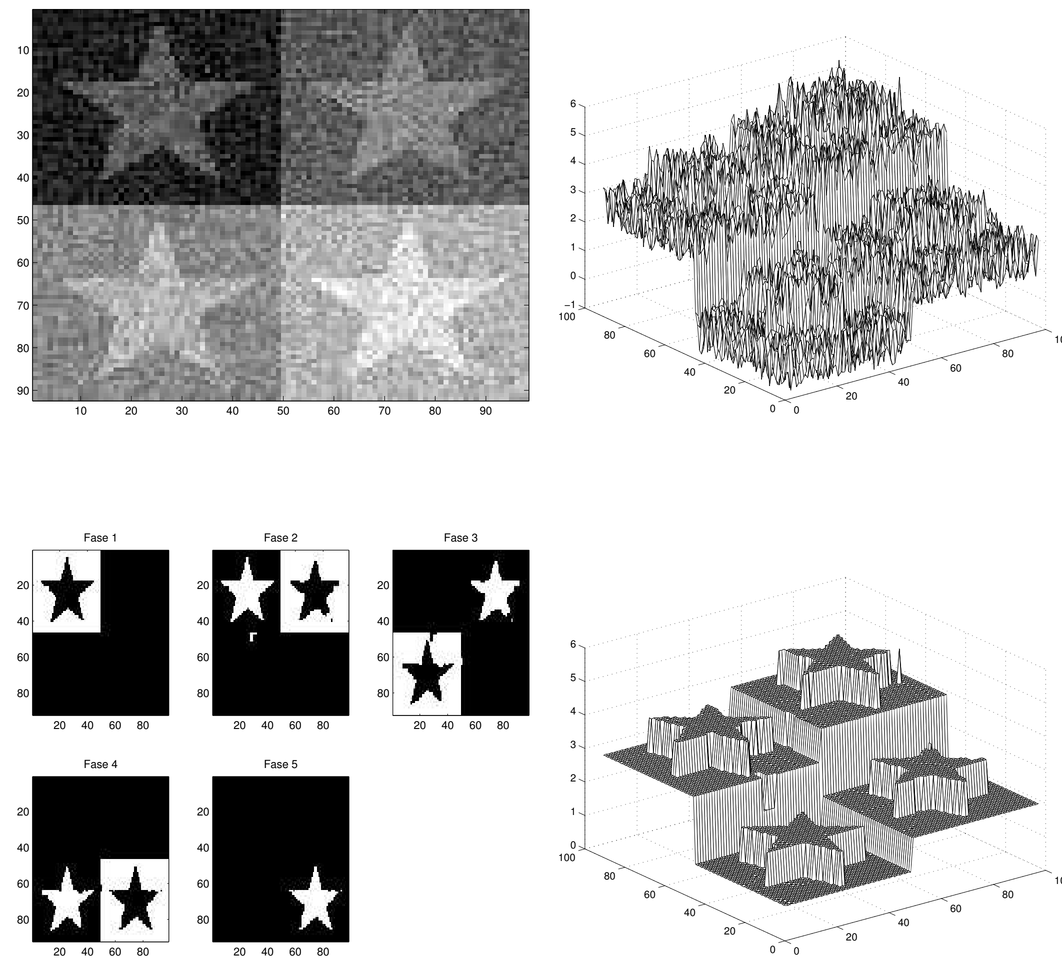
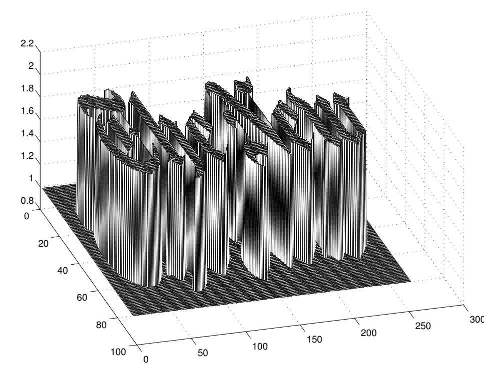


Figure 26:

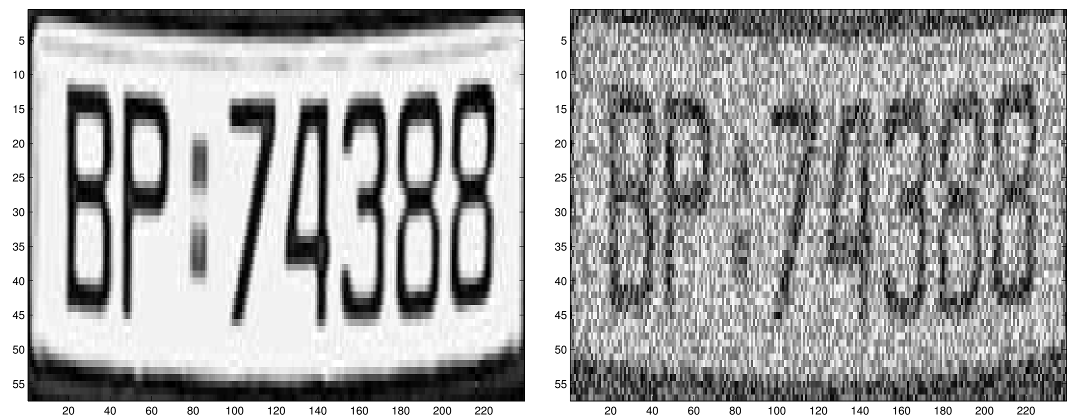
Segmentation

(a) Old newspaper.

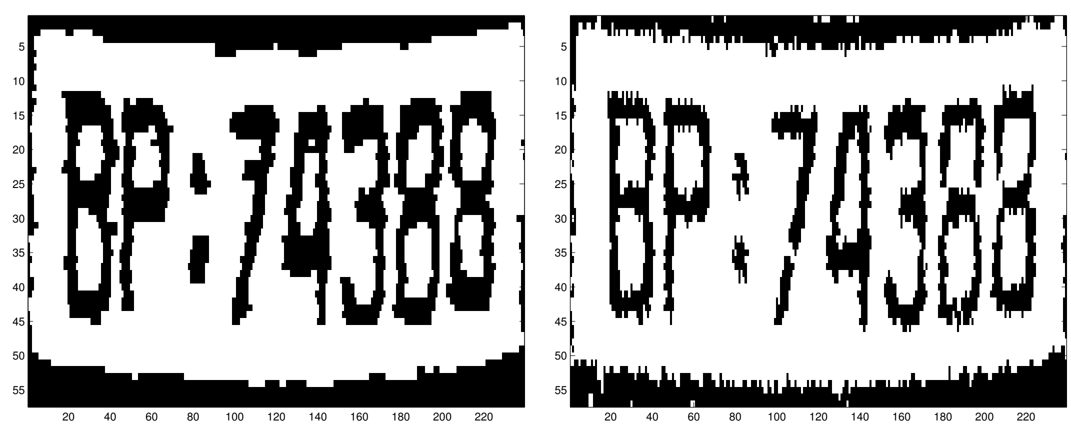


(b) A small partition of ϕ

Segmentation



(a) Car plate.



Segmentation

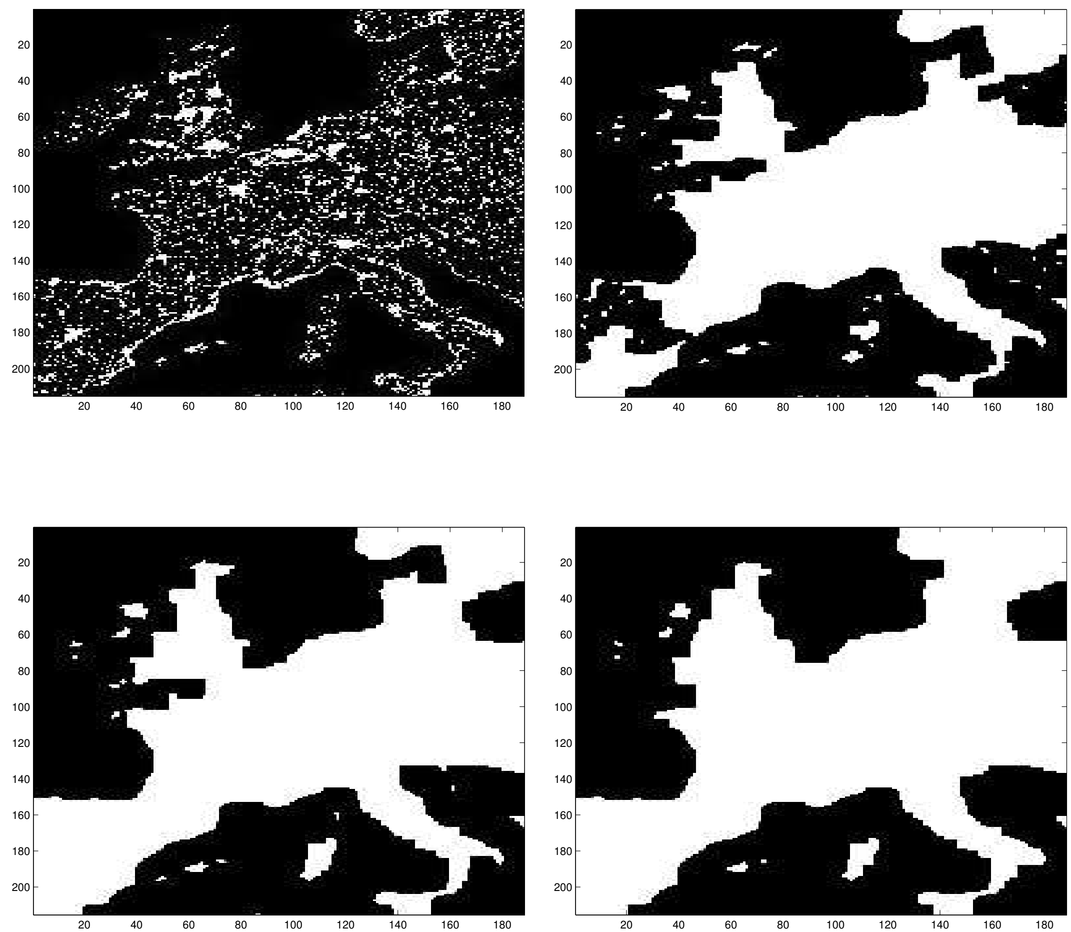
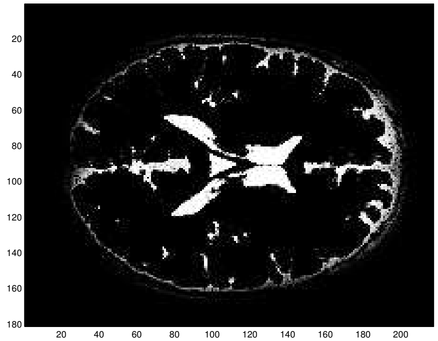
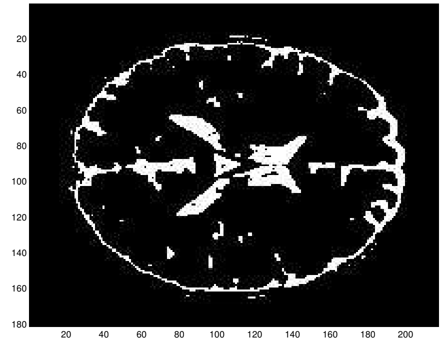


Figure 29: Scattered data with $\beta = 6, 7, 9$.

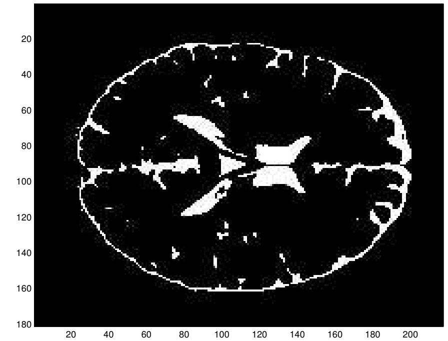
MRI segmentation



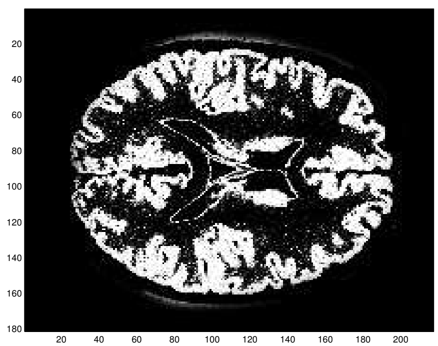
(a) SPM:phase



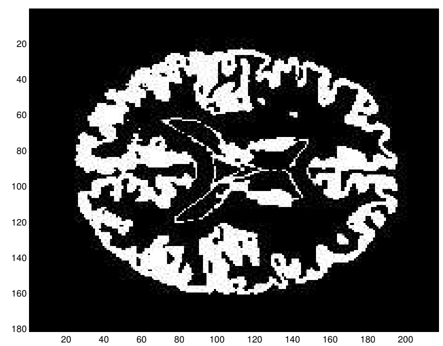
(b) PLSM:phase



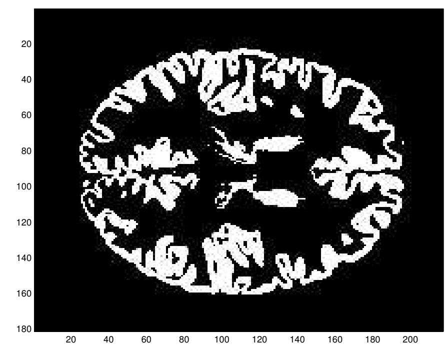
(c) Exact:phase



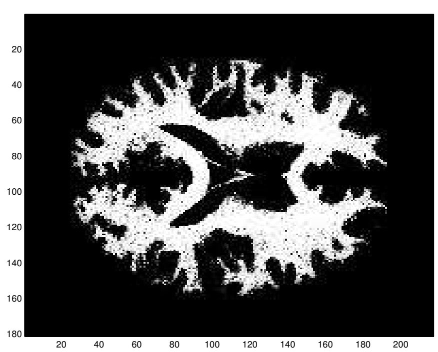
(d) SPM:phase

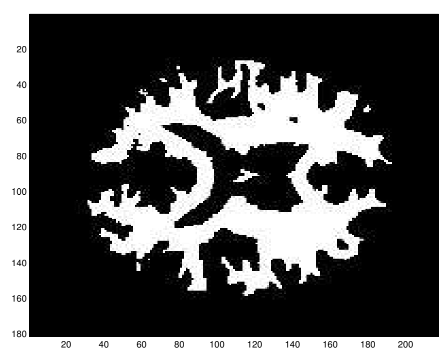


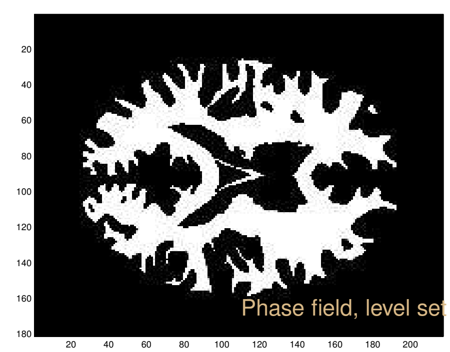
(e) PLSM:phase



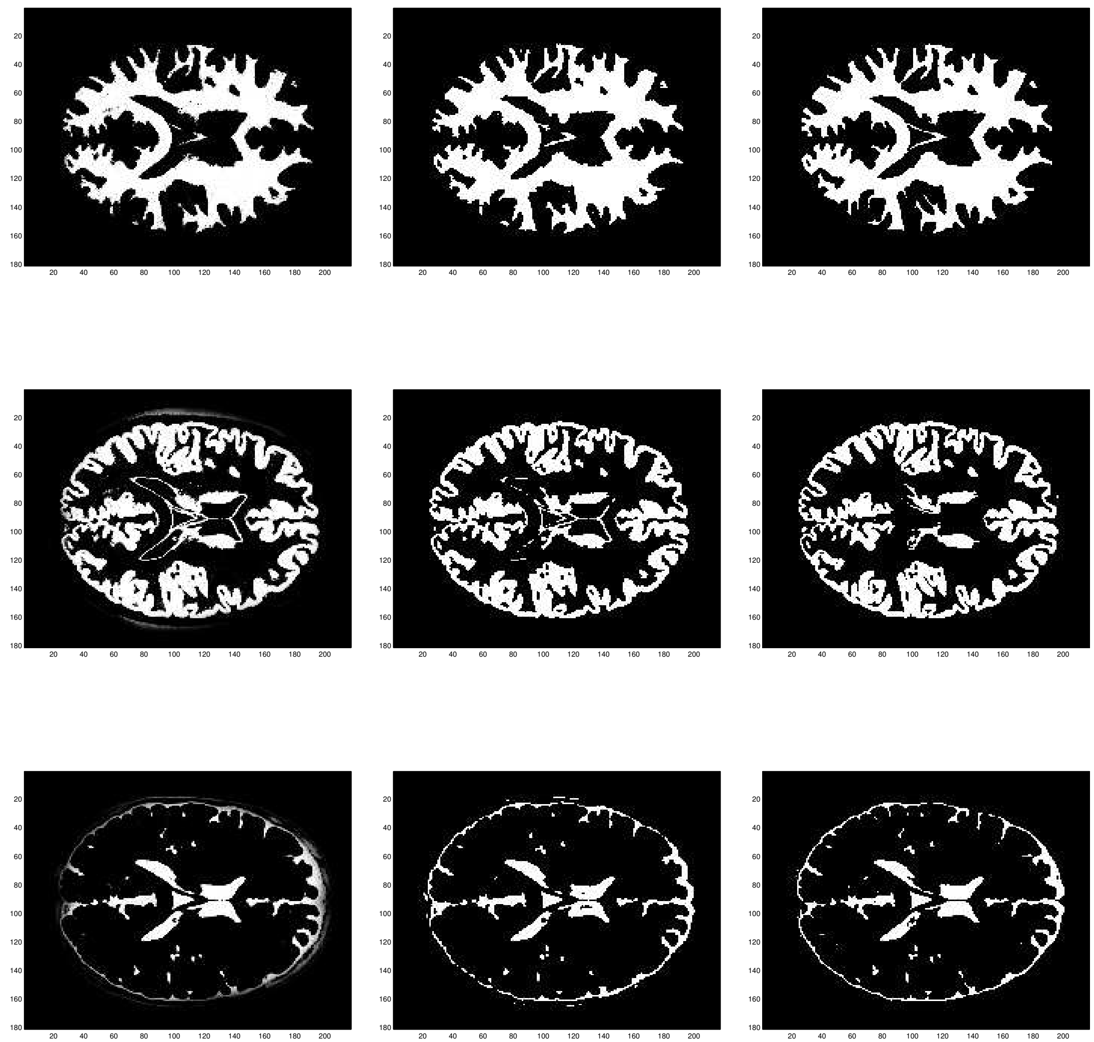
(f) Exact:phase



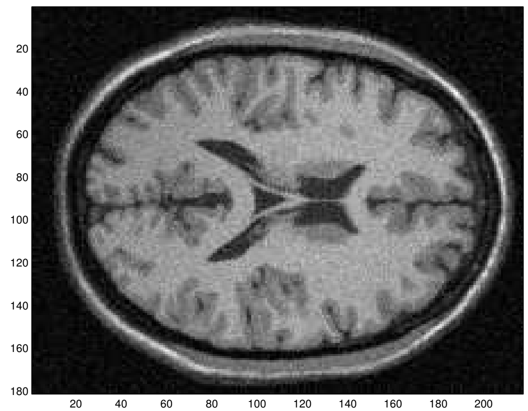




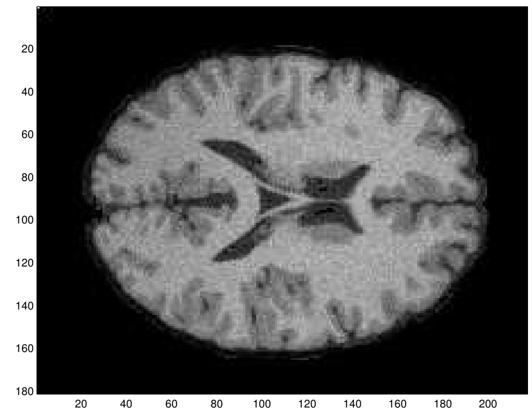
MRI segmentation



MRI segmentation

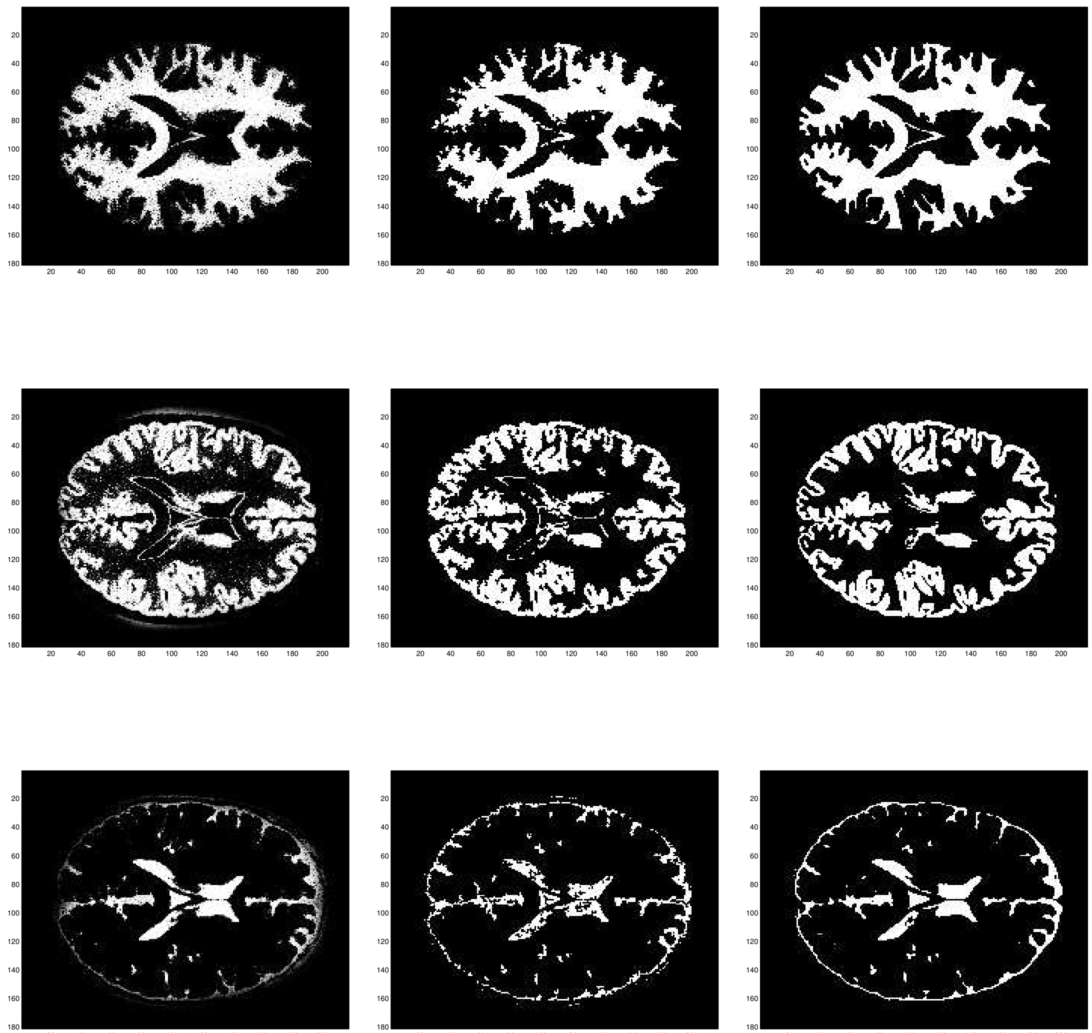


(a) The MRI image.

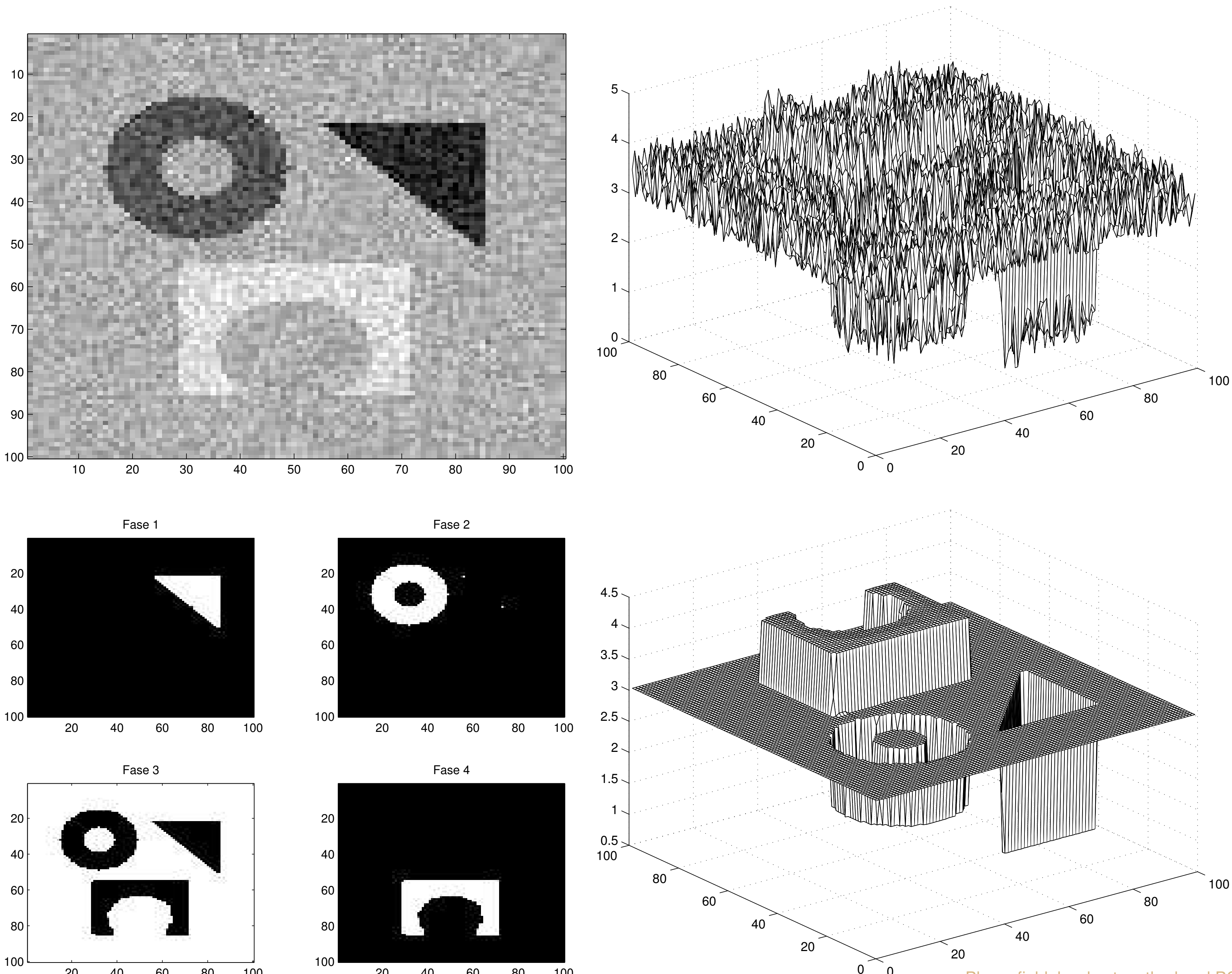


(b) The MRI image as input

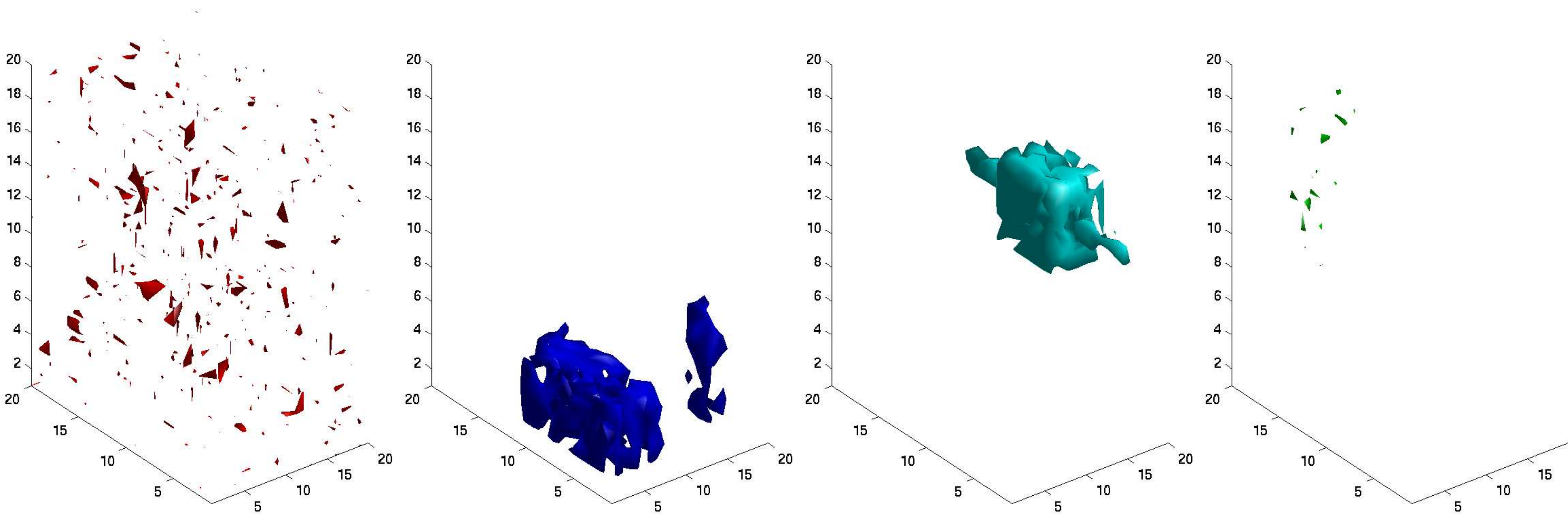
MRI segmentation



Segmentation: Good for complicated geometries

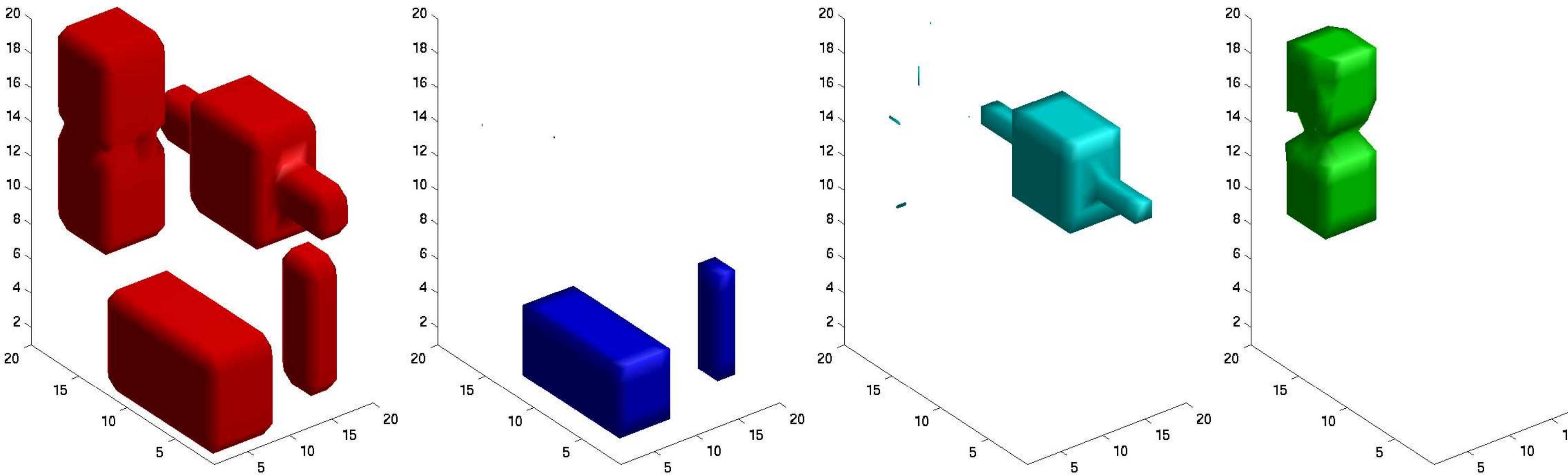


Segmentation: Extension to 3D



(a) 0.75 (b) 1.9165 (c) 3.0829 (d) 4.249

Segmentation: Extension to 3D



(e) 0.99823 (f) 1.99955 (g) 2.99022 (h) 3.99836

Figure 34: Segmentation of 3D objects

Another PCLSM: binary level set method

$$\phi = \begin{cases} 1, & x \in interior \\ -1, & x \in exterior \end{cases}$$

$$\phi^2 = 1$$

Use n level set functions $\phi_i^2 = 1$ to get 2^n regions.

Another PCLSM: binary level set method

$$u(\vec{x}) = \begin{cases} c_1, & \text{if } \phi_1(\vec{x}) = 1, \quad \phi_2(\vec{x}) = 1, \\ c_2, & \text{if } \phi_1(\vec{x}) = 1, \quad \phi_2(\vec{x}) = -1, \\ c_3, & \text{if } \phi_1(\vec{x}) = -1, \quad \phi_2(\vec{x}) = 1, \\ c_4, & \text{if } \phi_1(\vec{x}) = -1, \quad \phi_2(\vec{x}) = -1. \end{cases}$$

Thus, a piecewise constant function taking four different constant values can be written

$$\begin{aligned} u = {} & \frac{c_1}{4}(\phi_1 + 1)(\phi_2 + 1) - \frac{c_2}{4}(\phi_1 + 1)(\phi_2 - 1) \\ & - \frac{c_3}{4}(\phi_1 - 1)(\phi_2 + 1) + \frac{c_4}{4}(\phi_1 - 1)(\phi_2 - 1) \end{aligned} \tag{32}$$

Using (32), we can form the set of basis functions ψ_i as in the following

$$u = c_1 \underbrace{\frac{1}{4}(\phi_1 + 1)(\phi_2 + 1)}_{\psi_1} + c_2 \underbrace{(-1)\frac{1}{4}(\phi_1 + 1)(\phi_2 - 1)}_{\psi_2} + \ldots, \tag{33}$$

and we can write: $u = \sum_{i=1}^{4} c_i \psi_i$.

Binary level set methods

$$\min_{\substack{\mathbf{c},\ \phi \\ \phi_i^2=1}} \left\{ F(\mathbf{c}, \phi) = \int_\Omega |u - u_0|^2 dx + \beta \sum_{i=1}^{n} \int_\Omega |\nabla \psi_i| dx \right\}. \tag{34}$$

$$K_i(\phi_i) = \phi_i^2 - 1.$$

$$\mathbf{K} = (K_1, K_2, \cdots, K_n)$$

$$L(\mathbf{c}, \phi, \lambda) = F(\mathbf{c}, \phi) + \int_\Omega \lambda K(\phi)\, dx + \frac{r}{2} \int_\Omega |K(\phi)|^2 dx, \tag{35}$$

Binary level set methods

Algorithm 2 Choose initial values for $\vec{\phi}^0$ and $\vec{\lambda}^0$. For $k = 1, 2, \ldots,$ do:

- Update $\vec{\phi}^k$ by (29), to approximately solve

$$L(\vec{c}^{k-1}, \vec{\phi}^k, \vec{\lambda}^{k-1}) = \min_{\phi} L(\vec{c}^{k-1}, \vec{\phi}, \vec{\lambda}^{k-1}) \tag{36}$$

- Construct $u(\vec{c}^{k-1}, \vec{\phi}^k)$ by $u = \sum_{i=1}^{2^N} c_i^{k-1} \psi_i^k.$
- Update $\vec{c}^k$ by (31), to solve

$$L(\vec{c}^k, \vec{\phi}^k, \vec{\lambda}^{k-1}) = \min_{\vec{c}} L(\vec{c}, \vec{\phi}^k, \vec{\lambda}^{k-1}). \tag{37}$$

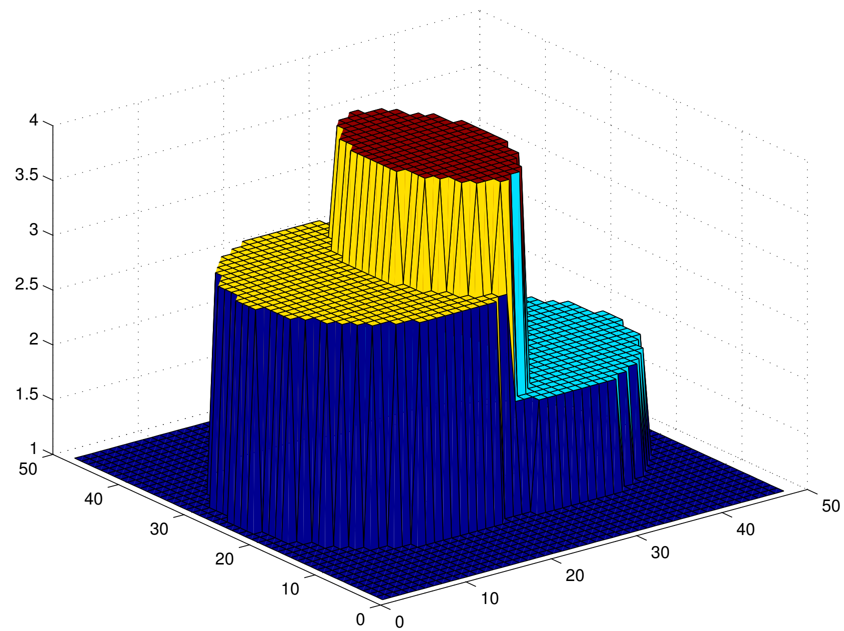
- Update the multiplier by

$$\vec{\lambda}^k = \vec{\lambda}^{k-1} + r\vec{K}(\vec{\phi}^k). \tag{38}$$

- If not converged: Set k=k+1 and go to step 1.

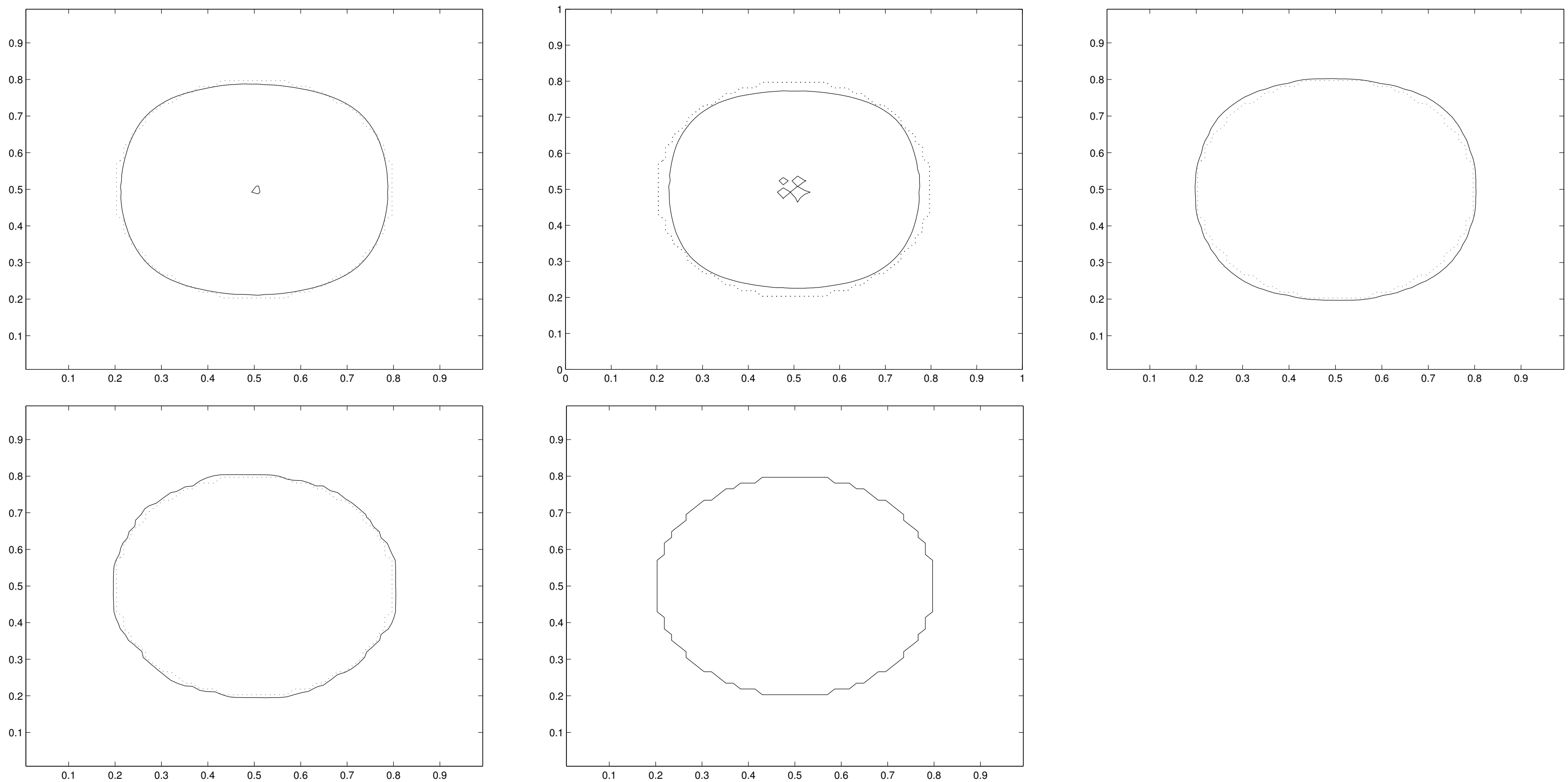
Elliptic inverse problems

$$-\nabla \cdot (q(x)\nabla u) = f, \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

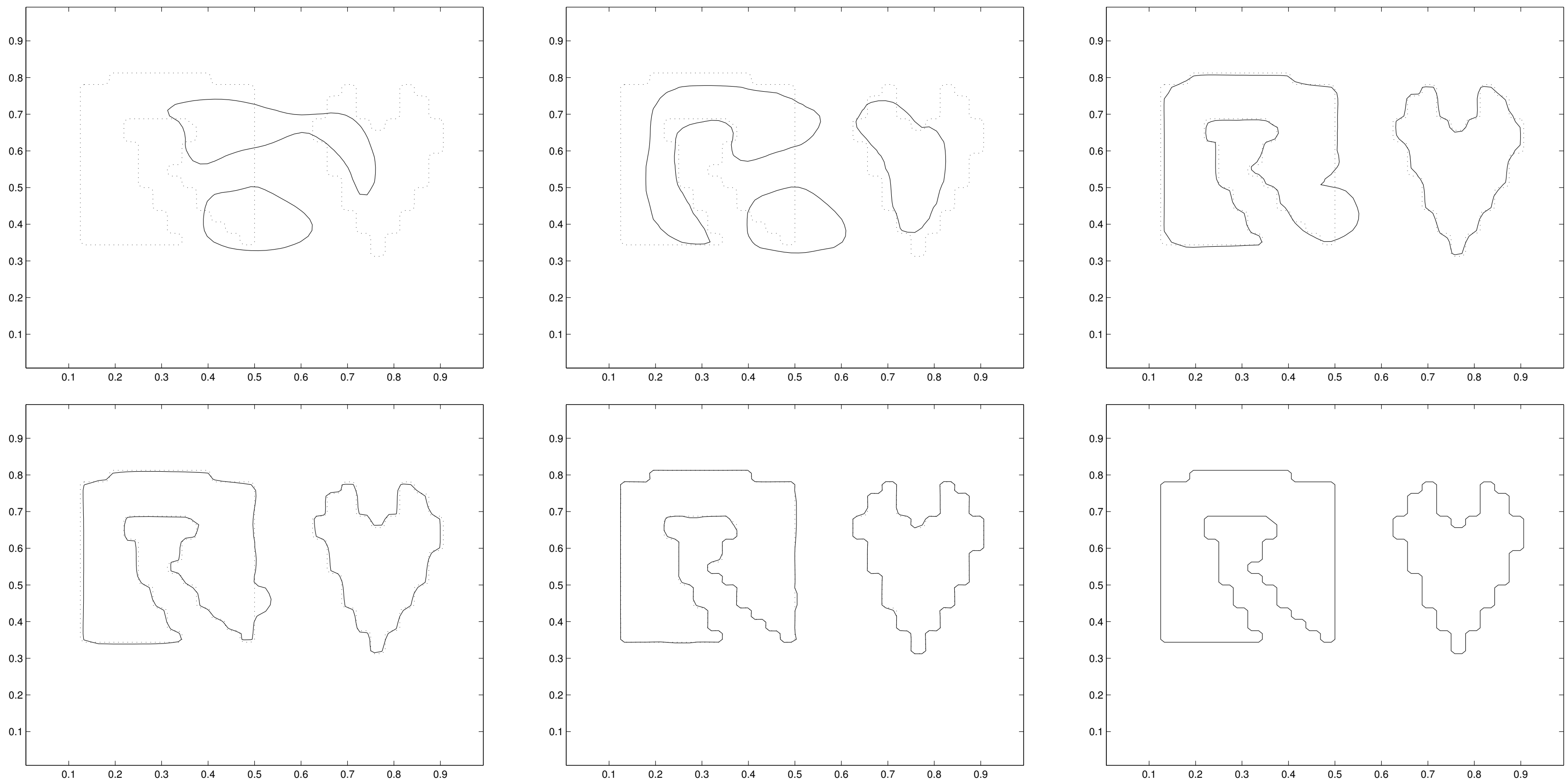


Our interests: q is discontinuous and is piecewise smooth. We want to use some information from u to recover $q(x)$.

Elliptic inverse problems



Elliptic inverse problems



Observations

- Can treat complicated geometries and sharp corners
- Discontinuous level set function, but smooth and convex minimization cost functional
- Remove Heaviside, get convexity and smoothness
- Disconnect from distance functions
- Different mechanism in moving the level function: global.
- Treat the same difficulties with another technique.

Fast methods

$$\begin{pmatrix} \phi^{new} \\ \lambda^{new} \end{pmatrix} = \begin{pmatrix} \phi^{old} \\ \lambda^{old} \end{pmatrix} + M^{-1} \begin{pmatrix} -\frac{\partial L}{\partial \phi}(\phi^{old}, \lambda^{old}) \\ rK(\phi^{old}). \end{pmatrix}$$

- Gradient method

$$M = \begin{pmatrix} \Delta t I & 0 \\ 0 & rI \end{pmatrix}$$

- Netwon or Quasi-Newton

$$H = \begin{pmatrix} \frac{\partial^2 L}{\partial \phi^2}(\phi^{old}, \lambda^{old}) & K'(\phi^{old}) \\ K'(\phi^{old}) & 0 \end{pmatrix}$$

- Gradient+Newton method

$$\tilde{M} = \alpha M + (1 - \alpha H), \quad \alpha \in [0, 1].$$

The MBO scheme

$$u_t = \epsilon \Delta u - \frac{1}{\epsilon} W'(u) \tag{39}$$

with $W(s) = (s^2 - 1)^2/2$.
If we use the splitting scheme to solve (39), we would need to solve the following two equations on $[t_n, t_{n+1}]$:

$$a)\ \phi_t = \epsilon \Delta \phi, \qquad b)\ \phi_t = -\frac{1}{\epsilon} W'(\phi) \tag{40}$$

The rescaled solution $\phi(x, t_n/\epsilon)$ of (43.a) is exactly the solution of (41). When $\epsilon \to 0^+$, the rescaled solution $\phi(x, t_n/\epsilon)$ of (43.b) has three values, i.e. 1, 0, -1. We drop the nonstable soultion 0 and get (42).

The MBO scheme

Algorithm 3 *(MBO scheme for two regions)*
Choose initial value $\phi(0) = \pm 1$ *and the time step* τ*. For* $n = 0, 1, 2, \cdots$ *and* $t_n = n\tau$*,*

- *Solve* $\tilde{\phi}(t), t \in [t_n, t_{n+1}]$ *from*

$$\tilde{\phi}_t = \Delta \tilde{\phi}, \qquad \tilde{\phi}(t_n) = \phi(t_n) \text{ in } \Omega, \quad \frac{\partial \tilde{\phi}}{\partial n} = 0 \text{ on } \partial\Omega. \tag{41}$$

- *Set*

$$\phi(t_{n+1}) = \begin{cases} -1 & \text{if } \tilde{\phi}(t_{n+1}) < 0, \\ 1 & \text{if } \tilde{\phi}(t_{n+1}) \geq 0. \end{cases} \tag{42}$$

The MBO scheme for segmentation

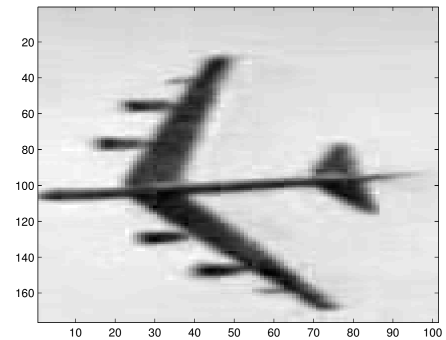
Once c_i are comouted, we need to solve the following minimization problem for segmentation:

$$\min_{K(\phi)=0} F(\phi).$$

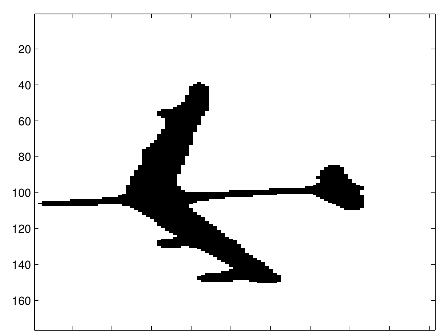
The splitting scheme is in this case:

$$a)\ \phi_t = \epsilon F'(\phi), \qquad b)\ \phi_t = -\frac{1}{\epsilon} W'(\phi) \tag{43}$$

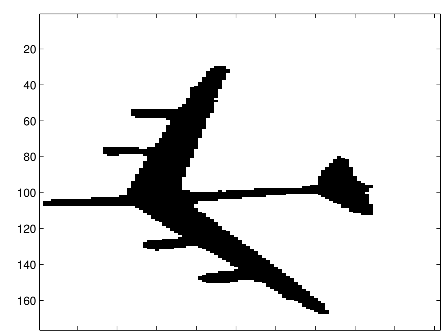
The MBO scheme for segmentation



(a) Input picture



(b) u, $\tau = 0.03$.



(c) $\tau = 0.01$.

Another fast variant

$$\frac{\phi^{n+1} - \phi^n}{\tau} = -\frac{1}{\epsilon} W'(\phi^n) \tag{44}$$

This is equivalent to

$$\min \frac{1}{2}|\phi - \phi^n|^2 + \frac{\tau}{2\epsilon}|K(\phi)|^2. \quad (W(\phi) = |K(\phi)|^2). \tag{45}$$

This is a polynormial of order $2n$. Similar problems need to be solved for Liquid Crystale problems. We have fast methods to solve this one. The method converes in about 10-20 iterations and the cost of the iteration per step is nearly the same as the explicit method.

Relationships

- Phase field:

$$-\epsilon\Delta u + \epsilon\Delta^2 u + \frac{1}{\epsilon}u(u^2-1) = 0 \text{ in } \Omega \quad u = g \text{ on } \partial\Omega.$$

$$\min_{u=g} \int_\Omega \frac{\epsilon}{2}(|\nabla u|^2 + |\Delta u|^2) + \frac{1}{4\epsilon}(u^2-1)^2 dx.$$

$$u \to u^* = \pm 1.$$

- Level set method: Level set function ϕ, $\phi(x) > 0$ indicates that x is in the interior, $\phi(x)$ indicates that x is in the exterior.

$$u = H(\phi).$$

- PCLSM:
  - Binary PCLSM: $u = \pm 1$.

$$|\partial\Omega_i| = \int_\Omega |\nabla \psi_i| \, dx \text{ and } |\Omega_i| = \int_\Omega \psi_i \, dx. \tag{46}$$

  - Multilayer PCLSM: $u = 1, 2, 3, \cdot, n$.

Relationships

- Advantages: Local and global mechanism.
- Level set method: velociyt for $\Gamma \to$ velocity for $\phi \to$ extend to $\Omega \to$ move ϕ. This mechanism is local.
- PCSLSM and phase field: Do not move the curve, move u so that at the end it goes to ± 1 or $1, 2, \cdot n$. Every point is moving during the iterations.
- Constraint: replace distance functions by other constraints.

References:

- Level set for inverse problems:
  - Survey: Tai and Chan, 2003. Burger and Osher, 2004.
  - Other papers by: Lysaker-Nielssen, Ascher, Hackl, Ring, Santosa, Dobson, Dorn, Lionheart, Mannseth ...
- Phase field:
  - Evans, Soner, Souganidis, Robistein, Sternberg, Keller, Liu, Du.
- Phase field combined with levelset:
  - MBO
  - Aubert, Song-Chan, Fedkiw-Gobu.
  - Lie, Lysaker and Tai, 2003, 2004
  - Esedohlu and Tsai, 2004.
  - Scherzer, Burger.

Conclusion

Thank you!