Adaptive Variational Multiscale Method: Basic A Posteriori Error Estimation Framework

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Goal

• We want to find computational methods for solving multiscale problems in a Galerkin finite element setting.
• We need an a posteriori estimation framework to measure the reliability of our solution.
• We also want to use the error bounds for adaptivity.
• We start with two scales in two dimensions.
Outline

• Model Problem
• Variational Multiscale Method
• Choice of Coarse and fine Spaces
• The Basic Idea of our Method
• Error Estimates
• Adaptive Strategy
• Numerical Examples
• Future Work
Model Problem

Poisson Equation. Find $u \in H^1_0(\Omega)$ such that

$$-\nabla \cdot a \nabla u = f \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega.$$ 

where $f \in H^{-1}(\Omega)$, $a > 0$ bounded, and $\Omega$ is a domain in $\mathbb{R}^d$, $d = 1, 2, 3$.

Weak form. Find $u \in H^1_0(\Omega)$ such that

$$(a \nabla u, \nabla v) = (f, v) \quad \text{for all} \ v \in H^1_0(\Omega).$$
Multiscale Problems

Below are three examples of multiscale problems.

The first one represents difficulties in the domain (cracks, holes, ...) the second one oscillations in $a$ and the third one oscillations in $f$. 
Variational Multiscale Method

- $H^1_0 = V_c \oplus V_f$, $u = u_c + u_f$, and $v = v_c + v_f$.

Find $u_c \in V_c$ and $u_f \in V_f$ such that

\[
(a \nabla u_c, \nabla v_c) + (a \nabla u_f, \nabla v_c) = (f, v_c) \quad \text{for all } v_c \in V_c, \\
(a \nabla u_f, \nabla v_f) = (f, v_f) - (a \nabla u_c, \nabla v_f) \\
:= (R(u_c), v_f) \quad \text{for all } v_f \in V_f.
\]
Figure 1: $u_c$, $u_f$, and $u_c + u_f$. 
Variational Multiscale Method

- The fine scale is driven by the coarse scale residual.
- Approximation to fine scale solution solved on each element analytically (Green’s functions).
- Fine scale information is then used to modify the coarse scale equation.

\[
(a \nabla u_c, \nabla v_c) + (a \hat{A}_f^{-1} R(U_c), \nabla v_c) = (f, v_c) \quad \forall v_c \in V_c.
\]
Choice of $V_c$ and $V_f$

We use the splits proposed by Vassilevski-Wang (1998) and also used by Aksoyulu-Holst (2004).

- Hierarchical basis, HB.
- Wavelet modified hierarchical basis, WHB.

The aim with WHB is to make $V_f$ more $L^2(\Omega)$ orthogonal to $V_c$ than in ordinary HB.

\[(Q_c^a v, w) = (v, w), \quad \text{for all } w \in V_c.\]

\[\varphi_{WHB} = (I - Q_c^a)\varphi_{HB}.\]
Choice of $V_c$ and $V_f$

Figure 2: HB-function and WHB-function with two Jacobi iterations.
Basic Idea

- Discretization of $V_f$ by (W)HB-functions ($V_f^h$).
- Solve localized fine scale problems for each coarse node (or some coarse nodes).
- Possibility to do this in parallel.
- A posteriori error estimation framework.
- Adaptive strategy for this setting.
Decouple fine Scale Equations

Remember the fine scale equations:

\[(a \nabla U_f, \nabla v_f) = (R(U_c), v_f), \quad \text{for all } v_f \in V^h_f.\]

Include a partition of unity,

\[(a \nabla U_f, \nabla v_f) = (R(U_c), v_f) = \sum_{i=1}^{n} (R(U_c), \varphi_i v_f),\]

let \(U_f = \sum_{i}^{n} U_{f,i}\) where

\[(a \nabla U_{f,i}, \nabla v_f) = (R(U_c), \varphi_i v_f).\]
Approximate Solution

Find $U_c \in V_c$ and $U_f = \sum_{i=1}^{n} U_{f,i}$ where $U_{f,i} \in V_{f}^h(\omega_i)$ such that

$$(a \nabla U_c, \nabla v_c) + (a \nabla U_f, \nabla v_c) = (f, v_c) \quad \text{for all } v_c \in V_c,$$

$$(a \nabla U_{f,i}, \nabla v_f) = (R(U_c), \varphi_i v_f) \quad \text{for all } v_f \in V_{f}^h(\omega_i).$$

- Since $\varphi_i$ has support on a star $S_i^1$ in node $i$ we solve the fine scale equations approximately on $\omega_i$ with $U_{f,i} = 0$ on $\partial \omega_i$. 
Figure 3: One, $S^1_i$, and two, $S^2_i$, layer stars.
Iterative or Direct

Iterative \( U_{f,i}^0 = 0, \)

\[
(a \nabla U_c^k, \nabla v_c) = (f, v_c) - (a \nabla U_f^{k-1}, \nabla v_c),
\]

\[
(a \nabla U_{f,i}^k, \nabla v_f) = (R(U_c^k), \varphi_i v_f),
\]

or in matrix form,

\[
A_c U_c^k = b_c(U_f^{k-1})
\]

\[
\hat{A}_f U_{f,i}^k = b_f(U_c^k)
\]
Iterative or Direct

Direct

\[(a \nabla U_c, \nabla v_c) + (\nabla \hat{A}_f^{-1} R(U_c), \nabla v_c) = (f, v_c)\]

or in matrix form,

\[(A_c + T)U_c = b - d,\]

where \(b_j = (f, \varphi_j),\)

\[T_{ij}\varphi_j + d_i = (\nabla \hat{A}_f^{-1}(R(\varphi_i)), \nabla \varphi_j).\]
Algorithm

\[
AU = b
\]

T_1 \quad T_2 \quad T_3 \quad ...

\[
(A + T)U = b - d
\]

R(U)
Error Estimation

We let \( e = u - U = u_c + \sum_{i=1}^{n} u_{f,i} - U_c - \sum_{i=1}^{n} U_{f,i} \) denote the error. We further let \( e_c = u_c - U_c \) and \( e_{f,i} = u_{f,i} - U_{f,i} \).

- Energy norm error estimate for primal solution, \( \| \nabla e \| \), in the case when \( a \) is a constant.
- Linear functional error estimate for the case when \( a \) is a constant.
- Application on the dual problem.
Energy norm estimate

We now focus on the case when $a = 1$. Remember the weak form for the exact solution, Find $u_c \in V_c$ and $u_f \in V_f$ such that

$$
(\nabla u_c, \nabla v_c) + (\nabla u_f, \nabla v_c) = (f, v_c) \quad \text{for all } v_c \in V_c,
$$

$$
(\nabla u_f, \nabla v_f) = (f, v_f) - (\nabla u_c, \nabla v_f) \quad \text{for all } v_f \in V_f.
$$

Since the first equation also holds for the approximate solution we have

$$
(\nabla e_c, \nabla v_c) + (\nabla e_f, \nabla v_c) = 0.
$$
Energy norm estimate

\[ \| \nabla e \|^2 = (\nabla e, \nabla e) = (\nabla e, \nabla e_f) \]
\[ = (\nabla e, \nabla e_f - P_f^h e) + (\nabla e, \nabla P_f^h e), \]

where \( P_f^h \) is the \( L^2 \) projection onto \( V_f^h \).

\[ (\nabla e, \nabla P_f^h e) = \sum_{i=1}^{n} (\nabla e_c, \nabla \varphi_i P_f^h e) + \sum_{\text{fine}} (\nabla e_{f,i}, \nabla P_f^h e) \]
\[ + \sum_{\text{coarse}} (\nabla u_{f,i}, \nabla P_f^h e) \]
Energy norm estimate

\[(\nabla e, \nabla P_f^h e) = \sum_{i=1}^n (\nabla e_c, \nabla \varphi_i P_f^h e) + \sum_{\text{fine}} (\nabla e_{f,i}, \nabla P_f^h e)\]

\[+ \sum_{\text{coarse}} (\nabla u_{f,i}, \nabla P_f^h e)\]

\[= \sum_{\text{fine}} (R(U_c), \varphi_i P_f^h e) - (\nabla U_{f,i}, \nabla P_f^h e)\]

\[+ \sum_{\text{coarse}} (R(U_c), \varphi_i P_f^h e)\]
Energy norm estimate

\[ \| \nabla e \|^2 = (\nabla e, \nabla e - P_f^h e) + \sum_{\text{coarse}} \left( R(U_c), \varphi_i P_f^h e \right) \]
\[ \quad + \sum_{\text{fine}} \left( R(U_c), \varphi_i P_f^h e \right) - (\nabla U_{f,i}, \nabla P_f^h e) \]
\[ = I + II + III \]
Energy norm estimate

I

\[(\nabla e, \nabla e - P^h_f e) \leq \| hR(U_c + U_f) \| \| \nabla e \| \]

II

\[
\sum_{\text{coarse}} (R(U_c), \varphi_i P^h_f e) \leq C \left( \sum_{\text{coarse}} \| H R(U_c) \|_{S^1_i} \right) \| \frac{1}{H} P^h_f e \|
\]

\[
\leq C \left( \sum_{\text{coarse}} \| H R(U_c) \|_{S^1_i} \right) \| \nabla e \|
Energy Norm Estimate

III
On the black board...
Energy Norm Estimate

\[ \| \nabla e \| \leq C \| hR(U_c + U_f) \| + C \sum_{\text{coarse}} \| H R(U_c) \|_{S_i^1} \]

\[ + C \sqrt{H} \sum_{\text{fine}} \| \Sigma_i \|_{\partial \omega_i} \]

- The first term is referred to as the truth mesh error (reference).
- The third term is the normal derivative of the fine scale solutions on \( \partial \omega_i \).
Dual Problem

The standard approach to get a bound of a linear functional of the error is to introduce a dual problem:

\[ \text{Find } \phi \in H^1_0 \text{ such that } \]

\[ -\Delta \phi = \psi. \]

We when get for \( \pi \phi \in V_h \),

\[ (e, \psi) = (e, -\Delta \phi) = (\nabla e, \nabla \phi) = (\nabla e, \nabla \phi - \pi \phi). \]

And after integration by parts we get

\[ (e, \psi) = (R(U), \phi - \pi \phi). \]
Dual Problem

• The dual solution $\phi$ need to be approximated but not in $V$.

• Regular refinement or higher order method allocate lots of memory.

Instead we solve the dual problem by local problems in each coarse node,

$$ (e, \psi) = \sum_{i=1}^{n} (R(U), \Phi_{f,i}) + (R(U), \phi_f - \Phi_f). $$
Dual Problem

The second term can be estimated in the following way,

$$(\nabla e, \nabla (\phi_f - \Phi_f)) \leq \|\nabla e\| \|\nabla (\phi_f - \Phi_f)\|$$

$$\leq \|\nabla e\| \|\nabla (\phi - (\Phi_c + \Phi_f))\|.$$

And we get the energy norm of the error in the dual solution which can be estimated.
Adaptive Strategy

\[ \| \nabla e \| \leq C \| hR(U_c + U_f) \| + C \sum_{\text{coarse}} \| H R(U_c) \|_{S^1_i} \]

\[ + C \sqrt{H} \sum_{\text{fine}} \| \Sigma_i \| \partial \omega_i \]

- We focus on the last two terms.
- We calculate these for each \( i \in \{ \text{coarse fine} \} \).
- Big values \( i \in \text{coarse} \rightarrow \) more local problems.
- Big values \( i \in \text{fine} \rightarrow \) more layers.
Numerical Examples

We start with a unit square containing a crack.

We let the coefficient $a = 1$ and solve, $-\triangle u = f$ with $u = 0$ on the boundary including the crack.
Numerical Examples

We solve the problem by using the adaptive algorithm with a refinement level of 10 % each iteration.

We plot the difference between our solution and a reference solution.
Numerical Examples

In this example we study a discontinuous coefficient $a$ in $-\nabla \cdot a \nabla u = f$. $a = 1$ (white) and $a = 0.05$ (blue).
Numerical Examples
Future Work

- Error estimates in the case when $a \neq 1$.
- Extended numerical tests in both 2D and 3D.
- More scales.
- Other equations (convection-diffusion, ...).
- Comparing results with classical Homogenization theory.
References
