Winter School on Parallel Computing:

Introduction to Domain Decomposition Methods

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Contents

What is domain decomposition

Many different interpretations within the PDE community:

- Parallel computing: data decomposition (independent of numerical method)
- Asymptotic analysis: separation of physical domain into regions with possibly different models
- Preconditioning methods: solution of a large linear system arising from the discretization of the PDE on the whole domain by DDM as a solver or a preconditioner:
  - Overlapping domain decomposition
  - Non-overlapping domain decomposition

We are interested in domain decomposition algorithms in the last sense.

Why use domain decomposition?

– Divide-and-conquer –

Domain decomposition – decomposition of the spatial domain into several subdomains. Search the global true solution through (*iterative*ly) solving subproblems while enforcing suitable continuity requirements between neighbor subdomains.

- Flexible – localized treatment of complex and irregular geometries, singularities etc.
- Efficient – often optimal convergence rate
- Easy to parallelize (*coarse grain parallelization*)

Motivations for the use of DDM

- Potential of parallelization.
- Ability to deal with equations on complicated geometries.
- Ability to deal with equations that exhibit different behaviour in different parts of the domain.
- Convergence properties (even on sequential machines!).

References

Survey/overview:

- T.F. Chan and T.P. Mathew, Domain decomposition algorithms, Acta Numerica pp. 61-143, 1994.
- B.F. Smith, P.E. Bjørstad and W.D. Gropp, *Domain Decomposition: parallel multilevel methods for elliptic partial differential equations.* Cambridge University Press, Cambridge, 1996.

Mathematical analysis:

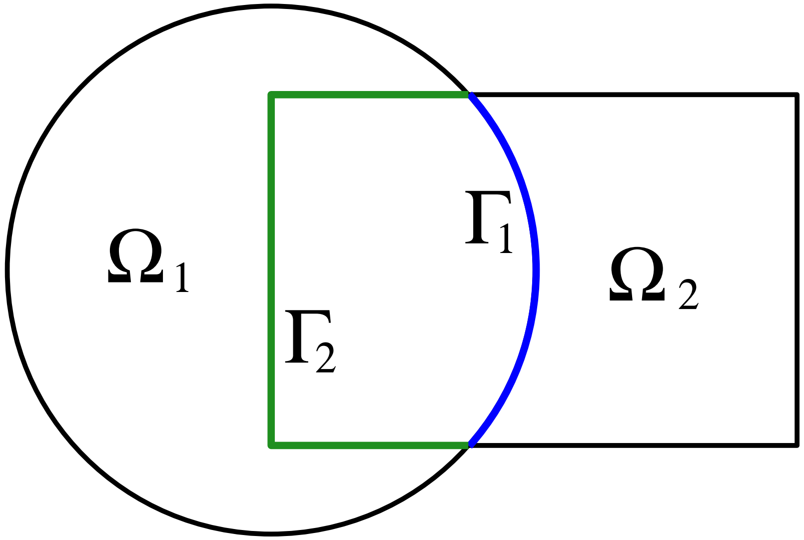
- M. Dryja and O.B. Widlund, Towards a unified theory of domain decomposition algorithms for elliptic problems, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, Houston, Texas 1989, T. Chan et. al., eds., SIAM Publications, pp. 3-21, 1990.
- J. Xu, Iterative methods by space decomposition and subspace correction, SIAM Rev. 34, pp. 581-613, 1992.

On line references: http://www.ddm.org

History: the first occurence

The first domain decomposition method, proposed by Schwarz in 1870.
We want to solve the following elliptic boundary value problem on the union of two subdomains $\Omega = \Omega_1 \cup \Omega_2$:

$$\begin{aligned} Au &= f, \quad \text{in } \Omega \\ u &= g, \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$



For Ω_i, $i = 1, 2$,

- Artificial boundary: Γ_i
- Real boundary: $\partial\Omega_i \backslash \Gamma_i$

Alternating Schwarz method

An iterative process where we generate a sequence of approximations $u_i^0, u_i^1, \ldots, u_i^n$ on each subdomain Ω_i.

The algorithm: Select an initial guess u_2^0.
For $n = 1, 2, 3, \ldots$,

$$\begin{aligned} A_1 u_1^n &= f_1 && \text{in} \quad \Omega_1, \\ u_1^n &= g && \text{on} \quad \partial\Omega_1 \backslash \Gamma_1, \\ u_1^n &= u_2^{n-1}|_{\Gamma_1} && \text{on} \quad \Gamma_1, \end{aligned} \tag{2}$$

and

$$\begin{aligned} A_2 u_2^n &= f_2 && \text{in} \quad \Omega_2, \\ u_2^n &= g && \text{on} \quad \partial\Omega_2 \backslash \Gamma_2, \\ u_2^n &= u_1^n|_{\Gamma_2} && \text{on} \quad \Gamma_2. \end{aligned} \tag{3}$$

Stopping criterion: (for example)

$$\|u^{n+1}|_{\Gamma_1} - u^n|_{\Gamma_1}\| \le \epsilon \;\; \text{and} \;\; \|u^{n+1}|_{\Gamma_2} - u^n|_{\Gamma_2}\| \le \epsilon$$

Γ_i can be replaced by Ω_i

Alternating Schwarz contd.

- We solve an elliptic boundary value problem on each subdomain Ω_i with boundary values:
  - u given by g on true boundaries $\partial\Omega_i$,
  - u given by updated values on artifical boundaries Γ_i
- The iterate u^{n+1} is defined by

$$u^{n+1}(x,y) = \begin{cases} u_2^{n+1}(x,y) & \text{if } (x,y) \in \Omega_2 \\ u_1^{n+1}(x,y) & \text{if } (x,y) \in \Omega \backslash \Omega_2 \end{cases}$$

- In the norm induced by the operator A, the iterates $\{u^n\}$ converge to the true solution u

$$\|u - u^n\| \leq \rho^n \left\|u - u^0\right\|$$

  with ρ depending on the choice of Ω_1 and Ω_2.
- One way of determining the convergence: the approximate solutions in both subdomains or on the artificial boundaries change by less than a given tolerance from the previous iteration.

Matrix formulation

Consider the decomposition of A_i as

$$A_i = (A_{\Omega_i} \;\; A_{\partial\Omega_i \setminus \Gamma_i} \;\; A_{\Gamma_i})$$

Assume (for simplicity) that $g \equiv 0$.

Let $I_{\Omega_i \to \Gamma_j}$ be a discrete operator which interpolates the nodes in the interior of Ω_i to the nodes on the curve Γ_j. Then

$$A_{\Omega_1} u_{\Omega_1}^n = f_1 - A_{\Gamma_1} I_{\Omega_2 \to \Gamma_1} u_{\Omega_2}^{n-1}$$

$$A_{\Omega_2} u_{\Omega_2}^n = f_2 - A_{\Gamma_2} I_{\Omega_1 \to \Gamma_2} u_{\Omega_1}^{n}$$

This is the block Gauss–Seidel method for the system

$$\begin{pmatrix} A_{\Omega_1} & A_{\Gamma_1} I_{\Omega_2 \to \Gamma_1} \\ A_{\Gamma_2} I_{\Omega_1 \to \Gamma_2} & A_{\Omega_2} \end{pmatrix} \begin{pmatrix} u_{\Omega_1} \\ u_{\Omega_2} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Matrix formulation contd.

Rearranging, we get the matrix form of the alternating Schwarz method :

$$u^n_{\Omega_1} \leftarrow u^{n-1}_{\Omega_1} + A^{-1}_{\Omega_1}(f_1 - A_{\Omega_1}u^{n-1}_{\Omega_1} - A_{\Gamma_1}I_{\Omega_2\to\Gamma_1}u^{n-1}_{\Omega_2})$$

$$u^n_{\Omega_2} \leftarrow u^{n-1}_{\Omega_2} + A^{-1}_{\Omega_2}(f_2 - A_{\Gamma_2}I_{\Omega_1\to\Gamma_2}u^{n}_{\Omega_1} - A_{\Omega_2}u^{n-1}_{\Omega_2})$$

Notice that:

- if $g \neq 0$ just substitute f_i with $f_i - A_{\partial\Omega_i\setminus\Gamma_i}g_i$
- the "block Gauss-Seidel" linear system is not symmetric since in general $A_{\Gamma_1}I_{\Omega_2\to\Gamma_1} \neq (A_{\Gamma_2}I_{\Omega_1\to\Gamma_2})$

Matching grids

Assume that the grids match in the overlap region.

Let us partition the matrix in two ways

$$\begin{pmatrix} A_{\Omega_1} & A_{\Omega\backslash\Omega_1} \\ \ldots & \ldots \end{pmatrix} = \begin{pmatrix} A_{\Omega_1} & A_{\Gamma_1} & A_{\Omega\backslash\bar{\Omega}_1} \\ \ldots & \ldots & \ldots \end{pmatrix}$$

and

$$\begin{pmatrix} \ldots & \ldots \\ A_{\Omega_2} & A_{\Omega\backslash\Omega_2} \end{pmatrix} = \begin{pmatrix} \ldots & \ldots & \ldots \\ A_{\Omega_2} & A_{\Gamma_2} & A_{\Omega\backslash\bar{\Omega}_2} \end{pmatrix}$$

If there is no coupling between nodes on opposite sides of Γ_i:

$$A_{\Omega\backslash\bar{\Omega}_i} = 0 \quad \Longrightarrow \quad A_{\Gamma_i} u^n_{\Gamma_i} = A_{\Omega\backslash\bar{\Omega}_i} u^n_{\Omega\backslash\bar{\Omega}_i}$$

Multiplicative Schwarz

Thus, we can remove the explicit dependence on the artificial Γ_i

$$u^n_{\Omega_1} \leftarrow u^{n-1}_{\Omega_1} + A^{-1}_{\Omega_1}(f_1 - A_{\Omega_1} u^{n-1}_{\Omega_1} - A_{\Omega\setminus\bar{\Omega}_1} u^{n-1}_{\Omega\setminus\bar{\Omega}_1})$$

$$u^n_{\Omega_2} \leftarrow u^{n-1}_{\Omega_2} + A^{-1}_{\Omega_2}(f_2 - A_{\Omega\setminus\bar{\Omega}_2} u^n_{\Omega\setminus\bar{\Omega}_2} - A_{\Omega_2} u^{n-1}_{\Omega_2})$$

This is the multiplicative Schwarz method.

To always use the most recent values, we can update $u^{n-1}_{\Omega_2}$ with values from $u^n_{\Omega_1}$ to get $\tilde{u}^{n-1}_{\Omega_2}$

$$u^n_{\Omega_1} \leftarrow u^{n-1}_{\Omega_1} + A^{-1}_{\Omega_1}(f_1 - A_{\Omega_1} u^{n-1}_{\Omega_1} - A_{\Omega\setminus\bar{\Omega}_1} u^{n-1}_{\Omega\setminus\bar{\Omega}_1})$$

$$u^n_{\Omega_2} \leftarrow \tilde{u}^{n-1}_{\Omega_2} + A^{-1}_{\Omega_2}(f_2 - A_{\Omega\setminus\bar{\Omega}_2} u^n_{\Omega\setminus\bar{\Omega}_2} - A_{\Omega_2} \tilde{u}^{n-1}_{\Omega_2})$$

However, if $A^{-1}_{\Omega_2}$ is solved exactly, the methods are equal.

Additive Schwarz

Similarly, we can define the analogue of the block Jacobi algorithm

$$u^n_{\Omega_1} \leftarrow u^{n-1}_{\Omega_1} + A^{-1}_{\Omega_1}(f_1 - A_{\Omega_1}u^{n-1}_{\Omega_1} - A_{\Omega\backslash\bar{\Omega}_1}u^{n-1}_{\Omega\backslash\bar{\Omega}_1})$$

$$u^n_{\Omega_2} \leftarrow u^{n-1}_{\Omega_2} + A^{-1}_{\Omega_2}(f_2 - A_{\Omega\backslash\bar{\Omega}_2}u^{n-1}_{\Omega\backslash\bar{\Omega}_2} - A_{\Omega_2}u^{n-1}_{\Omega_2})$$

This is the additive Schwarz method.

Error equation

Consider the linear system

$$\mathbf{Au} = \mathbf{f},$$

where $\mathbf{A}$ is symmetric and positive definite.

Suppose we have an approximate solution $\mathbf{u}^{\text{old}}$. Define error $\mathbf{e}$:

$$\mathbf{e} \equiv \mathbf{u} - \mathbf{u}^{\text{old}}.$$

Define residual $\mathbf{r}$:

$$\mathbf{r} \equiv \mathbf{f} - \mathbf{Au}^{\text{old}}.$$

We thus have

$$\mathbf{Ae} = \mathbf{A}(\mathbf{u} - \mathbf{u}^{\text{old}}) = \mathbf{f} - \mathbf{Au}^{\text{old}} = \mathbf{r}.$$

Residuals

In the multiplicative Schwarz algorithm

$$f_1 - A_{\Omega_1} u_{\Omega_1}^{n-1} - A_{\Omega\backslash\bar{\Omega}_1} u_{\Omega\backslash\bar{\Omega}_1}^{n-1}$$

and

$$f_2 - A_{\Omega\backslash\bar{\Omega}_2} u_{\Omega\backslash\bar{\Omega}_2}^{n} - A_{\Omega_2} u_{\Omega_2}^{n-1}$$

are residuals in the interior of Ω_1 and Ω_2.
In other words,

$$u_{\Omega_1}^{n} = u_{\Omega_1}^{n-1} + A_{\Omega_1}^{-1} \cdot \text{<residual in } \Omega_1\text{>}$$

$$u_{\Omega_2}^{n} = u_{\Omega_2}^{n-1} + A_{\Omega_2}^{-1} \cdot \text{<residual in } \Omega_2\text{>}$$

Similarly for the additive Schwarz algorithm.

Multiplicative Schwarz: two-step algorithm

Other handling of matrices gives a two-step algorithm:

$$u^{n+\frac{1}{2}} \leftarrow u^n + \begin{bmatrix} A_{\Omega_1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} (f - Au^n)$$

$$u^{n+1} \leftarrow u^{n+\frac{1}{2}} + \begin{bmatrix} 0 & 0 \\ 0 & A_{\Omega_2}^{-1} \end{bmatrix} (f - Au^{n+\frac{1}{2}})$$

for the whole of u.

Schwarz methods: final form

Introduce the following matrix notation

$$u_{\Omega_1} = R_1 u = [I \quad 0] \begin{bmatrix} u_{\Omega_1} \\ u_{\Omega \backslash \Omega_1} \end{bmatrix}, \qquad u_{\Omega_2} = R_2 u = [0 \quad I] \begin{bmatrix} u_{\Omega \backslash \Omega_2} \\ u_{\Omega_2} \end{bmatrix}$$

So, we have

$$A_{\Omega_i} = R_i A R_i^T \quad \Longrightarrow \quad B_i = R_i^T (R_i A R_i^T)^{-1} R_i$$

We obtain the two Schwarz methods

$$\mathbf{u}^{n+1} \leftarrow \mathbf{u}^n + (\mathbf{B}_1 + \mathbf{B}_2 - \mathbf{B}_2 \mathbf{A} \mathbf{B}_1)(\mathbf{f} - \mathbf{A} u^n) \qquad \text{(multiplicative)}$$

$$\mathbf{u}^{n+1} \leftarrow \mathbf{u}^n + (\mathbf{B}_1 + \mathbf{B}_2)(\mathbf{f} - \mathbf{A} u^n) \qquad \text{(additive)}$$

We will see next how the two methods can be interpreted as preconditioners for the linear system $\mathbf{A}\mathbf{u} = \mathbf{f}$.

Preconditioning

If we can find $\mathbf{e} = \mathbf{A}^{-1}\mathbf{r}$, then

$$\mathbf{u} \;=\; \mathbf{u}^{\text{old}} + \mathbf{e}.$$

However this is as difficult as solving the original system $\mathbf{A}\mathbf{u} = \mathbf{f}$.
Nevertheless, if we can find a SPD $\mathbf{B}$ where

$$\mathbf{B} \simeq \mathbf{A}^{-1},$$

then

$$\mathbf{u}^{\text{new}} = \mathbf{u}^{\text{old}} + \mathbf{B}\mathbf{r}$$

will be a better approximation of the true solution $\mathbf{u}$.
We call $\mathbf{B}$ a preconditioner of $\mathbf{A}$. This is equivalent to solving the following preconditioned system of linear equations

$$\mathbf{B}\mathbf{A}\mathbf{u} = \mathbf{B}\mathbf{f}.$$

Requirements to $\mathbf{B}$:

- $\mathbf{y} = \mathbf{B}\mathbf{w}$ is inexpensive to evaluate.
- $\mathbf{B}$ is spectrally equivalent to $\mathbf{A}^{-1}$.

Interpretation of $\mathbf{B}_i$

Properties of B_i:

- B_i restricts the residual to one subdomain,
- solves the problem $\Longrightarrow$ correction,
- and extends the the correction back to the whole domain.

Observe that (for $e = u - u^n$, $r = f - Au^n$) :

$$R_i^T (R_i A R_i^T)^{-1} R_i r = B_i r = (B_i A) A^{-1} r = (B_i A) e$$

is a vector in the subspace spanned by the rows of R_i.

Corrections as projections

Assume A is symmetric positive definite. Observe that $P_i = B_i A$ satisfies

1. $P_i^2 = R_i^T(R_iAR_i^T)^{-1}R_iAR_i^T(R_iAR_i^T)^{-1}R_iA = P_i$
2. $AP_i = AR_i^T(R_iAR_i^T)^{-1}R_iA = P_i^T A$

Moreover, any e_i in the span of R_i^T can be written $e_i = R_i^T w$

$$\min_w \|e - R_i^T w\|_A^2 \quad \Longrightarrow \quad -2R_iA(e - R_i^T w) = 0 \quad \Longleftrightarrow \quad R_iAR_i^T w = R_iAe$$

and the local correction is given by

$$e_i = R_i^T w = R_i^T(R_iAR_i^T)^{-1}R_iAe = B_ir$$

Thus $P_i = B_iA$ is an orthogonal projection of the error into the subspace spanned by R_i^T.

Projections, from continuous formulation

The two equations can be written

$$\begin{aligned} A(u^{n+1/2} - u^n) &= -Au^n + f, && \text{in } \Omega_1 \\ A(u^{n+1} - u^{n+1/2}) &= -Au^{n+1/2} + f, && \text{in } \Omega_2 \end{aligned}$$

Introduce the error $e^n = u - u^n$, and write the equation in weak form

$$\begin{aligned} a(u^{n+1/2} - u^n, v) &= f(v) - a(u^n, v) = a(e^n, v) && \forall v \in H_0^1(\Omega_1) \\ a(u^{n+1} - u^{n+1/2}, v) &= f(v) - a(u^{n+1/2}, v) = a(e^{n+1/2}, v) && \forall v \in H_0^1(\Omega_2) \end{aligned}$$

We define a projection $P_i u$ as

$$a(P_i u, v) = a(u, v), \quad \forall v \in H_0^1(\Omega_i).$$

Thus, in each substep the corrections are projections onto the corresponding subspace $H_0^1(\Omega_i)$.

Many overlapping subdomains

For the multiplicative Schwarz algorithm each iteration involves the following substeps

$$\begin{aligned}
\mathbf{u}^{n+1/p} &= \mathbf{u}^{n} + \mathbf{B}_1(\mathbf{f} - \mathbf{A}\mathbf{u}^{n}), \\
\mathbf{u}^{n+2/p} &= \mathbf{u}^{n+1/p} + \mathbf{B}_2(\mathbf{f} - \mathbf{A}\mathbf{u}^{n+1/p}), \\
&\vdots \\
\mathbf{u}^{n+1} &= \mathbf{u}^{n+(p-1)/p} + \mathbf{B}_p(\mathbf{f} - \mathbf{A}\mathbf{u}^{n+(p-1)/p}),
\end{aligned}$$

where $\mathbf{B}_i = \mathbf{R}_i^T \mathbf{A}_i^{-1} \mathbf{R}_i$ and $\mathbf{A}_i = \mathbf{R}_i \mathbf{A} \mathbf{R}_i^T$. Thus, in each substep:

- Calculate residual: $\mathbf{f} - \mathbf{A}\mathbf{u}^{n+i-1/p}$;
- Restriction: $\mathbf{R}_i$;
- Subdomain solve: $\mathbf{A}_i^{-1}$;
- Interpolation (and correction): $\mathbf{R}_i^T$.

Many domains contd.

The algorithm can be written in compact matrix form

$$\mathbf{u}^{n+1} = \mathbf{u}^n + (I - (I - \mathbf{B}_p\mathbf{A}) \dots (\mathbf{I} - \mathbf{B}_1\mathbf{A}))\mathbf{A}^{-1}(\mathbf{f} - \mathbf{A}\mathbf{u}^n)$$

Properties of the multiplicative Schwarz method:

- Multiplicative Schwarz method is a generalization of the block Gauss-Seidel iteration (with overlapping blocks).
- The substeps in each iteration must be carried out in sequence.
- It can be symmetrized by traversing back through all the subdomains after the subdomain number p has been reached.
- It is not necessary to create $\mathbf{B}_i \in \mathbb{R}^{n\times n}$ explicitly in its matrix form. We are only interested in calculating $\mathbf{y} = \mathbf{B}_i\mathbf{w}$.
- The successive correction of the error:

$$\mathbf{e}^{n+1} = \Pi_{i=1}^{p}(\mathbf{I} - \mathbf{P}_{a_i}).\mathbf{e}^n$$

Many domains contd.

Some issues:

- The residual at substep i needs only be updated in $\hat{\Omega}_i$, using values from $\hat{\Omega}_i$ and its immediate neighbors.
- The restriction and interpolation matrices $\mathbf{R}_i$ and $\mathbf{R}_i^T$ are never formed in practice.
- In practice, inexact subdomain solves $\tilde{\mathbf{A}}_i^{-1}$ may replace the exact subdomain solves $\mathbf{A}_i^{-1}$.
- Multicoloring of subdomains can promote parallelism.
  No subdomain has neighbor of the same color $\Longrightarrow$ Subdomains with the same color can carry out subdomain solves concurrently.

Many domains contd.

For the additive Schwarz method, substeps in each iteration are

$$\begin{aligned}
\mathbf{u}^{n+1/p} &= \mathbf{u}^{n} + \mathbf{B}_1(\mathbf{f} - \mathbf{A}\mathbf{u}^{n}),\\
\mathbf{u}^{n+2/p} &= \mathbf{u}^{n+1/p} + \mathbf{B}_2(\mathbf{f} - \mathbf{A}\mathbf{u}^{n}),\\
&\vdots\\
\mathbf{u}^{n+1} &= \mathbf{u}^{n+(p-1)/p} + \mathbf{B}_p(\mathbf{f} - \mathbf{A}\mathbf{u}^{n}).
\end{aligned}$$

All the above substeps can be carried out simultaneously, in parallel

$$\mathbf{u}^{n+1} = \mathbf{u}^{n} + \sum_{i=1}^{p} \mathbf{B}_i(\mathbf{f} - \mathbf{A}\mathbf{u}^{n}).$$

Many domains contd.

Properties of additive Schwarz method:

- Analogous with the block Jacobi iteration.
- Symmetric.
- Subdomain solves can be carried out concurrently – readily parallelizable!
- Error projection :
  - we project simultaneously the error e^n on each subdomain
  - we accumulate the local corrections on the whole domain:

$$\mathbf{e}^{n+1} = (I - \sum_{i=1}^{p} \mathbf{P}_{a_i}).\mathbf{e}^n$$

Schwarz methods as preconditioners

Additive Schwarz method:

$$\mathbf{B}^{\text{as}} = \sum_{i=1}^{p} \mathbf{B}_i,$$

where $\mathbf{B}_i = \mathbf{R}^{-1}\mathbf{A}_i^{-1}\mathbf{R}$.

Multiplicative Schwarz method:

$\mathbf{B}^{\text{ms}}$ is defined through

$$\mathbf{B}^{\text{ms}}\mathbf{w} = \mathbf{y}^{2p}$$

where $\mathbf{y}^0 = 0$ and
for $i = 1, 2, \ldots, p$,

$$\mathbf{y}^i = \mathbf{y}^{i-1} + \mathbf{B}_i(\mathbf{w} - \mathbf{A}\mathbf{y}^{i-1}),$$

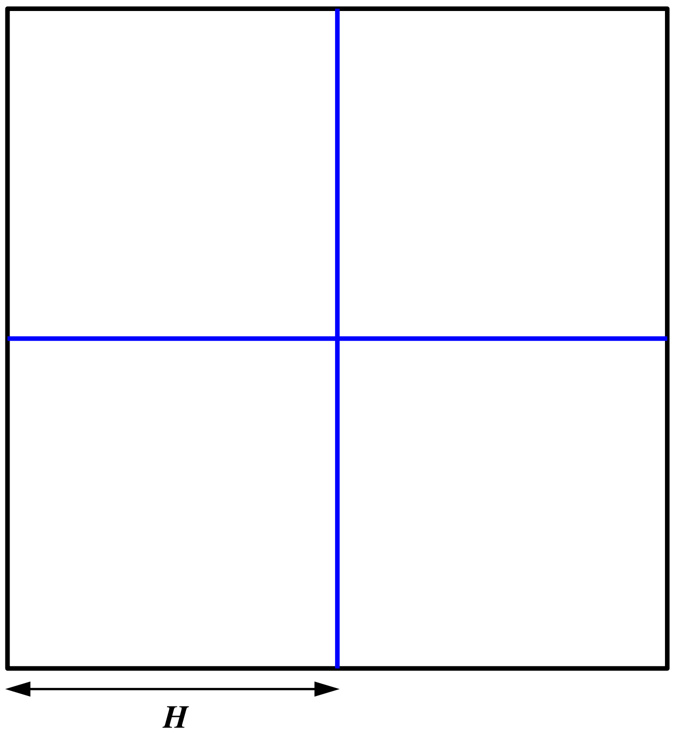
and for $i = p+1, p+2, \ldots, 2p$,

$$\mathbf{y}^i = \mathbf{y}^{i-1} + \mathbf{B}_{2p+1-i}(\mathbf{w} - \mathbf{A}\mathbf{y}^{i-1}).$$

Grid generation: step 1

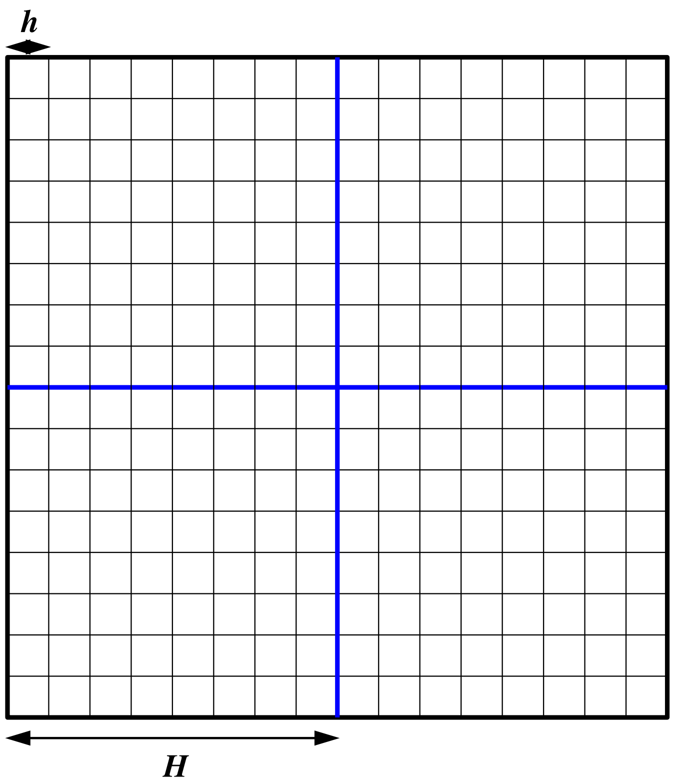
One way of grid generation for Ω:

1. Start with a coarse partition with characteristic grid size H. (Non-overlapping partition $\Omega_i, i = 1, 2, \ldots, p$).



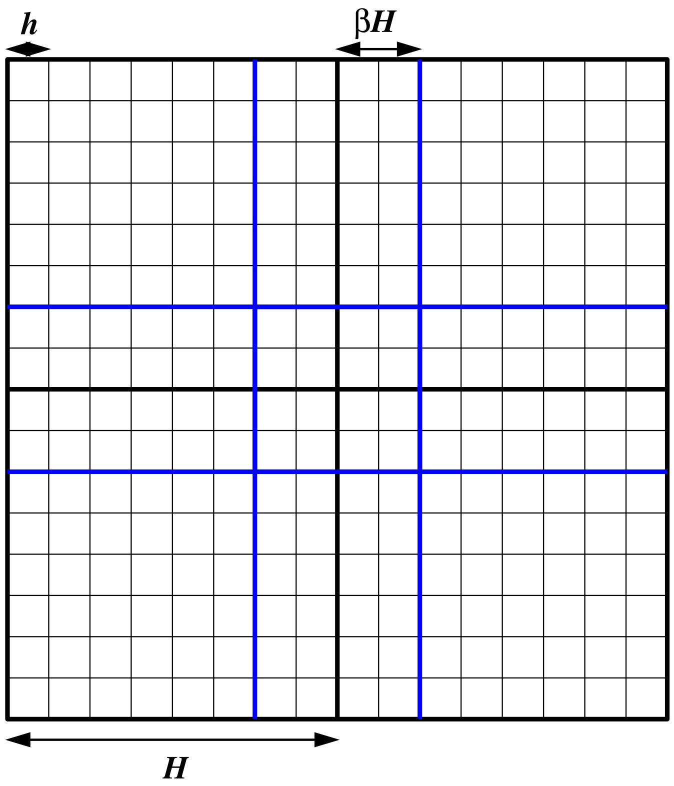
Grid generation: step 2

2. Refine the coarse grid to produce a global fine grid which has characteristic grid size $h < H$

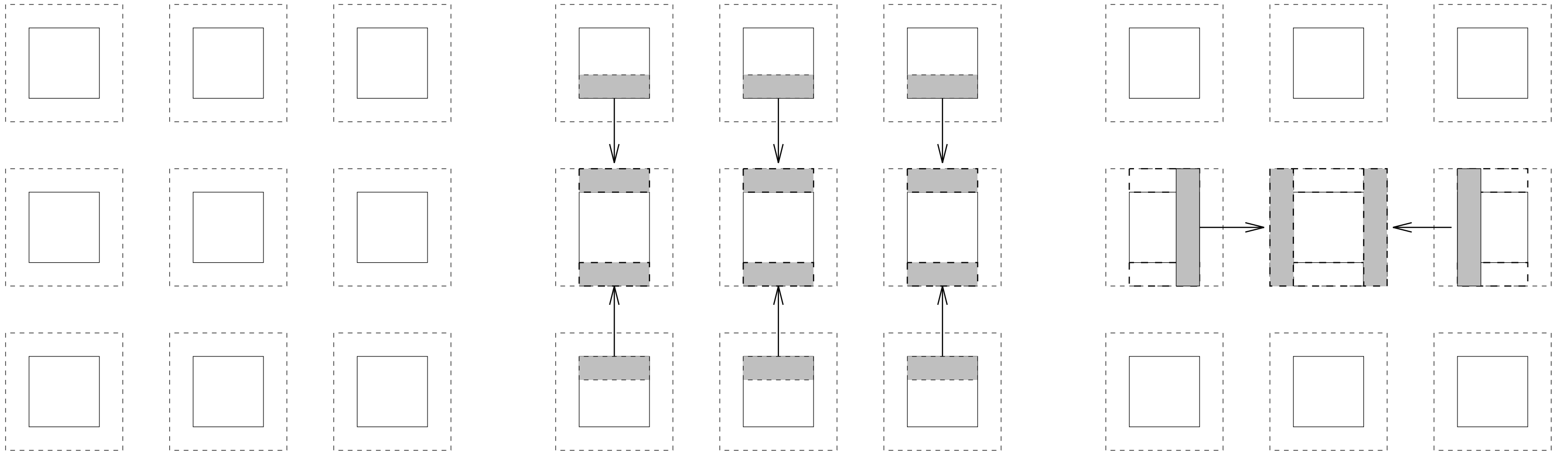


Grid generation: step 3

3. Enlarge each (open) element Ω_i of the coarse grid to produce overlapping subdomains $\hat{\Omega}_i, i = 1, 2, \ldots, p$, with artificial boundaries Γ_i. (We only consider the case of matching grids.)



Example: 9-subdomain decomposition



Extension of a local domain by it neighbor values.

Numerical experiment

We consider the following 2D Poisson equation in the unit square $\Omega = [0,1]^2$,

$$
\begin{aligned}
-\nabla^2 u &= 2\pi^2 \sin(\pi x)\sin(\pi y) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

The linear system of equations, arising from the FEM discretization is solved by the PCG method with $\mathbf{B}^{\text{as}}$ or $\mathbf{B}^{\text{ms}}$ as the preconditioner. (We have a fixed overlap $\beta = \frac{1}{6}$.)

	p=4		p=16		p=64	
Fine grid	$\mathbf{B}^{\text{as}}$	$\mathbf{B}^{\text{ms}}$	$\mathbf{B}^{\text{as}}$	$\mathbf{B}^{\text{ms}}$	$\mathbf{B}^{\text{as}}$	$\mathbf{B}^{\text{ms}}$
$h = 1/48$	37	20	110	58	377	193
$h = 1/96$	41	22	124	65	423	221
$h = 1/192$	45	24	138	72	478	248

The number of PCG iterations needed to get a reduction of the residual in L^2-norm by a factor of 10^4.

Numerical experiment contd.

Problem: Convergence deteriorates as p increases (H decreases).

Reason: The only global communication of information between subdomains are through overlapping regions. Too slow!

Assume a smooth error, $e = \bar{e} + e_{\text{small}}$. For any standard difference scheme, $\sum_i A_{ij} = 0$ away from the boundary. The local correction is

$$B_i(f - Au^n) = B_i A e^n = B_i A(\bar{e}^n + e^n_{\text{small}}) = B_i A e^n_{\text{small}}$$

Thus, local correction is only effective near the boundary.
$\longrightarrow$ We need another method for correcting smooth errors!

Two-level methods: motivation

Consider the model problem

$$Au = f, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega.$$

The explicit solution can be written

$$u(x) = \int G(x, y) f(y)\, dy,$$

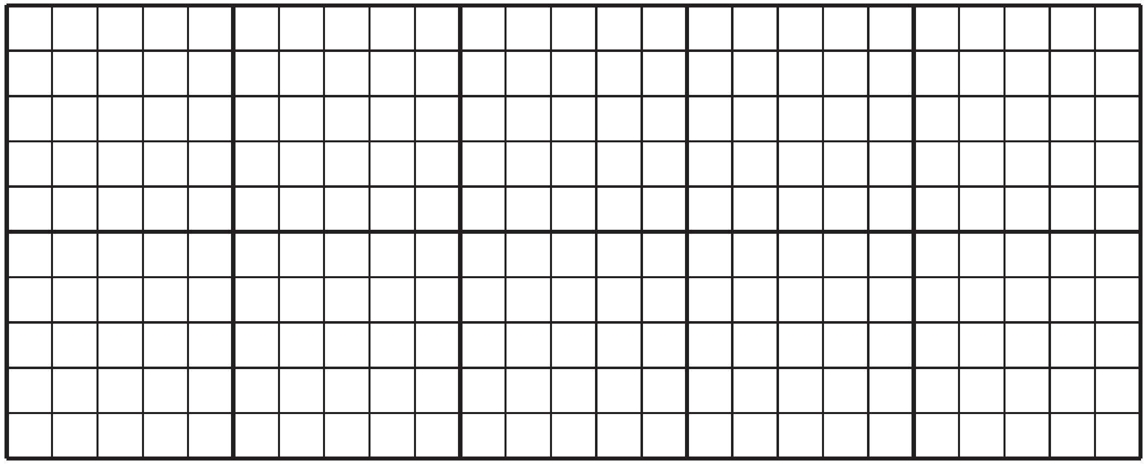
where $G(x, y)$ is an *Green's* function.

The Green's function $G(x, y)$ for fixed x has a peak at $y = x$ and rapidly decays outside. Thus $u(x)$ depends strongly on $f(y)$ for y close to x and only weakly for values far away.

$\longrightarrow$ We can resolve local influence accurately and long-range influence less accurately!

Two-level mesh

Let us consider a two-level, nested mesh



Now we introduce matrix representation of two operators

- R – restriction from fine to coarse grid
- R^T – interpolation from coarse to fine grid

Coarse grid correction

Now we can write a coarse grid correction:

$$u_F \leftarrow u_F + R^T A_C^{-1} R(f - A_F u_F)$$

What happens?

- Interpolate residual to coarse grid $r_C \leftarrow R(f - A_F u_F)$
- Solve coarse grid problem $e_C \leftarrow A_C^{-1} r_C$
- Interpolate back to fine grid $e_F \leftarrow R^T e_C$
- Correct fine grid solution $u_F \leftarrow u_F + e_F$

What have we achieved ?

Obtained a *coarse grid preconditioner* $B_C = R^T A_C^{-1} R$

Two-level error reduction

Observation:

- $B_C = R^T A_C^{-1} R$ has a large null space (because rank B_C is equal rank A_C)
- High frequency parts of $f - A_F u_F$ are in the null space of B_C

Assuming that we have a fine grid preconditioner B_F we can introduce a two-step (or two-level) algorithm

$$u_F^{n+1/2} \leftarrow u_F^n + B_C(f - A_F u_F^n)$$

$$u_F^{n+1} \leftarrow u_F^{n+1/2} + B_F(f - A_F u_F^{n+1/2})$$

Conclusion:

- B_C corrects *low frequency* errors. (The error corresponding to $\bar{e}$..)
- B_F corrects *high frequency* errors. (The error corresponding to e_{small}..)

Connections to multigrid

- Domain decomposition can be viewed as a generalisation of multigrid
- In multigrid, B_F is usually called smoothers
- In multigrid, B_F is usually a simple iterative scheme, typically Jacobi or Gauss–Seidel
- One can define Schwarz smoothers, e.g., the additive Schwarz smoother:

$$u \leftarrow \sum_i R_i^T A_{\Omega_i}^{-1} R_i r$$

There are subcases:

- If R_i returns a point: classical Jacobi or Gauss–Seidel
- No overlap between subdomains: block Jacobi or Gauss–Seidel

Two-level Schwarz methods

Let B_F be either additive or multiplicative Schwarz. Write $B_0 = B_C$.

1. Two-level additive Schwarz:

$$u \leftarrow \left(R^T A_C^{-1} R + \sum_{i=1}^{p} R_i^T A_{\Omega_i}^{-1} R_i \right) r = \sum_{i=0}^{p} B_i r$$

2. Two-level multiplicative Schwarz:

$$\begin{aligned} u &\leftarrow B_0 r \\ u &\leftarrow v + B_1(r - A_F v) \\ &\vdots \\ u &\leftarrow v + B_p(r - A_F v) \end{aligned}$$

Two-level Schwarz contd.

3. Two-level hybrid Schwarz I:

$$
\begin{aligned}
u &\leftarrow B_1 r \\
&\vdots \\
u &\leftarrow v + B_p(r - A_F v) \\
u &\leftarrow v + B_0 r
\end{aligned}
$$

4. Two-level hybrid Schwarz II:

$$
\begin{aligned}
u &\leftarrow \sum_{i=1}^{p} B_i r \\
u &\leftarrow v + B_0(r - A_F v)
\end{aligned}
$$

Numerical experiment revisited

The same problem as in experiment one. However, we introduce coarse grid correction in the Schwarz preconditioners.

Fine grid	p=4		p=16		p=64	
	$\mathbf{B}_H^{\text{as}}$	$\mathbf{B}_H^{\text{ms}}$	$\mathbf{B}_H^{\text{as}}$	$\mathbf{B}_H^{\text{ms}}$	$\mathbf{B}_H^{\text{as}}$	$\mathbf{B}_H^{\text{ms}}$
$h = 1/48$	9	6	12	5	7	4
$h = 1/96$	10	6	13	5	9	5
$h = 1/192$	11	7	15	6	10	5

Number of PCG iterations needed to get a reduction of the residual in L^2-norm by a factor of 10^4.

Convergence behaviour of Schwarz methods

- The number of colors affects the convergence rate of the multiplicative Schwarz method.
- The number of iterations for the additive Schwarz method is twice the number of iterations needed by multiplicative Schwarz method.
- With an overlap size proportional to H, the number of iterations is bounded independently of the entire domain mesh h and H/h.
- The convergence is better with a large overlap.

Convergence behaviour contd.

- Used as preconditioners of the Krylov solvers, the Schwarz methods allow to improve the convergence properties of these methods and to limit the number of communications necessary to parallel implementation,
- and, in particular, the additive Schwarz method presents the advantage of a high rate of parallelism.
- Each application of the preconditioner consists in one iteration of the Schwarz method with a zero as initial guess.
  (*for a Krylov subspace method*)