Finite element modelling of structural mechanics problems

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Outline

- Devices to avoid transverse shear "locking" of flexural (structural) elements
  - Mixed formulations
  - Reduced integration
  - ANS – Assumed Natural Coordinate Strains (based on a three-field formulation)
- Devices to overcome shear "locking" in continuum elements
  - Selective Reduced Integration (SRI)
  - Incompatible elements
  - EAS – Enhanced Assumed Strains (based on a three-field Hu-Washizu formulation)

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“Locking” of flexural elements

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Idealization

- In general the domain is considered to be a continuum, a rigid multibody system or a set of discrete elements.
- In continuum problems FE approximations are based on approximation of the displacement, stress and strain fields at each material point in the domain.
- Even though the continuum approach is general, for structural mechanics problems, there are many instances where it is difficult or impossible to obtain viable solutions economically.
- If one or two dimensions of the domain are small compared to the others, the FE approximations for structural mechanics problems may often be better understood from a physical (structural mechanics), rather than mathematical, standpoint.

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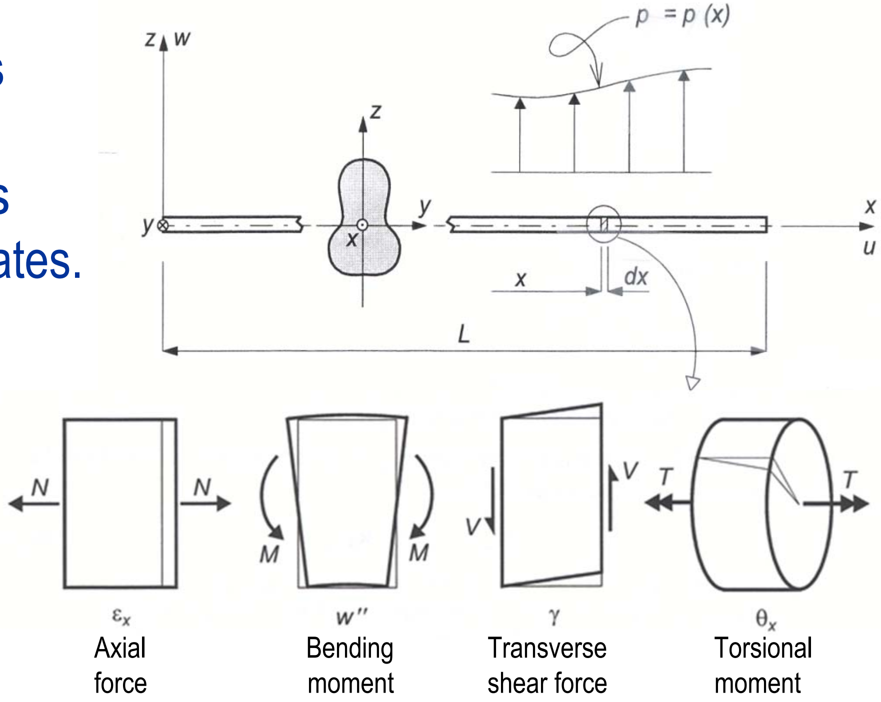
1D elements

- If the longitudinal or axial dimension is much larger than the other two dimensions (known as transverse dimensions), the element may be parameterized as a one-dimensional (1D) or line element.

- Although the intrinsic dimensionality is one, line elements may be used in one, two or three space dimensions upon transformation to global coordinates.

- We distinguish between two main categories of 1D elements:
  - Bar elements — resist axial force along its longitudinal axis
  - Beam elements — resist axial force, bending moments, transverse shear forces, and torsion



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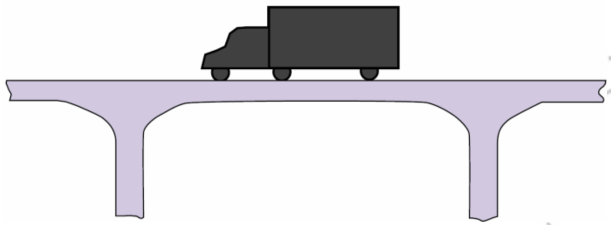
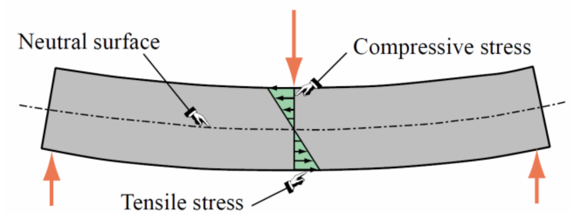
1D elements cont.

- Bar elements are used to model trusses, cables, chains and ropes.
- In principle a bar element is a 1D continuum element.
- In contrast to the bar that can only resist axial stretching or compression, a beam resists transverse loads mainly through bending action.



- Bending produces compressive longitudinal stresses on one side and tensile stresses on the opposite side.



- If attempting to model a beam with a standard 3D FE model there are two aspects which may cause difficulty:
  - One is purely numerical and associated with large round-off errors when attempting to solve the simultaneous equations.
  - The other is a form of "locking" in interactions between bending, shear and axial behavior when low-order elements are used.

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1D elements cont.

- 1D mathematical models of structural beams are constructed on the basis of beam theories.
- Because beams are actually 3D bodies, all models necessarily involve some form of approximation to the underlying physics.
- This is achieved by "filtering out" physical details that are not relevant to the analysis process.
- For example, a continuum material model filters out the aggregate, crystal, molecular and atomic levels of matter.
- Engineers are typically interested in a few integrated quantities, such as maximum deflection and maximum bending moments.
- Consequently, picking a mathematical model is equivalent to choosing an information filter.

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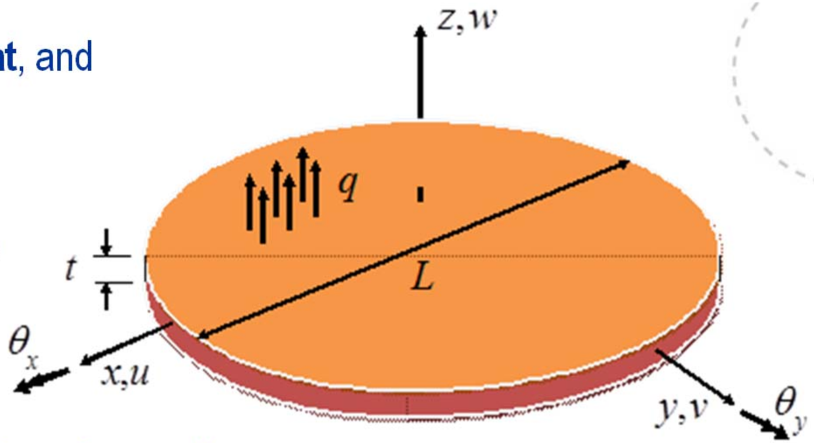
2D elements

- When one dimension is small compared to the other two dimensions the element may be parameterized as a two-dimensional (2D) or surface element.

- We distinguish between two main categories of 2D elements:
  - Plate element — if the surface is initially flat, and
  - Shell element — if the surface is curved

- If the plate/shell element is subjected to transverse loading the analyst must choose which plate theory to apply:
  - Thick: $t/L > 1/3$ — 3D continuum theory
  - Moderately thick: $1/3 > t/L > 1/10$ — Mindlin plate theory
  - Thin: $1/10 > t/L$ — Kirchhoff plate theory



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Beam models

- FE modeling of beams are usually based on:
  - Euler-Bernoulli (EB) beam theory – also called classical or engineering beam theory
    $\Rightarrow$ C^1 - (Hermitian) elements
  - Timoshenko beam theory – also called Mindlin-Reissner beam theory
    $\Rightarrow$ C^0 - elements
- Mathematically, the main difference is that the EB beam requires increased order of continuity compared to the Timoshenko beam.
- The application of the EB theory is usually restricted to situations where dimensions along the axis of the beam are at least ten times those of the transverse (cross-section) dimensions: $\boldsymbol{t / L} < 1/10$
- In contrast to the EB theory, the Timoshenko theory includes transverse shear deformations and is applicable when the length to cross-section dimensions are above five (when smaller the continuum theory becomes viable) : $\boldsymbol{t / L} < 1/5$

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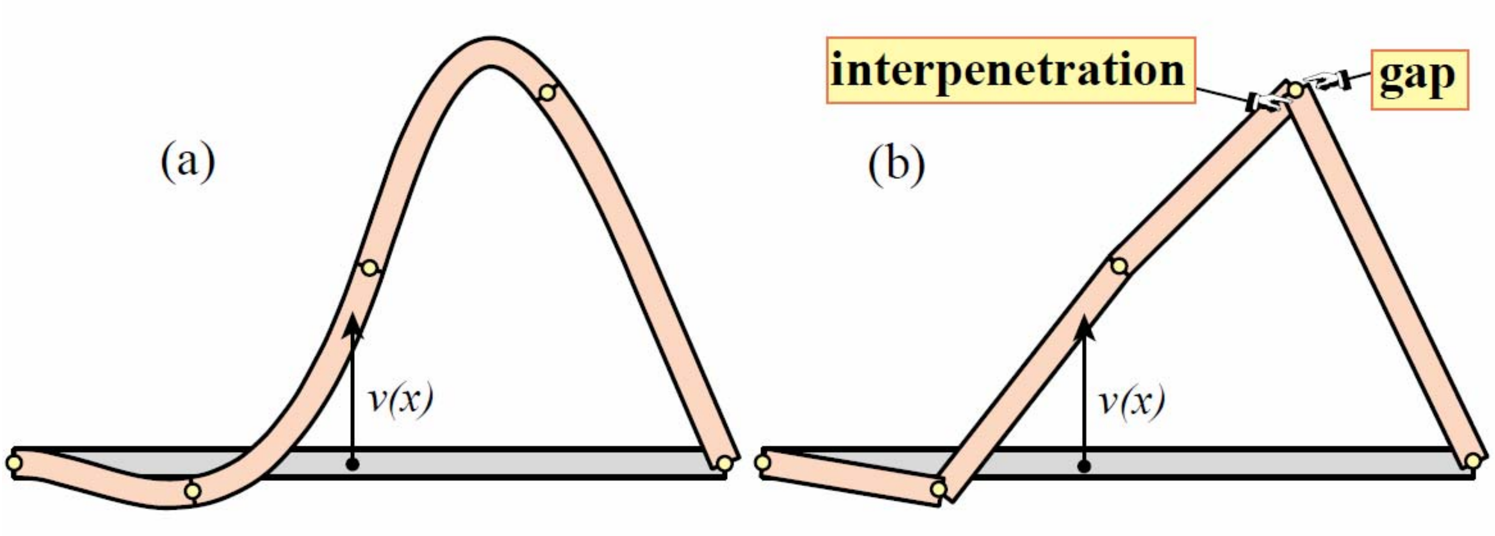
Interpretation of C^1-continuity

(deflections grossly exaggerated for visibility)



Piecewise cubic interpolation provides required C^1 - continuity

Piecewise linear interpolation gives unacceptable C^1 - continuity

Basics – plane EB beam

Displacements :

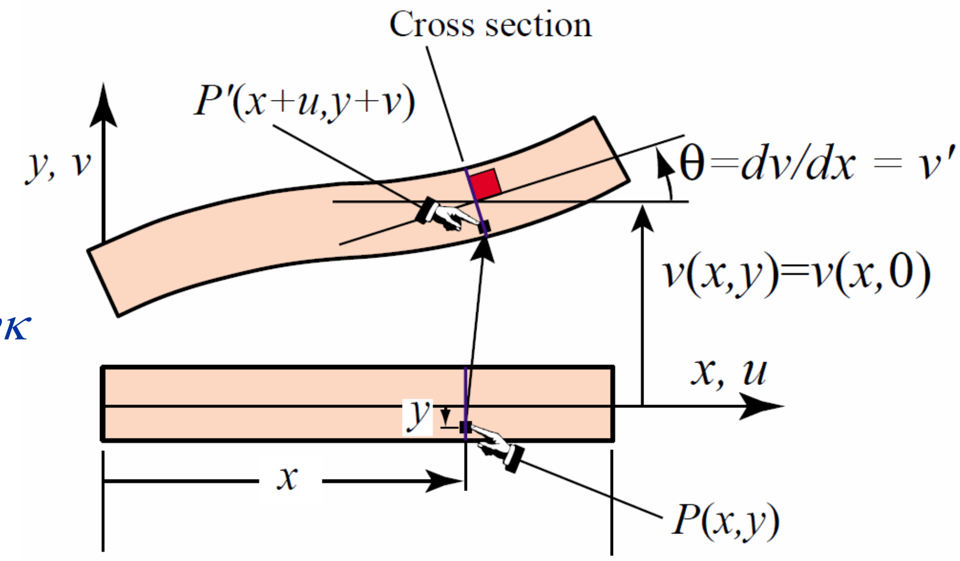
Axial: $u(x,y) = -y\dfrac{dv}{dx} = -yv' = -y\theta$

Transverse: $v(x,y) = v(x)$

Axial (bending) strain and stress :

Strain: $e = \dfrac{du}{dx} = -y\dfrac{d^2v}{dx^2} = -yv'' = -y\kappa$

Stress: $\sigma = Ee = -Eyv'' = -Ey\kappa$



Bending moment :

$$M = -\int_A y\sigma\, dA = E\frac{d^2v}{dx^2}\overbrace{\int_A y^2 dA}^{I} = EI\kappa$$

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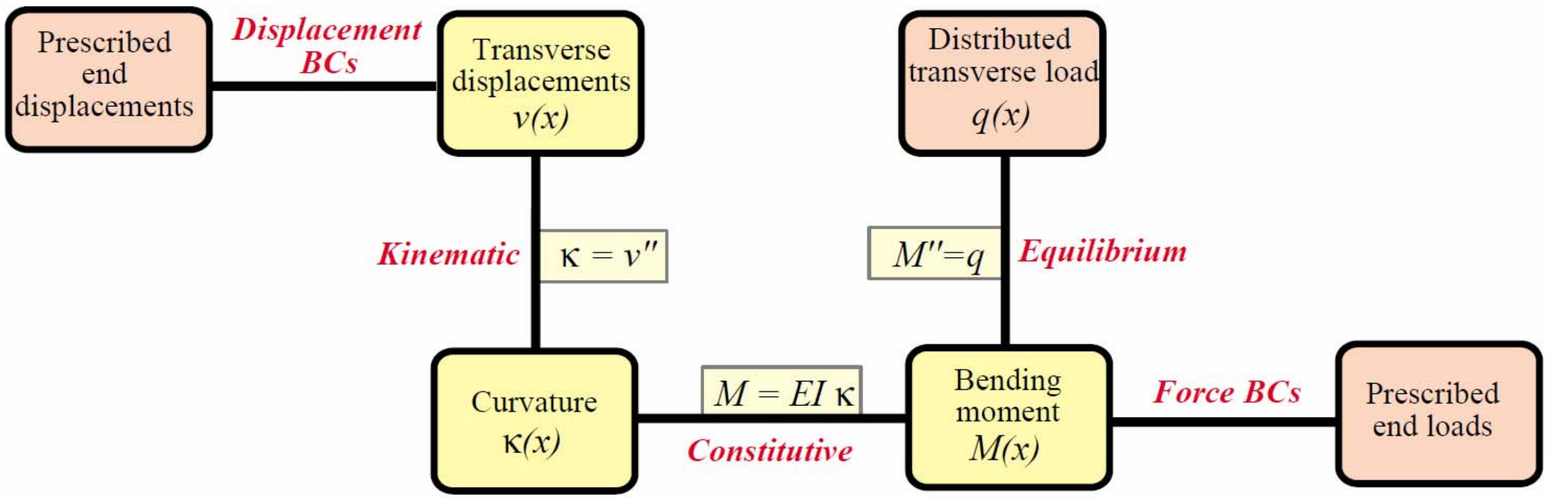
Weak form – plane EB beam



Internal energy due to bending:

$$U = \frac{1}{2}\int_V \sigma e \, dV = \frac{1}{2}\int_L M\kappa \, dx = \frac{EI}{2}\int_L \kappa^2 \, dx = \frac{EI}{2}\int_L \left(v''\right)^2 \, dx$$

External energy due to transverse load *q*:

$$W = \int_L qv \, dx$$

$\Rightarrow$ **Total potential energy:**

$$\Pi = U - W$$

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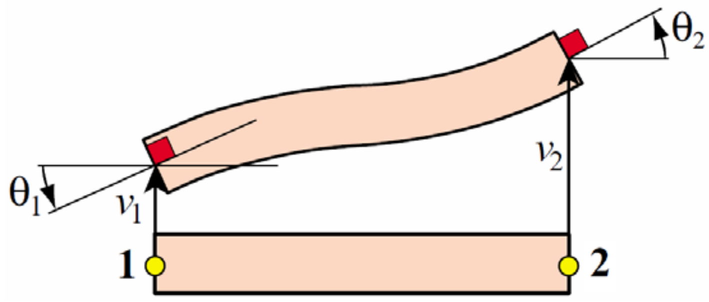
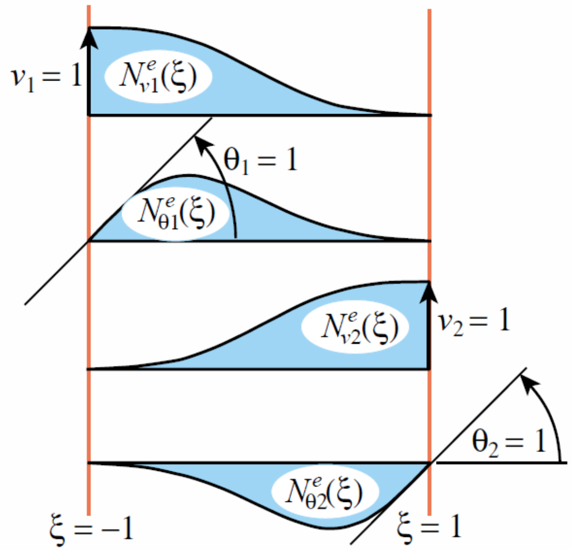
FE approx. – plane EB beam

To meet the C^1-continuity requirement, Hermitian cubic shape functions are used to approximate the transverse displacements:

$$v^e(\xi) = \begin{bmatrix} N^e_{v1} & N^e_{\theta 1} & N^e_{v2} & N^e_{\theta 1} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \mathbf{N}^e \tilde{\mathbf{u}}^e$$



where
$$\begin{cases} N^e_{v1} = \dfrac{1}{4}(1-\xi)^2(2+\xi) \\ N^e_{\theta 1} = \dfrac{\ell}{8}(1-\xi)^2(1+\xi) \\ N^e_{v2} = \dfrac{1}{4}(1+\xi)^2(2-\xi) \\ N^e_{\theta 2} = \dfrac{\ell}{8}(1+\xi)^2(1-\xi) \end{cases}$$



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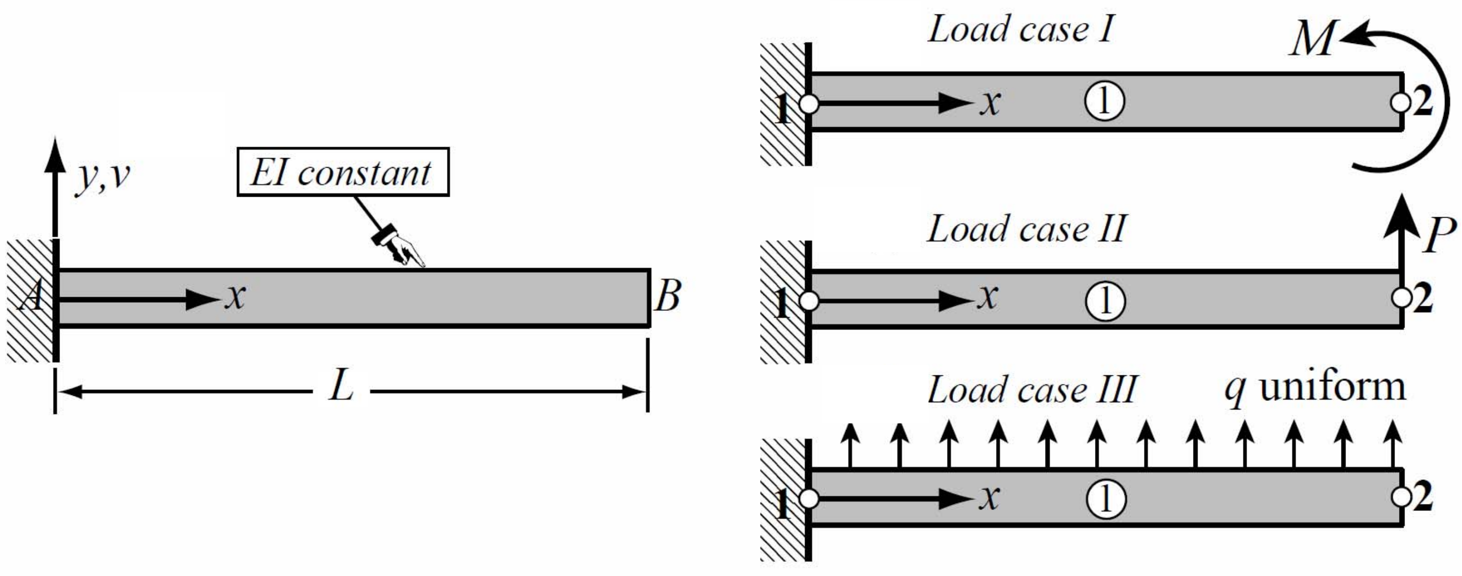
Example – plane EB beam

- Cantilever beam problem discretized with one single EB beam element:



- Since the STRONG FORM solution for the transverse displacement v for load cases I and II are quadratic and cubic polynomials in x, respectively, they are both included in the span of the element shape functions $\mathbf{N}^e$

$\Rightarrow$ The FE solution coincide with the analytical solution for load cases I and II, respectively.

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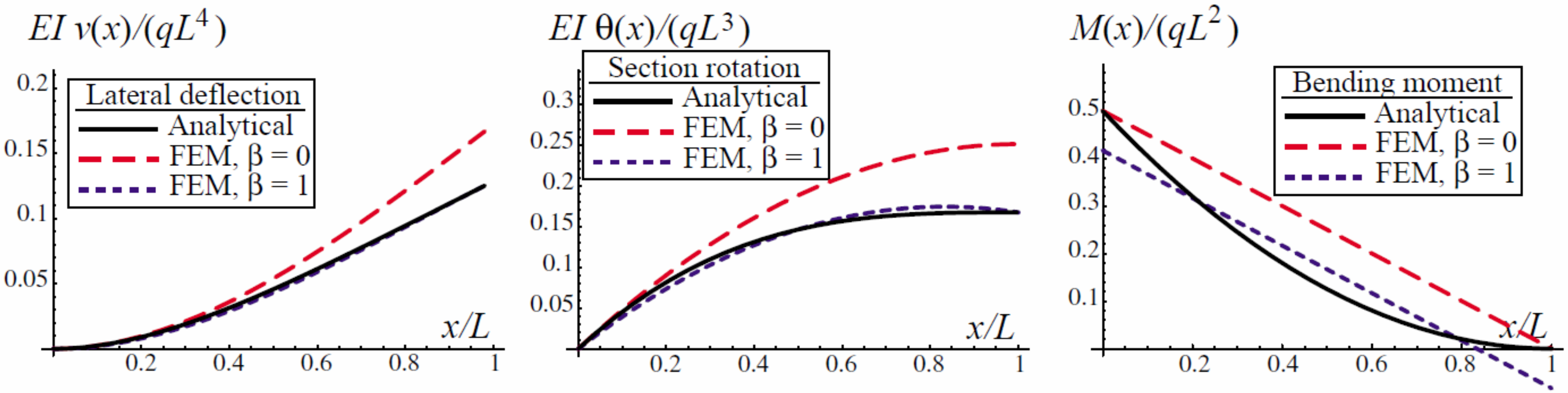
Example – plane EB beam



- The results for load case III are more interesting since now the exact deflection is a quartic polynomial in *x*, which lies beyond the span of $\mathbf{N}^e$
- The FE solution for the normalized transverse displacement, section rotation and bending moment are compared to the SF analytical solution.
- In the above figures, $\beta = 1$ corresponds to FE solution obtained with consistent load vectors.
- While the transverse displacement and section rotation compares very well, the bending moment gives a linear fit to the parabolic function corresponding to the SF solution to *M*.

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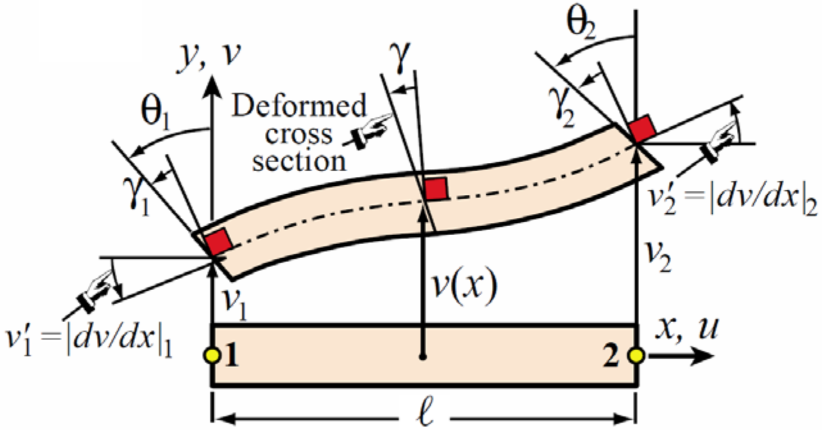
Basics – Timoshenko beam



Displacements :

Axial: $u(x,y) = -y\theta$

Transverse: $v(x,y) = v(x)$

Axial (bending) strain and stress :

Strain: $e = \dfrac{du}{dx} = -y\dfrac{d\theta}{dx} = -y\theta'$

Stress: $\sigma = Ee = -Ey\theta'$

Transverse (shear) strain and stress :

Strain: $\gamma = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = v' - \theta$

Stress: $\tau = G\gamma = G\left(v' - \theta\right)$

Bending moment :

$$M = -\int_A y\sigma \, dA = E\theta' \overbrace{\int_A y^2 dA}^{I} = EI\theta'$$

Transverse shear force :

$$V = \int_A \tau \, dA = GA_s\gamma$$

NOTE: To account for the distortion of the cross-section due to transverse shear the area A is replaced by a modified area A_s

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Weak form – Timoshenko beam

Internal energy due to bending:

$$U_b = \frac{1}{2}\int_V \sigma e \, dV = \frac{EI}{2}\int_L (\theta')^2 \, dx = \frac{1}{2}\left[\tilde{\mathbf{u}}^e\right]^T \left[\mathbf{k}_b\right]\left[\tilde{\mathbf{u}}^e\right]$$

Internal energy due to transverse shear:

$$U_s = \frac{1}{2}\int_V \tau\gamma \, dV = \frac{GA_s}{2}\int_L (v' - \theta)^2 \, dx = \frac{1}{2}\left[\tilde{\mathbf{u}}^e\right]^T \left[\mathbf{k}_s\right]^T \left[\tilde{\mathbf{u}}^e\right]$$

External energy due to transverse load *q*:

$$W = \int_L qv \, dx$$

$\Rightarrow$ **Total potential energy:**

$$\Pi = U_b + U_s - W$$

- Since the highest derivatives of transverse displacement *v* and section rotation θ in the weak form are only first order, both fields may be interpolated by C^0 functions.

- Use of an equal-order interpolation the transverse displacement and section rotation is expressed as:

$$\begin{bmatrix} v \\ \theta \end{bmatrix} = \sum_a N_a(\xi) \begin{bmatrix} v_a \\ \theta_a \end{bmatrix} = \mathbf{N}^e \tilde{\mathbf{u}}^e$$

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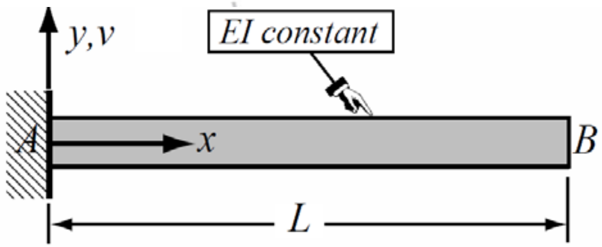
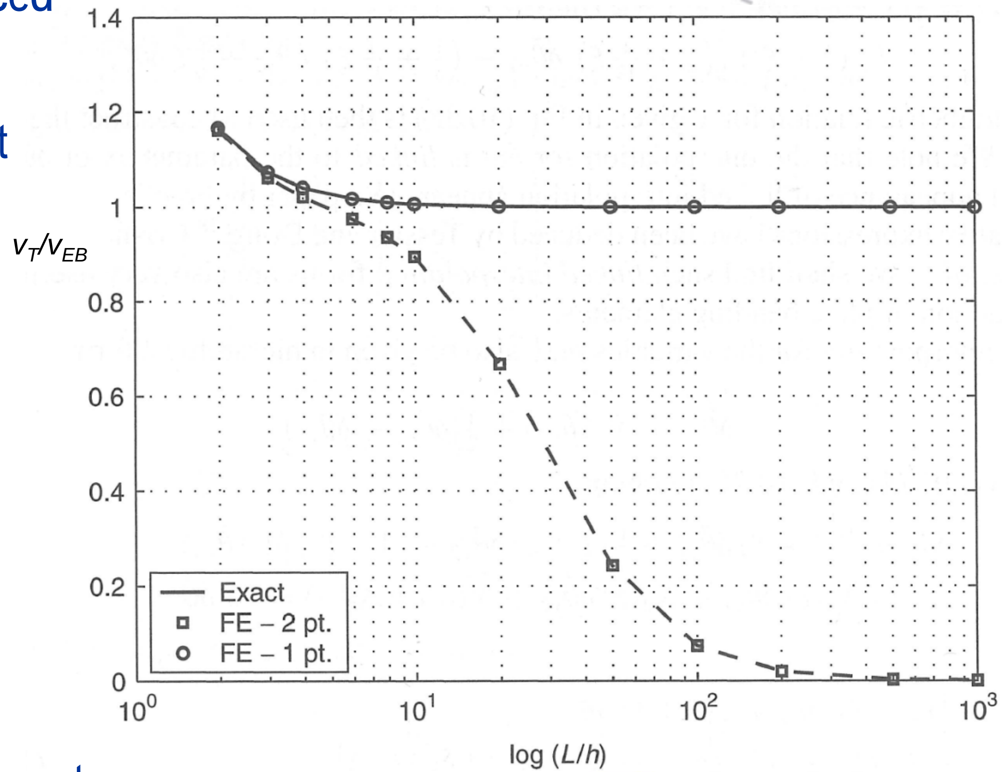
Example – Timoshenko beam

- Uniformly loaded cantilever beam discretized with 20 linear elements with equal-order interpolation.
- FE results are obtained with **full integration** (2-point Gauss quadrature) and **reduced integration** (1-point quadrature).
- v_T / v_{EB} denote ratio of tip displacement for Timoshenko beam theory to that of EB beam theory.
- The use of full integration leads to a solution which **locks** as the beam becomes slender, whereas reduced integration shows no locking for the range plotted.
- This example demonstrate the main deficiency of low-order/equal-order C^0 interpolation of Timoshenko beam elements.





Devices to alleviate “locking”

- As demonstrated in the example, use of Timoshenko theory can lead to **locking** effects when the theory is applied to cases where the EB theory also could be used.
- However, it is desirable to have a single formulation which remains valid throughout the range of length to cross-section considerations and for this the Timoshenko theory should be used.
- How can FE approximations which are free from transverse shear locking be developed for the Timoshenko beam as well as plate and shell elements based on Mindlin plate theory.
- As demonstrated in the example, one device is to apply **reduced integration**, a more general approach is to apply a **mixed formulation**.

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Mixed form – Timoshenko beam

- In the sequel a three-field Hu-Washizu variational form will be used to construct mixed FE approximations for the Timoshenko beam theory.
- The three-field weak form involves two displacement components, *v* and θ, two forces, *V* and *M*, and two strains, γ and *κ*, as given by:

$$\delta\Pi\left(v,\theta,V,M,\gamma,\kappa\right)=\int_L\left[\delta\theta' M+\left(\delta v'-\delta\theta\right)V\right]dx+\int_L\left[\delta\kappa\left(\bar{M}-M\right)+\delta\gamma\left(\bar{V}-V\right)\right]dx$$
$$+\int_L\left[\delta M\left(\theta'-\kappa\right)+\delta V\left(v'-\theta-\gamma\right)\right]dx-\delta W$$

- To obtain exact interelement nodal displacements for the mixed formulation we let:

$$M\left(\xi\right)=N_1\left(\xi\right)M_1+N_2\left(\xi\right)M_2+\hat{M}\left(\xi\right)\quad \text{and}\quad V\left(\xi\right)=\frac{M_2-M_1}{L}+\hat{V}\left(\xi\right)$$

where $\hat{M}$ and $\hat{V}$ are particular solutions which satsify the governing equilibrium

equations : $\boxed{\dfrac{d\hat{V}}{dx}=\hat{V}'=-q \quad \text{and} \quad \dfrac{d\hat{M}}{dx}=\hat{M}'=\hat{V}}$

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Mixed form – Timoshenko beam

- In addition we enforce the strain approximations to satisfy the constitutive equations, that for the linear elastic problem reads:

$$\kappa = \theta' = \frac{M}{EI} \quad \text{and} \quad \gamma = \frac{V}{GA_s}$$

- Integrating by parts we obtain the corresponding reduced mixed form:

$$\delta\Pi(v,\theta,M) = -\int_L \left[\delta M \frac{M}{EI} + \delta V \frac{V}{GA_s} \right] dx - \delta W$$

$$= \begin{bmatrix} \tilde{\mathbf{u}}^T & \tilde{\mathbf{M}}^T \end{bmatrix} \left(\begin{bmatrix} \mathbf{0} & \mathbf{G}^T \\ \mathbf{G} & -(\mathbf{V}+\mathbf{W}) \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{u}} \\ \tilde{\mathbf{M}} \end{bmatrix} - \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_M \end{bmatrix} \right)$$

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Mixed form – Timoshenko beam

- Eliminating $\tilde{\mathbf{M}}$ at the element level we obtain the element "stiffness" matrix and the consistent load vector (which may be assembled as usual):

$$\mathbf{k}^e = \mathbf{G}^T \left(\mathbf{V} + \mathbf{W}\right)^{-1} \mathbf{G}$$

$$\mathbf{f}^e = \mathbf{f}_u + \mathbf{G}^T \left(\mathbf{V} + \mathbf{W}\right)^{-1} \mathbf{f}_M$$

where

$$\mathbf{V} = \frac{L}{6EI}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ \mathbf{W} = \frac{1}{2GA_s L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ \mathbf{f}_u = \frac{qL}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{f}_M = \frac{qL^2}{24EI}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- REMARKS:
  - If the material behavior is non-linear the mixed formulation becomes more complex, and may for some cases not be possible to obtain.
  - When inertia effects are included the interelement solution is no longer exact.

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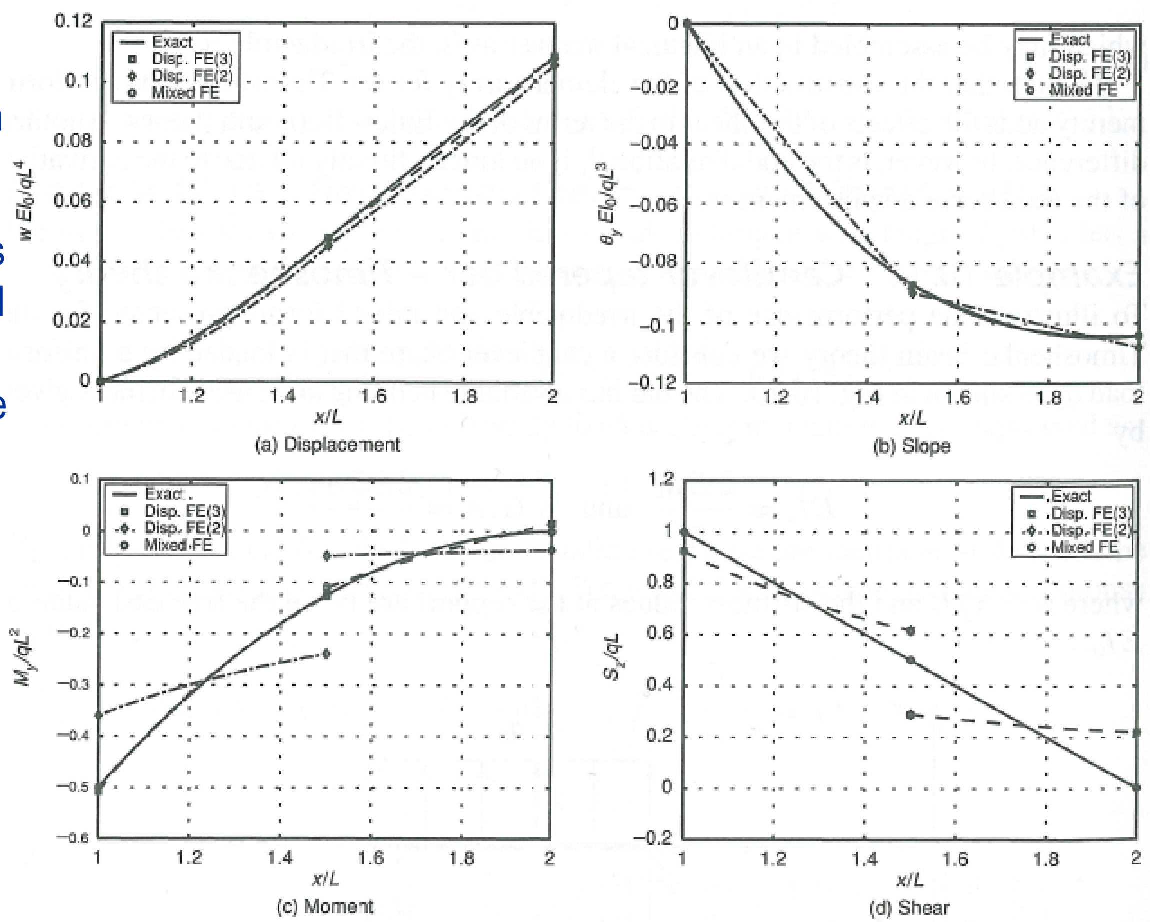
Example – Timoshenko beam

- Uniformly loaded tapered cantilever beam discretized with 2 elements of equal length.
- The bending and shear stiffness varies linearly between the fixed and the free end such that the values at the fixed end are twice the values at the free end.
- The results for the mixed form are compared to the one-field displacement solution:

  $[\text{Disp.FE}\,(3)]$: Cubic v and quadratic θ.

  $[\text{Disp.FE}\,(2)]$: Quadratic v and linear θ.



(a) Displacement (b) Slope (c) Moment (d) Shear

- The results for the mixed solution are exact whereas those for the other solutions have error in all quantities, although those for the displacement and slope are quite small.

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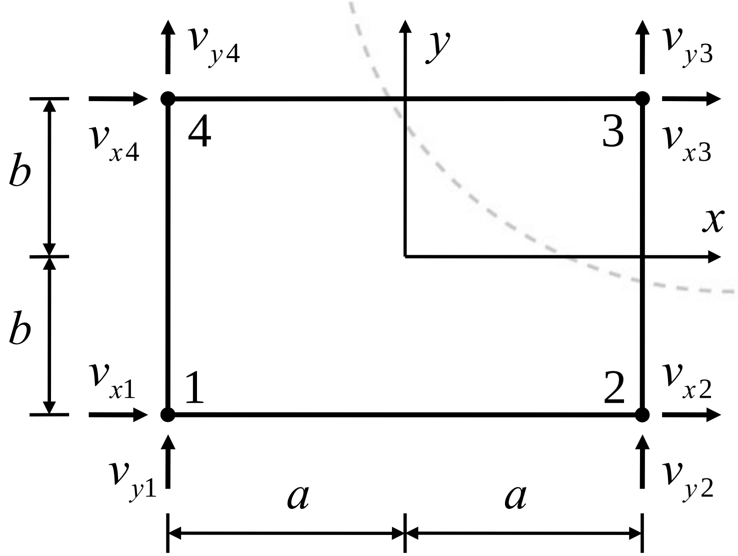
Devices to overcome shear “locking”

Q4 bilinear quadrilateral

- The displacement field u and the associated strain field $\boldsymbol{\varepsilon}$ for the four-node bilinear rectangle having eight dofs may either be established directly in terms of bilinear shape functions in non-dimensional coordinates (ξ,η),

$$N_i(\xi,\eta)=\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$$



- or in terms of generalized coordinates where the displacement field is expressed in terms of physical coordinates *x* and *y*:

$$u(x,y)=1\cdot q_{x1}+x\cdot q_{x2}+y\cdot q_{x3}+xy\cdot q_{x4}=\mathbf{N}_{\text{qo}}\mathbf{q}_x$$

$$v(x,y)=1\cdot q_{y1}+x\cdot q_{y2}+y\cdot q_{y3}+xy\cdot q_{y4}=\mathbf{N}_{\text{qo}}\mathbf{q}_y$$

where $\begin{cases}\mathbf{N}_{\text{qo}}=\begin{bmatrix}1 & x & y & xy\end{bmatrix}\\ \mathbf{q}_x^T=\begin{bmatrix}q_{x1} & q_{x2} & q_{x3} & q_{x4}\end{bmatrix}\\ \mathbf{q}_y^T=\begin{bmatrix}q_{y1} & q_{y2} & q_{y3} & q_{y4}\end{bmatrix}\end{cases}$

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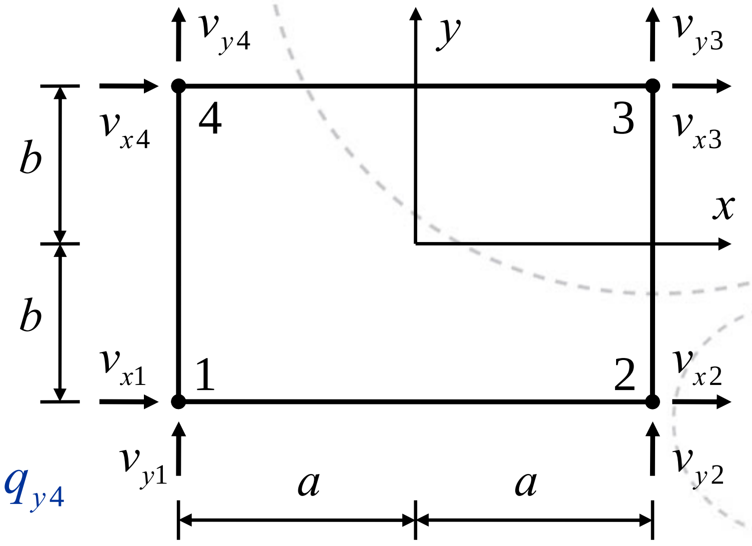
Strain fields – Q4 FE

- The associated strain field is obtained as:

$$\varepsilon_x = \frac{\partial u}{\partial x} = 1 \cdot q_{x2} + y \cdot q_{x4}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 1 \cdot q_{y3} + x \cdot q_{y4}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 1 \cdot \left(q_{x3} + q_{y2}\right) + x \cdot q_{x4} + y \cdot q_{y4}$$



- In matrix form this becomes:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix} \begin{bmatrix} \mathbf{q}_x \\ \mathbf{q}_y \end{bmatrix} = \mathbf{B}_{\mathrm{q}} \mathbf{q}$$

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Displacement patterns Q4 FE

- Physical interpretation of the eight linearly independent displacement patterns:
- ***Rigid body modes:***

❶ Translation in *x*-direction: $u = 1 \cdot q_{x1} \Rightarrow \varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$

❷ Translation in *y*-direction: $v = 1 \cdot q_{y1} \Rightarrow \varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$

❸ Rigid-body rotation: $u = y \cdot q_{x3}, v = x \cdot q_{y2}$ and $q_{x3} = -q_{y2} \Rightarrow \varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$

- ***Constant-strain modes:***

❹ Strain in *x*-direction: $u = x \cdot q_{x2} \Rightarrow \varepsilon_x = q_{x2}, \quad \varepsilon_y = \gamma_{xy} = 0$

❺ Strain in *y*-direction: $v = y \cdot q_{y3} \Rightarrow \varepsilon_y = q_{y3}, \quad \varepsilon_x = \gamma_{xy} = 0$

❻ Shear strain: $u = y \cdot q_{x3}, v = x \cdot q_{y2} \Rightarrow \gamma_{xy} = q_{x3} + q_{y2}, \ \varepsilon_x = \varepsilon_y = 0$

- ***Bending-modes:***

❼ In *x*-direction: $u = xy \cdot q_{x4} \Rightarrow \varepsilon_x = y \cdot q_{x4}, \ \varepsilon_y = 0, \ \gamma_{xy} = x \cdot q_{x4}$

❽ In *y*-direction: $v = xy \cdot q_{y4} \Rightarrow \varepsilon_x = 0, \ \varepsilon_y = x \cdot q_{y4}, \ \gamma_{xy} = y \cdot q_{y4}$

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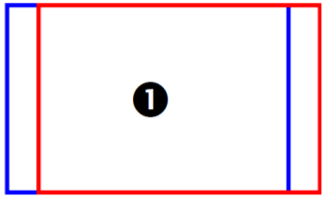
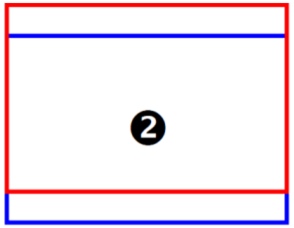
Displacement patterns Q4 FE

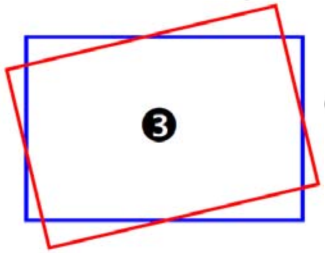
- ***Rigid body modes:***



$u = 1 \cdot q_{x1}$



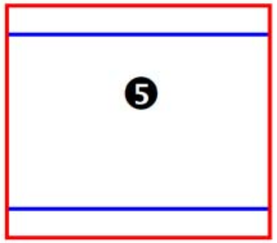
$v = 1 \cdot q_{y1}$



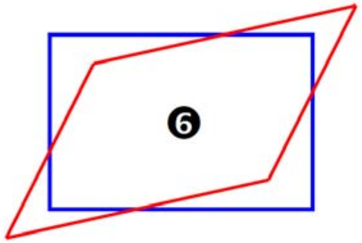
$u = y \cdot q_{x3}, v = x \cdot q_{y2}$

- ***Constant-strain modes:***

$u = x \cdot q_{x2}$



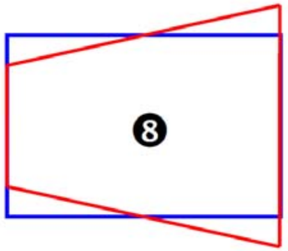
$v = y \cdot q_{y3}$



$u = y \cdot q_{x3}, v = x \cdot q_{y2}$

- ***Bending-modes:***

$u = xy \cdot q_{x4}$



$v = xy \cdot q_{y4}$

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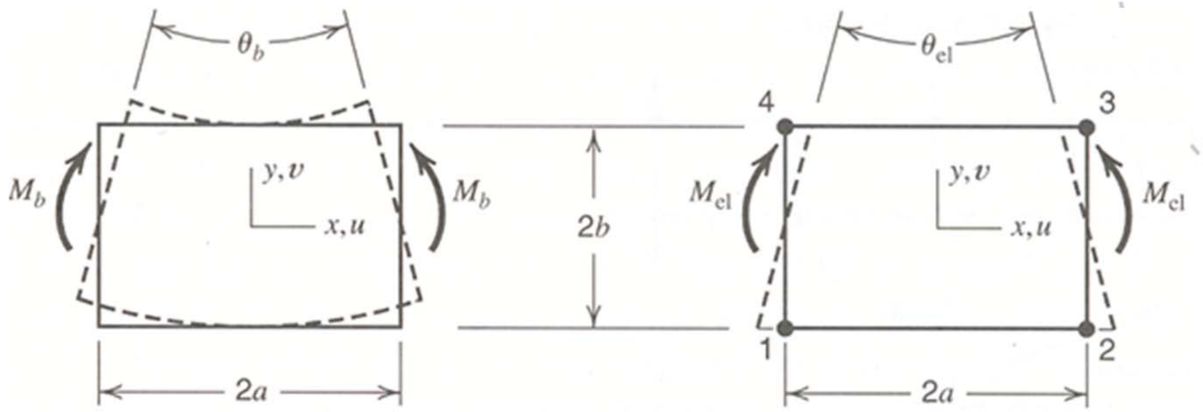
Defects of the Q4 in bending



- When the Q4 element is bent, its top and bottom sides remain straight, and each node has only a horizontal displacement of magnitude:

$$\tilde{u}_2 = \tilde{u}_4 = -\tilde{u}_1 = -\tilde{u}_3 = \frac{\theta_{el} b}{2}$$

- The corresponding generalized coordinate is obtained by substituting the nodal point coordinates into the expression for the generalized displacement pattern describing bending in *x*-direction (mode ❼):

$$\left.\begin{aligned} u(x,y) &= xy \cdot q_{x4} \\ \tilde{u}_3 = u(a,b) &= ab \cdot q_{x4} = -\frac{\theta_{el} b}{2} \end{aligned}\right\} \Rightarrow \boxed{q_{x4} = -\frac{\theta_{el}}{2a}}$$

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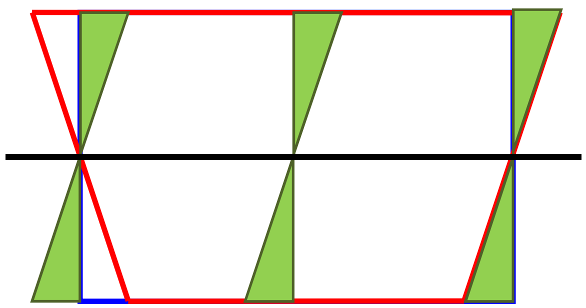
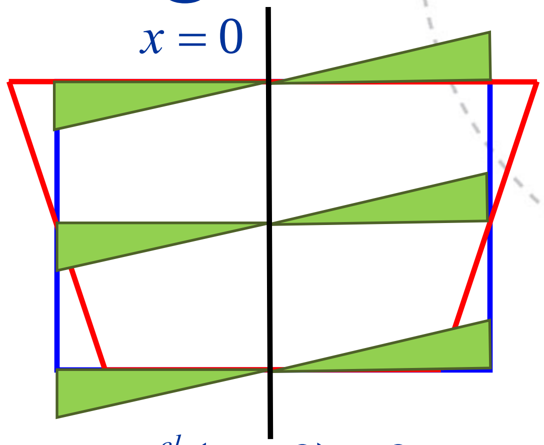
Q4 in pure bending



$\varepsilon_x^{el}(y=0)=0$



$\gamma_{xy}^{el}(x=0)=0$

- From $\boldsymbol{\varepsilon} = \mathbf{B}_\mathrm{q}\mathbf{q}$, the FE approximation of the strain field reads:

$$\varepsilon_x^{el} = y \cdot q_{x4} = -\frac{\theta_{el} y}{2a}, \qquad \varepsilon_y^{el} = 0 \qquad \text{and} \qquad \gamma_{xy}^{el} = x \cdot q_{x4} = -\frac{\theta_{el} x}{2a}$$

- While from beam bending theory strains in pure bending reads:

$$\varepsilon_x = -\frac{\theta_b y}{2a}, \qquad \varepsilon_y = \nu \frac{\theta_b y}{2a} \qquad \text{and} \qquad \gamma_{xy} = 0$$

$$\Rightarrow \boxed{\varepsilon_x^{el} = \varepsilon_x \quad \text{while} \quad \varepsilon_y^{el} = \varepsilon_y \quad \text{only if} \quad \nu = 0}$$

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Parasitic shear – shear “locking”

- We observe that the Q4 when exposed to pure bending displays ***spurious shear strains*** γ_{xy}^{el} as well as the expected bending strain ε_x^{el}
- This ***parasitic shear absorbs strain energy***, so that if the Q4 element is exposed to bending the bending deformations becomes smaller than expected:

$\Rightarrow$ The Q4 element exhibit ***shear locking*** behaviour

- The ratio between M_{el} and M_b reads.

$$\frac{M_{el}}{M_b} = \frac{1}{1+\nu}\left[\frac{1}{1-\nu} + \frac{1}{2}\left(\frac{a}{b}\right)^2\right]$$

- The $(a/b)^2$ term is present only because of parasitic shear. The ratio $M_{el}/M_b \rightarrow \infty$ as the aspect ratio $a/b \rightarrow \infty$. Hence, the ***FE model exhibit shear locking*** behaviour when the aspect ratio is large. In practice, however, we avoid elements of large aspect ratio, such that the FE mesh does not “lock”. It is, however, overly stiff in bending.

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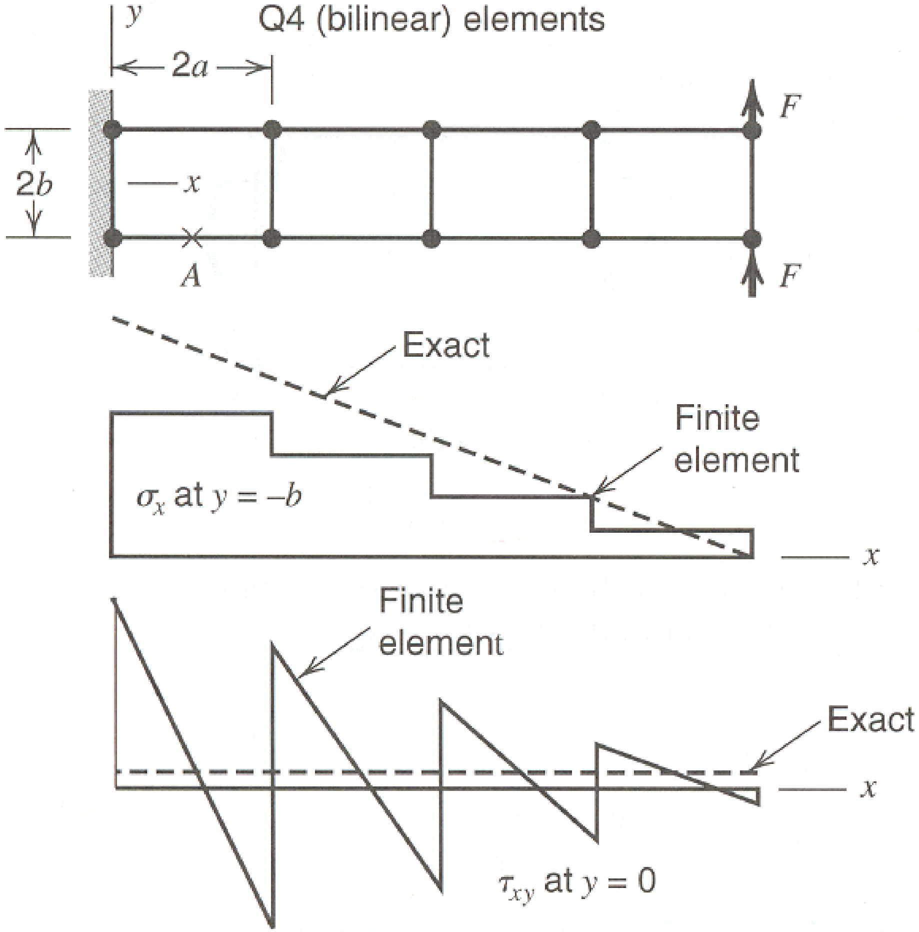
Example – Cantilever beam

- Tip loaded cantilever beam modeled with 4 equally sized Q4 elements (ν = 0.3).
- As may be expected the FE stress σ_x^{el} is constant (independent of *x*) within each element.
- Except at element centres (where $\tau_{xy}^{el} = 0$), the FE shear stress is dominated by the parasitic shear effect.
- The FE bending stress σ_x^{el} is about 2/3 of its correct value

  $\Rightarrow$ 1/3 of the bending strain energy is absorbed by the parasitic shear effect



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Selective Reduced Integration

- One device to avoid parasitic shear in pure bending is applying Selective Reduced Integration (SRI).
- For an isotropic/orthotropic linear elastic material the ***constitutive matrix***, **D**, may be ***splitted into a normal strain part***, $\mathbf{D}_\varepsilon$, ***and a shear strain part***, $\mathbf{D}_\gamma$:

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_{33} \end{bmatrix} = \mathbf{D}_\varepsilon + \mathbf{D}_\gamma$$

- Thus, the ***stiffness matrix***, **k**, may also be ***splitted into a normal strain part***, $\mathbf{k}_\varepsilon$, ***and a shear strain part***, $\mathbf{k}_\gamma$:

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV = \int_V \mathbf{B}^T \mathbf{D}_\varepsilon \mathbf{B} \, dV + \int_V \mathbf{B}^T \mathbf{D}_\gamma \mathbf{B} \, dV = \mathbf{k}_\varepsilon + \mathbf{k}_\gamma$$

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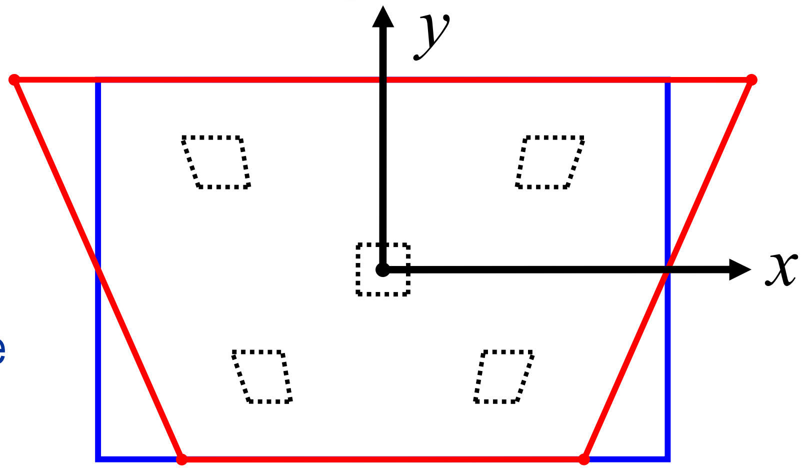
Selective Reduced Integration

- We observed that the ***parasitic shear strain coincide with the correct solution along x = 0,*** while the shear strain in the bending field rotated 90º to that shown above is correct along the line y = 0.



- The ***shear locking*** behaviour is ***related to*** the ***shear strain energy expressed through the element shear stiffness*** $\mathbf{k}_\gamma$.

- Since the shear strain is zero at the element centre ***spurious shear strain energy in pure bending may be avoided by evaluating the shear strain contribution,*** $\mathbf{k}_\gamma$***, at the element centre*** (i.e. 1x1 Gauss rule), while the normal strain contribution, $\mathbf{k}_\varepsilon$, is integrated by a 2x2 Gauss rule (full integration)

- The above is referred to as ***Selective Reduced Integration (SRI)***, since reduced integration is applied to the shear part of the strain energy, only.

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Incompatible elements – Q6

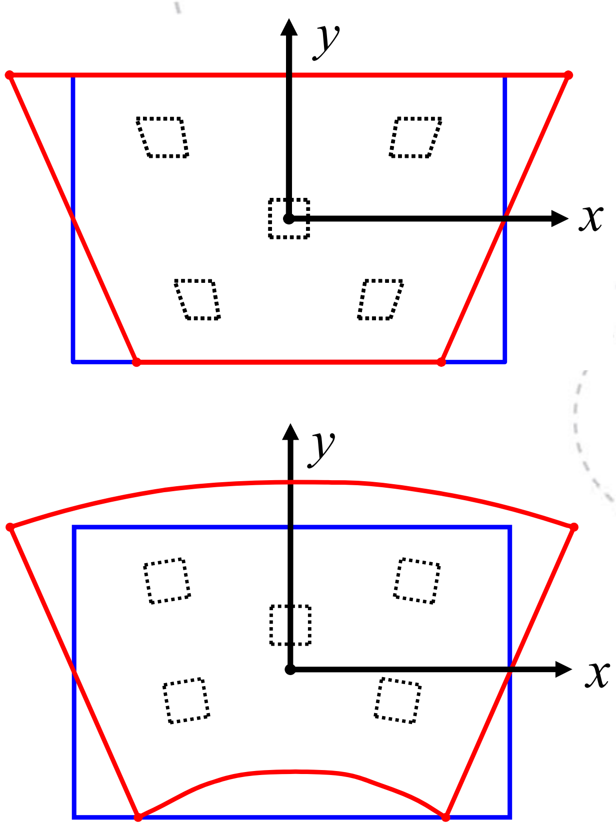
- The main reason why the Q4 element is overly stiff when bent is that the element ***cannot produce*** the desired ***quadratic displacement modes*** associated with pure bending.

- A remedy for this trouble is to ***augment*** the ***compatible displacement field*** for *u* and *v* ***with two additional modes referred to as incompatible displacement modes*** that describes a state of constant curvature (allow edges of the element to become curved).

$$u = \sum_{i=1}^{4} N_i \tilde{u}_i + \left(1-\xi^2\right) a_1 + \left(1-\eta^2\right) a_2$$

$$v = \sum_{i=1}^{4} N_i \tilde{v}_i + \left(1-\xi^2\right) a_3 + \left(1-\eta^2\right) a_4$$

- The additional modes are referred to as incompatible displacement modes, so-called ***internal nodeless d.o.f.*** that are ***<u>not</u> associated with any node*** nor are they connected to d.o.f. of any other element



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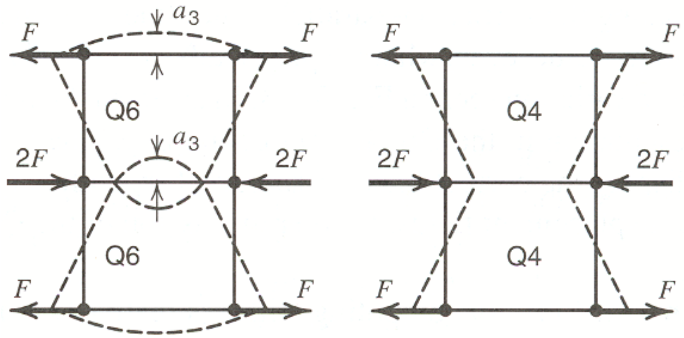
Incompatible elements – Q6

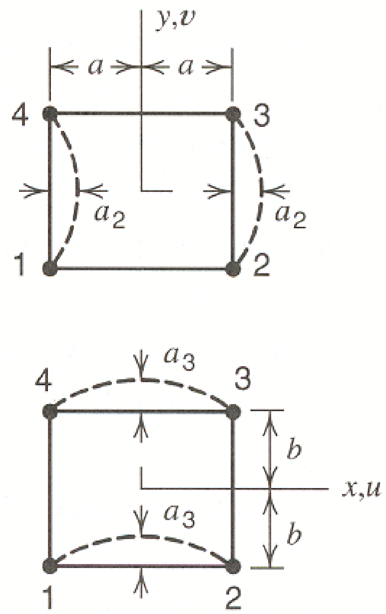
- Q6 is called “***incompatible***” because ***overlap or gap may appear between adjacent elements.***



- The ***Q6*** element ***can model pure bending with either an x-, or y-parallell neutral axis***; indeed the generalized dof can be non-zero simultaneously

- ***If*** the ***element is rectangular***, the ***shear strain*** in the Q6 element becomes

$$\gamma_{xy} = \sum_{i=1}^{4}\left(\frac{\partial N_i}{\partial y}\tilde{u}_i + \frac{\partial N_i}{\partial x}\tilde{v}_i\right) - \frac{2y}{b^2}a_2 - \frac{2x}{a^2}a_3$$



- ***In pure bending***, the negative terms $2ya_2 / b^2$ and $2xa_3 / a^2$ are equal in magnitude to positive terms produced by the compatible modes (summation terms), thus permitting ***shear strains*** to ***vanish***, as is proper.

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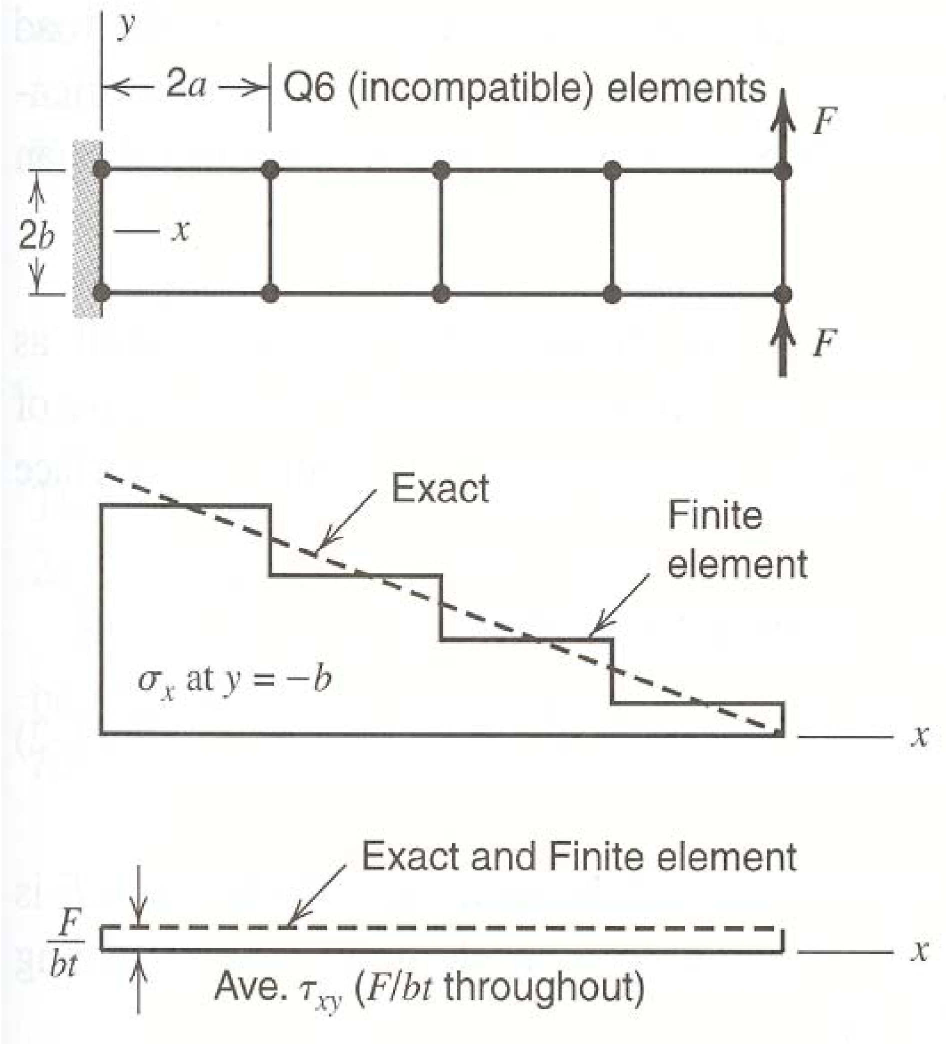
Example – Cantilever beam

- Tip loaded cantilever beam modeled with 4 equally sized Q6 rectangular incompatible elements ($\nu = 0$).
- Transverse tip displacement is almost exact (< 1 % error).
- Bending stress σ_x^{el} is exact along the vertical (*y*-parallel) centerline of each element.
- Average transverse shear stress τ_{xy}^{el} is exact everywhere without the spurious variation in the *x*-direction as observed with the compatible element .



Incompatible element – QM6

- The Q6 element does not satisfy the completeness criterion unless the Jacobian is constant (i.e. rectangular or parallelogram shaped element).
- A remedy for this defect has been proposed by Taylor and Wilson:
- The strain energy of the element reads:

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\ dV = \frac{1}{2}\left(\int_V \boldsymbol{\sigma}^T \mathbf{B}\ dV\right)\tilde{\mathbf{u}} + \frac{1}{2}\left(\int_V \boldsymbol{\sigma}^T \mathbf{B}_a dV\right)\mathbf{a}$$

- We observe that the compatible Q4 element fulfil completeness irrespective of the shape of the element:

$$\Rightarrow \boxed{\begin{array}{l}\text{The incompatible (Q6) element will also fulfil completeness if the strain energy} \\ \text{associated with the incompatible modes vanish for all constant strain states}\end{array}}$$

$$\Rightarrow \quad \frac{1}{2}\left(\int_V \boldsymbol{\sigma}_o^T \mathbf{B}_a dV\right)\mathbf{a} = \frac{1}{2}\boldsymbol{\sigma}_o^T\left(\int_V \mathbf{B}_a dV\right)\mathbf{a} = 0$$

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Incompatible element – QM6

- Thus if the Q6 element satisfy the requirement:

$$\frac{1}{2}\left(\int_V \boldsymbol{\sigma}_o^T \mathbf{B}_a dV\right)\mathbf{a} = \frac{1}{2}\boldsymbol{\sigma}_o^T\left(\int_V \mathbf{B}_a dV\right)\mathbf{a} = 0 \quad \Rightarrow \quad \boxed{\int_V \mathbf{B}_a dV = 0}$$

  the Q6 element fulfill completeness.

- A more general technique for constructing incompatible elements satisfying completeness has been proposed by Wilson and Ibrahimbegovic:

$$\boxed{\mathbf{B}_a^m = \mathbf{B}_a + \mathbf{B}_{ac}} \;\Rightarrow\; \int_V \mathbf{B}_a^m dV = \int_V \left(\mathbf{B}_a + \mathbf{B}_{ac}\right) dV = 0 \;\Rightarrow\; \mathbf{B}_{ac} = -\frac{1}{V}\int_V \mathbf{B}_a dV$$

- The modified strain-displacement matrix $\mathbf{B}_a^m = \mathbf{B}_a - \frac{1}{V}\int_V \mathbf{B}_a dV$ replaces $\mathbf{B}_a$.

- The modified Q6 element is referred to as the QM6 element which fulfill completeness for all element shapes (i.e. the strong form of the patch test).

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Example – Cantilever beam

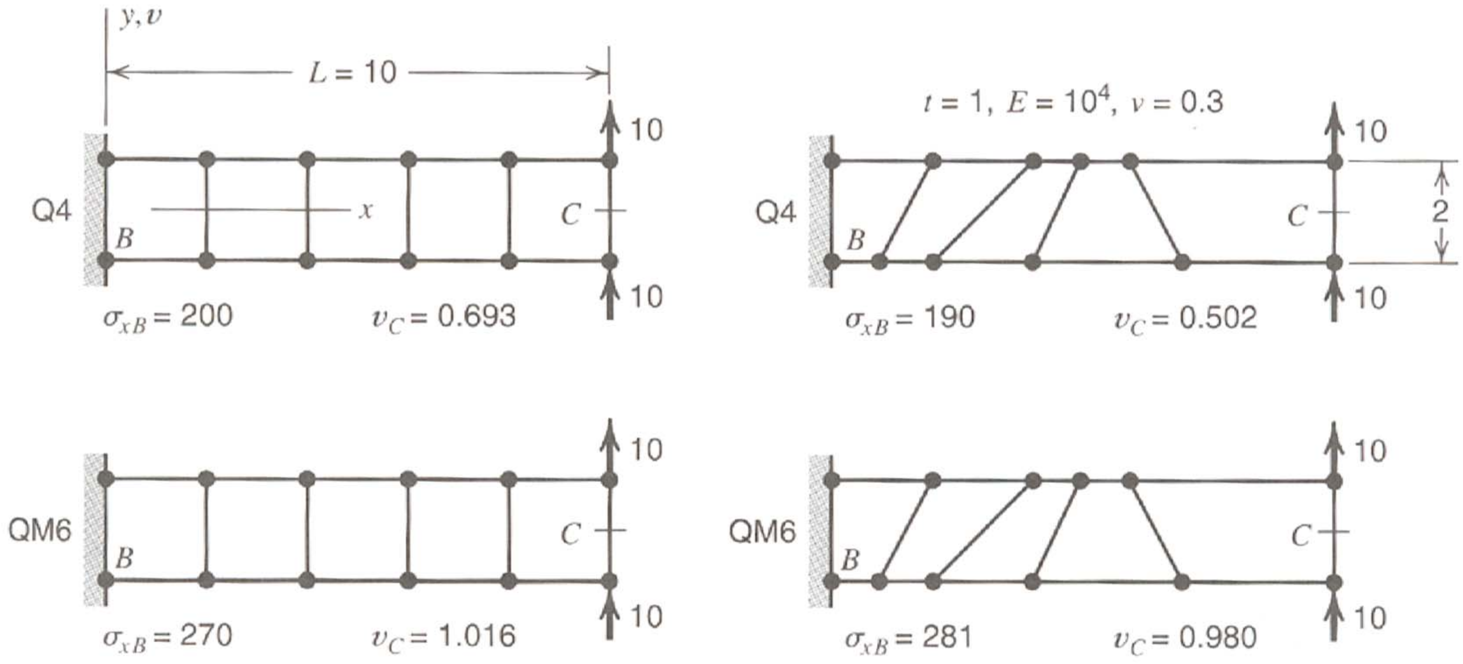

Figure 6.11-1. Stresses and deflections in FE models of a cantilever beam. Four-node elements and four-point Gauss quadrature are used. Exact $\sigma_{xB} = 300$; exact $v_C = 1.031$.

- Distortion test of the Q4 and the QM6 element.
- The compatible Q4 element is far too stiff and yields displacements and stresses that are 30% too low for rectangular shaped elements, end even worse when the element is distorted.
- In contrast, the incompatible QM6 element yields displacements and stresses that are almost exact when the elements are distorted.

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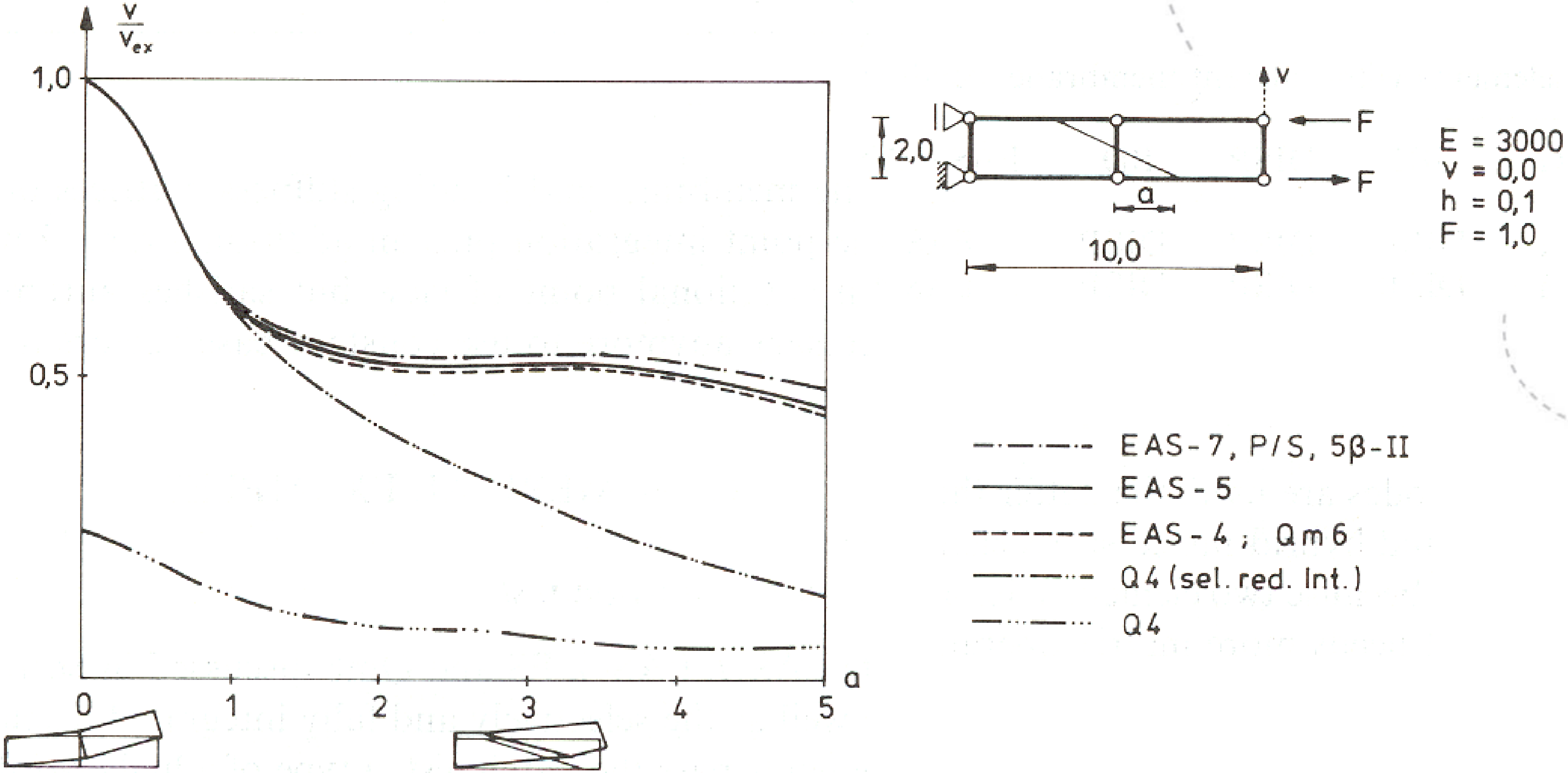
Example – Distortion test



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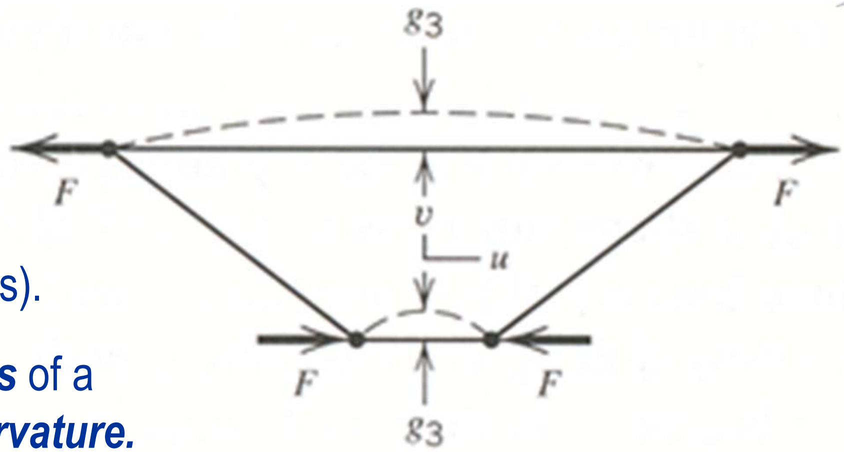
Remarks – QM6

- The ***QM6 element can represent pure bending exactly, but only if the element is rectangular***. The accuracy decline rapidly with increasing shape distortion.
- The following example with ***a trapezoidal shaped QM6*** element may be used to ***illustrate the defect*** of the QM6 element ***when it is not rectangular shaped.***
- ***When a bending load is applied*** to a trapezoidal shaped QM6 element, the incompatible displacement mode is activated so that ***top and bottom edges become arcs*** (as shown by the dashed lines).



- Under pure bending, ***top and bottom edges*** of a beam ***should have the same radius of curvature.***
- However, ***the arcs at the top and bottom edges of a trapezoidal shaped element have much different radius of curvature***, and the discrepancy increases as the amount of shape distortion increases (i.e. the element shape deviate from the rectangle).

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