

# A Binary Level Set Model and some Applications to Mumford-Shah Image Segmentation

Johan Lie<sup>\*</sup>, Marius Lysaker<sup>†</sup> and Xue-Cheng Tai<sup>‡</sup>

**Abstract**—In this work we propose a PDE based level set method. Traditionally interfaces are represented by the zero level set of continuous level set functions. We instead let the interfaces be represented by discontinuities of piecewise constant level set functions. Each level set function can at convergence only take two values, i.e. it can only be 1 or -1, and is thus related to phase-field methods. Some of the properties of standard level set methods are preserved in the proposed method, while others are not. Using this new level set method for interface problems, we need to minimize a smooth convex functional under a quadratic constraint. The level set functions are discontinuous at convergence, but the minimization functional is smooth. We show numerical results using the method for segmentation of digital images.

**Index Terms**—image segmentation, image processing, PDE, variational, level set, piecewise constant level set functions.

## I. INTRODUCTION

THE level set method proposed by Osher and Sethian [1] is a versatile tool for tracing interfaces separating a domain  $\Omega$  into subdomains. Interfaces are treated as the zero level set of some functions. Moving the interfaces can be done by evolving the level set functions instead of directly moving the interfaces. This idea is now used on a broad spectrum of problems, including image analysis, reservoir simulation, inverse problems, computer vision and optimal shape design [2]–[8]. For a recent survey on the level set methods see [9]–[12]. Newton type of fast methods have been successfully used for these level set formulations in [7]. In this work, we propose a variant of the level set method. This variant extends the level set models proposed in [13], [14] and it is also closely related to the phase-field methods [15]–[18]. The proposed method can be used for various inverse problems, but in this paper we restrict ourselves to segmentation of digital images. For a given digital image  $u_0 : \Omega \rightarrow \mathbb{R}$ , the aim is to separate  $\Omega$  into a set of subdomains  $\Omega_i$  such that  $\Omega = \cup_{i=1}^n \Omega_i$  and  $u_0$  is nearly a constant in each  $\Omega_i$ . The essential contribution of this work is to propose a mathematical mechanism to use the phase field idea for image segmentation and related inverse problems. The new idea we propose is to

use a binary level set function to represent piecewise constant (or polynomial) functions. The way we measure the length and area of the subdomains seems to be new. Moreover, using a constrained minimization approach to find the regions also seems to be different from the other approaches, c.f. [13]–[17]. We especially mention that the method proposed here is truly variational. In order to study the fundamental properties of the proposed models, we have chosen to use the most stable algorithm, i.e. the steepest decent method, in the tests given here.

One general image segmentation model was proposed by Mumford and Shah in [19]. Numerical approximations are thoroughly treated in [20], [21]. Using this model, the image  $u_0$  is decomposed into  $\Omega = \cup_i \Omega_i \cup \Gamma$ , where  $\Gamma$  is a curve separating the different domains. Inside each  $\Omega_i$ ,  $u_0$  is approximated by a smooth function. The optimal partition of  $\Omega$  is found by minimizing the Mumford-Shah functional (8). This is explained in the next section. Following the Mumford-Shah formulation for image segmentation, Chan and Vese [2], [22] solved the minimization problem using level set methods. The interface  $\Gamma$  is traced by the level set functions. Motivated by the Chan-Vese (CV) approach, we will in this article solve the segmentation problem in a different way, i.e. by introducing a piecewise constant level set function. Instead of using the zero level of a function to represent the interface between subdomains, we let the interface be represented implicitly by the discontinuities of a level set function. A two-phase segmentation is accomplished by requiring the level set function  $\phi$  to take the value 1 in one of the regions and  $-1$  in the other region, by enforcing  $\phi$  to satisfy  $\phi^2 = 1$ . In order to divide the domain into several subdomains, we use a set of functions  $\phi_i$  satisfying  $\phi_i^2 = 1$ . Using  $N$  level set functions, we can identify  $2^N$  phases. See also a recent work [23], where we have developed a technique which only use one discontinuous level set function for representing multiple phases. Another related techniques are considered in [18], [24].

The rest of this article is structured as follows. In §II we give a brief review of the traditional level set method, and the phase-field methods. Our model is formulated in §III. In §IV we apply this model for image segmentation. The segmentation problem is formulated as a minimization problem with a smooth cost functional under a quadratic constraint. Our constraint is analogous to the potential used in phase-field methods [18]. The minimization functional is essentially the Mumford-Shah functional associated with the proposed level set model. We propose two algorithms for solving the segmentation problem. In §V we show numerical

This work is supported by the Norwegian Research Council, Center of Mathematics for Applications (CMA) and Center for Integrated Petroleum Research (CIPR).

<sup>\*</sup>Department of Mathematics, University of Bergen, Johannes Brunsgate 12, N-5008, Norway, e-mail: johanl@mi.uib.no, web page: <http://www.mi.uib.no/BBG>

<sup>†</sup>Simula Research Laboratory, Martin Linges v 17, Fornebu P.O.Box 134, 1325 Lysaker, Norway, e-mail: mariul@simula.no

<sup>‡</sup>Department of Mathematics, University of Bergen, Johannes Brunsgate 12, N-5008, Norway, e-mail: tai@mi.uib.no, web page: <http://www.mi.uib.no/~tai>

examples, before we conclude the paper in §VI.

## II. SOME EXISTING METHODS

The main idea behind the level set formulation is to represent an interface  $\Gamma(t)$  bounding a possibly multiply connected region in  $\mathbb{R}^n$  by a Lipschitz continuous function  $\phi$ , changing sign at the interface, i.e.

$$\begin{cases} \phi(x, t) > 0, & \text{if } x \text{ is inside } \Gamma(t), \\ \phi(x, t) = 0, & \text{if } x \text{ is at } \Gamma(t), \\ \phi(x, t) < 0, & \text{if } x \text{ is outside } \Gamma(t). \end{cases} \quad (1)$$

In numerical implementations, often regularity is imposed on  $\phi$  to prevent the level set function of being too steep or flat near the interface. This is normally done by requiring  $\phi$  to be the signed distance function to the interface

$$\begin{cases} \phi(x, t) = d(\Gamma(t), x), & \text{if } x \text{ is inside } \Gamma(t), \\ \phi(x, t) = 0, & \text{if } x \text{ is at } \Gamma(t), \\ \phi(x, t) = -d(\Gamma(t), x), & \text{if } x \text{ is outside } \Gamma(t), \end{cases} \quad (2)$$

where  $d(\Gamma(t), x)$  denotes Euclidian distance between  $x$  and  $\Gamma(t)$ . We emphasize that requiring (2) is a technicality to prevent instabilities in numerical implementations. Having defined the level set function  $\phi$  as in (2), there is an one to one correspondence between the curve  $\Gamma$  and the function  $\phi$ . The distance function  $\phi$  obeys the Eikonal equation

$$|\nabla\phi| = 1. \quad (3)$$

The solution of (3) is not unique in the distributional sense. Finding the unique vanishing viscosity solution of (3) can be done by solving the following initial value problem to steady state

$$\phi_t + \text{sgn}(\tilde{\phi})(|\nabla\phi| - 1) = 0, \quad (4)$$

$$\phi(x, 0) = \tilde{\phi}(x). \quad (5)$$

In the above,  $\tilde{\phi}$  may not be a distance function. When the steady state of equation (4) is reached,  $\phi$  will be a distance function having the same zero level curve as  $\tilde{\phi}$ . This is commonly known as the reinitialization procedure. For numerical computations this procedure is crucial, and many numerical finite difference schemes exist. See [3], [9], [10], [25] for some details. Alternatively, the problem (3) can be efficiently solved using fast marching methods [26], [27].

The interface  $\Gamma(t)$  is implicitly moved according to the nonlinear PDE

$$\frac{\partial\phi}{\partial t} + \mathbf{v}(\phi) \cdot \nabla\phi = 0, \quad (6)$$

where  $\mathbf{v}(\phi)$  is a given velocity field. This vector field can depend on geometry, position, time and internal or external physics. Usually only the velocity normal to the interface  $v_N$  is needed, and  $\phi$  is then moved according to the modified equation

$$\frac{\partial\phi}{\partial t} + v_N(\phi)|\nabla\phi| = 0. \quad (7)$$

### A. Level Set Methods and Image Segmentation

The active contour (snake) model evolves a curve  $\Gamma(t)$  in order to detect objects in an image  $u_0$  [28]. The curve is moved from an initial position  $\Gamma(0)$  in the direction normal to the curve, subject to constraints in the image. An edge detector function  $g(\nabla u_0)$  indicates when  $\Gamma(t)$  is situated at the boundary of an object. One limitation of the original snake model is that the curve is represented explicitly, thus topological changes like merging and breaking of the curve may be hard to handle. To address this problem, a level set formulation of the active contour model was introduced in [29]. Later, Chan and Vese introduced a level set model for active contour segmentation, with the very important property that the stopping criteria is independent of  $\nabla u_0$  [2]. This means that boundaries not defined by gradients are detectable. The evolution of the curve is based on the general Mumford-Shah formulation of image segmentation, by minimization of

$$F^{MS}(u, \Gamma) = \int_{\Omega \setminus \Gamma} |u - u_0|^2 dx + \beta|\Gamma| + \nu \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx. \quad (8)$$

In the above,  $|\Gamma|$  is the length of  $\Gamma$ . A minimizer of this functional is smooth in  $\Omega \setminus \Gamma$ . The piecewise constant Mumford-Shah formulation of image segmentation is to find a partition of  $\Omega$  such that  $u$  in  $\Omega_i$  equals a constant  $c_i$ , and  $\Omega = \cup_i \Omega_i \cup \Gamma$ . The two last terms in (8) are regularizers measuring curve-length of the curves bounding the phases, and smoothness of  $u$  in  $\Omega \setminus \Gamma$ .  $\beta$  and  $\nu$  control the amount of regularization and smoothness. Based on (8), Chan and Vese [2] proposed the following minimization problem for a two-phase segmentation

$$\min_{c_1, c_2, \phi} \left\{ \int_{\Omega} |u_0 - c_1|^2 H(\phi) dx + \int_{\Omega} |u_0 - c_2|^2 (1 - H(\phi)) dx + \nu \int_{\Omega} H(\phi) dx + \beta \int_{\Omega} \delta(\phi) |\nabla\phi| dx \right\}. \quad (9)$$

Here  $\phi$  is the level set function satisfying (1),  $H(\phi)$  is the Heaviside function:  $H(\phi) = 1$  if  $\phi \geq 0$  and  $H(\phi) = 0$  if  $\phi < 0$ , and  $\delta(\phi)$  is the Dirac-delta function which by definition only collects the zero level set of  $\phi$ . Since  $\delta(\phi)$  is a singular function, a regularized approximation  $\delta_\epsilon(\phi)$  must be used in computations [2]. Finding a minimum of (9) is done by introducing an artificial time variable, and moving  $\phi$  in the steepest descent direction to steady state

$$\begin{aligned} \phi_t &= \delta_\epsilon(\phi) \left( -(u_0 - c_1)^2 + (u_0 - c_2)^2 - \nu + \beta \nabla \cdot \left( \frac{\nabla\phi}{|\nabla\phi|} \right) \right), \\ \phi(0) &= \phi_0. \end{aligned} \quad (10)$$

The recovered image is a piecewise constant approximation to  $u_0$ . This level set framework was later generalized to multiple phase segmentation using multiple level set functions [22].

In this paper we solve the piecewise constant Mumford-Shah segmentation using a slightly different approach. We separate the connection between the level set function and the distance function. This means that we get rid of the reinitialization procedure. In our approach, we impose a quadratic constraint on the level set functions, i.e.  $\phi_i^2 = 1$ . Our approach is truly variational, i.e. the equations we need to solve are

coming from the Euler-Lagrange equations for some smooth convex functions.

### B. Phase-field Methods and Image Segmentation

The phase field methods (Van der Waals-Cahn-Hilliard methods) can be used for determining stable configurations of dynamical systems, for example for a fluid where the energy  $W$  is a function of its density distribution  $u$ . The stable configuration can be found by searching for  $\lim_{\epsilon \rightarrow 0} \inf_u E(u)$  from

$$E(u) = \int_{\Omega} \left[ \epsilon |\nabla u(x)|^2 + \frac{1}{\epsilon} W(u(x)) \right] dx \quad (11)$$

where  $u$  is constrained by  $\int_{\Omega} u(x) dx = m$ , the total mass of the fluid.  $W : \mathbf{R} \rightarrow \mathbf{R}^+$  is a nonnegative function having the same number of minima as phases in the fluid, with  $W = 0$  at each minimum.  $W$  is supposed to grow at least linearly at infinity.

This model was adapted to image segmentation in [18]. In that work the functional

$$\begin{aligned} \bar{E} = & \int_{\Omega} (u - u_0)^2 dx + \epsilon \lambda^2 \int_{\Omega} f(|\nabla u(x)|) dx \\ & + \frac{\eta^2}{\epsilon} \int_{\Omega} W(u(x), \mu, \sigma) dx \end{aligned} \quad (12)$$

is minimized, and the solution to the image segmentation problem is given by  $\lim_{\epsilon \rightarrow 0} \operatorname{argmin}_u \bar{E}$ . In the above,  $f(s)$  is a function not necessarily convex. The first term is a fidelity term, ensuring that the recovered image  $u$  is close to the original image  $u_0$ . The second term is a restoration term. If  $f(s) = s^2$ , the restoration is Laplacian and thus linear. Other, especially non-convex choices give better restoration properties, but a sound theoretical foundation does not exist in this case [18]. The last term of (12) is a classification term making sure every pixel is correctly labeled, and thus classified into a specific phase  $c_i$ . Each phase is characterized by a Gaussian distribution  $N(\mu_i, \sigma_i)$ . Thus, some basic knowledge of the mean  $\mu_i$  and variance  $\sigma_i$  of each of the phases is required. The potential  $W$  is non-negative and it only equals zero when  $u = c_i$ . The constant parameters  $\lambda$  and  $\mu$  are weighting the amount of restoration and classification. As demonstrated in [18], the phase-field method is a quick method for supervised segmentation. In our approach, we identify the  $c_i$  values and the level set functions separately. We use a different mechanism to guarantee that the level set function approaches some pre-set fixed values.

### III. OUR APPROACH

To introduce our main idea, let us first assume that the interface is enclosing  $\Omega_1 \subset \Omega$ . By standard level set methods the interior of  $\Omega_1$  is represented by points  $\mathbf{x} : \phi(\mathbf{x}) > 0$ , and the exterior of  $\Omega_1$  is represented by points  $\mathbf{x} : \phi(\mathbf{x}) < 0$ , as in (1). We instead use a discontinuous level set function  $\phi$ , with  $\phi(\mathbf{x}) = 1$  if  $\mathbf{x}$  is an interior point of  $\Omega_1$  and  $\phi(\mathbf{x}) = -1$  if  $\mathbf{x}$  is an exterior point of  $\Omega_1$ , i.e.

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \operatorname{int}(\Omega_1), \\ -1 & \text{if } \mathbf{x} \in \operatorname{ext}(\Omega_1). \end{cases} \quad (13)$$

Thus  $\Gamma$  is implicitly defined as the discontinuity of  $\phi$ . This representation can be used for various applications where subdomains need to be identified. In order to use this idea for image segmentation, we use (13). Let us assume that  $u_0$  is an image consisting of two distinct regions  $\Omega_1$  and  $\Omega_2$ , and that we want to construct a piecewise constant approximation  $u$  to  $u_0$ . Let  $u(\mathbf{x}) = c_1$  in  $\Omega_1$ , and  $u(\mathbf{x}) = c_2$  in  $\Omega_2$ . If  $\phi(\mathbf{x}) = 1$  in  $\Omega_1$ , and  $\phi(\mathbf{x}) = -1$  in  $\Omega_2$ ,  $u$  can be written as the sum

$$u = \frac{c_1}{2}(\phi + 1) - \frac{c_2}{2}(\phi - 1). \quad (14)$$

The formula (14) can be generalized to represent functions with more than two constant values by using multiple functions  $\{\phi_i\}$  following the essential ideas of the level set formulation used in [4], [22]. A function having four constant values can be associated with two level set functions  $\{\phi_i\}_{i=1}^2$  satisfying  $\phi_i^2 = 1$ . More precisely, a function given as

$$\begin{aligned} u = & \frac{c_1}{4}(\phi_1 + 1)(\phi_2 + 1) - \frac{c_2}{4}(\phi_1 + 1)(\phi_2 - 1) \\ & - \frac{c_3}{4}(\phi_1 - 1)(\phi_2 + 1) + \frac{c_4}{4}(\phi_1 - 1)(\phi_2 - 1), \end{aligned} \quad (15)$$

is a piecewise constant function of the form

$$u(\mathbf{x}) = \begin{cases} c_1, & \text{if } \phi_1(\mathbf{x}) = 1, \quad \phi_2(\mathbf{x}) = 1, \\ c_2, & \text{if } \phi_1(\mathbf{x}) = 1, \quad \phi_2(\mathbf{x}) = -1, \\ c_3, & \text{if } \phi_1(\mathbf{x}) = -1, \quad \phi_2(\mathbf{x}) = 1, \\ c_4, & \text{if } \phi_1(\mathbf{x}) = -1, \quad \phi_2(\mathbf{x}) = -1. \end{cases}$$

Introducing basis functions  $\psi_i$  as in the following

$$\begin{aligned} u = & c_1 \underbrace{\frac{1}{4}(\phi_1 + 1)(\phi_2 + 1)}_{\psi_1} \\ & + c_2 \underbrace{(-1) \frac{1}{4}(\phi_1 + 1)(\phi_2 - 1)}_{\psi_2} + \dots, \end{aligned} \quad (16)$$

we see that  $u$  can be written as

$$u = \sum_{i=1}^4 c_i \psi_i. \quad (17)$$

For more general cases, we can use  $N$  level set functions to represent  $2^N$  phases. To simplify notation, we define the vectors  $\boldsymbol{\phi} = \{\phi_1, \phi_2, \dots, \phi_N\}$  and  $\mathbf{c} = \{c_1, c_2, \dots, c_{2^N}\}$ . For  $i = 1, 2, \dots, 2^N$ , let  $(b_1^{i-1}, b_2^{i-1}, \dots, b_N^{i-1})$  be the binary representation of  $i - 1$ , where  $b_j^{i-1} = 0 \vee 1$ . Furthermore, set

$$s(i) = \sum_{j=1}^N b_j^{i-1}, \quad (18)$$

and write  $\psi_i$  as the product

$$\psi_i = \frac{(-1)^{s(i)}}{2^N} \prod_{j=1}^N (\phi_j + 1 - 2b_j^{i-1}). \quad (19)$$

Then a function  $u$  having  $2^N$  constant values can be written as the weighted sum

$$u = \sum_{i=1}^{2^N} c_i \psi_i. \quad (20)$$

If the level set functions  $\phi_i$  satisfy  $\phi_i^2 = 1$  and  $\psi_i$  are defined as in (17) or (19), then  $\text{supp}(\psi_i) = \Omega_i$ ,  $\psi_i = 1$  in  $\Omega_i$ , and  $\text{supp}(\psi_i) \cap \text{supp}(\psi_j) = \emptyset$  when  $j \neq i$ . This ensures non-overlapping phases, and in addition  $\bigcup_i \text{supp}(\psi_i) = \Omega$ , which prevents vacuums.  $\psi_i$  is the characteristic function of the set  $\Omega_i$ .

If the level set functions satisfy  $\phi_i^2 = 1$ , then we can use the basis functions  $\psi_i$  to calculate the length of the boundary of  $\Omega_i$  and the area inside  $\Omega_i$ , i.e.

$$|\partial\Omega_i| = \int_{\Omega} |\nabla\psi_i| dx, \quad \text{and} \quad |\Omega_i| = \int_{\Omega} \psi_i dx. \quad (21)$$

The first equality of (21) shows that the length of the boundary of  $\Omega_i$  equals the total variation of  $\psi_i$ . See [30] for more explanations about the total variations of functions that might have discontinuities. In numerical computations, we use the approximation

$$\int_{\Omega} |\nabla\psi_i| dx \doteq \int_{\Omega} \sqrt{(\psi_i)_x^2 + (\psi_i)_y^2 + \epsilon} dx, \quad (22)$$

for a small  $\epsilon$  and the derivatives  $(\psi_i)_x, (\psi_i)_y$  are approximated by finite differences.

In fact, measuring the length of boundaries by this representation is more accurate than using  $\int_{\Omega} \delta(\phi_i) |\nabla\phi_i| dx$  which is done in [2], [22]. This is due to the fact that their regularizer does not treat all edges equally, by measuring some edges once and other edges twice, i.e. some edges are treated as more important than other edges, as pointed out by Chan and Vese [22]. Our method on the contrary counts every edge twice, and thus all the edges are treated equally. A simple example illustrating the difference between the two regularizers is shown in Fig. 1. Using  $\int_{\Omega} \delta(\phi_i) |\nabla\phi_i| dx$  as the regularizer, the length of the dashed lines in Fig. 1(b) are counted once while the thick line is counted twice. Using our approach, the length of all the lines are counted twice.

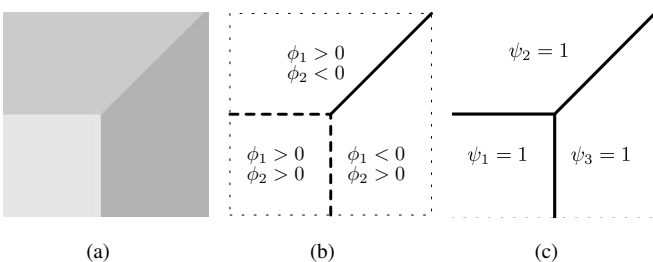


Fig. 1. (a) A simple image consisting of three phases. (b) If  $\int_{\Omega} \delta(\phi_i) |\nabla\phi_i| dx$  is used as a regularizer, the different edges are not treated in a similar fashion. The edge with the (thick) dashed line is measured once and the other (thick) edges are measured twice. Moreover, in general it is impossible to determine how each edge is supposed to be measured. (c) Using our representation, all edges are measured two times.

Before we explain our model in more detail, we mention two related works. Song and Chan solved the segmentation problem in a very elegant way in [14]. They minimize the functional (9) by a discrete algorithm, by using the fact that only the sign of the function  $\phi$  is needed in (9) and not  $\phi$  itself. The result is a very quick algorithm, but it is

not variational. Gibou and Fedkiw proposed to link k-Means clustering methods with level set methods, and thus managed to construct an algorithm with the speed of clustering methods and the robustness of level set methods [13].

#### IV. MINIMIZATION PROBLEM

We have now introduced a way to represent a piecewise constant function  $u$  by using the binary level set functions. Based on this we propose to minimize the following functional to find a segmentation of a given image  $u_0$

$$F(\phi, \mathbf{c}) = \frac{1}{2} \int_{\Omega} |u - u_0|^2 dx + \beta \sum_{i=1}^{2^N} \int_{\Omega} |\nabla\psi_i| dx. \quad (23)$$

In the above,  $\beta$  is a nonnegative parameter controlling the regularizing,  $u$  is a piecewise constant function depending on  $\phi$  and  $\mathbf{c}$ , as in (20). The first term of (23) is a least square functional, measuring how well the piecewise constant image  $u$  approximates  $u_0$ . The second term is a regularizer measuring the length of the edges in the image  $u_0$ . Considering the constraints imposed on the level set functions, we find that the segmentation problem is the following constrained minimization problem

$$\min_{\phi, \mathbf{c}} F(\phi, \mathbf{c}), \quad \text{subject to} \quad \phi_i^2 = 1, \forall i. \quad (24)$$

Recall that  $\phi$  is a vector having  $N$  elements  $\phi_i$ . For notational simplicity, we introduce a vector  $\mathbf{K}(\phi)$  of the same dimension as  $\phi$  with  $K_i(\phi) = \phi_i^2 - 1$ . It is easy to see that

$$\phi_i^2 = 1, \forall i \Leftrightarrow \mathbf{K}(\phi) = \mathbf{0}. \quad (25)$$

This leads to two related iterative algorithms for image segmentation, presented in the next section.

We see that the functional (24) is related to the functional (12) used in phase-field segmentation [18]. Both functionals have the same fidelity term. The potential  $W$  of (12) is analogous to our constraint  $\mathbf{K}$ , and both methods have a regularization term. We use two related methods to solve the minimization problem (24), a projection Lagrangian approach, and an augmented Lagrangian approach. If we set  $f(s) = s$  in (12) we see that this is very similar to our regularization term.

##### A. Projection Lagrangian Algorithm

The Lagrangian functional involves both  $F$  and the constraint  $\mathbf{K}$

$$L(\phi, \mathbf{c}, \boldsymbol{\lambda}) = F(\phi, \mathbf{c}) + \sum_{i=1}^N \int_{\Omega} \lambda_i K_i dx. \quad (26)$$

Here  $\boldsymbol{\lambda} = \{\lambda_i\}_{i=1}^N$  is a vector of functions of the same dimension as  $\phi$ , called the Lagrange multipliers. Note that when the constraint is fulfilled, the Lagrangian term vanishes, and  $L = F$ . We search for a saddle point of (26), which in turn will give a minimizer of (24). At a saddle point of  $L$  we must have

$$\frac{\partial L}{\partial \phi_i} = 0, \quad \frac{\partial L}{\partial \mathbf{c}_i} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda_i} = 0, \quad \forall i. \quad (27)$$

The saddle point is sought by minimizing  $L$  with respect to  $\phi$  and  $\mathbf{c}$ , and maximizing with respect to  $\lambda$ . By maximizing  $\lambda$  the constraint must be fulfilled at convergence, otherwise the Lagrangian term of (26) will not vanish. From the definition of  $L$ , we see that

$$\begin{aligned} \frac{\partial L}{\partial \phi_i} &= \frac{\partial F}{\partial \phi_i} + \sum_{j=1}^N \lambda_j \frac{\partial K_j}{\partial \phi_i} \\ &= (u - u_0) \sum_{j=1}^{2^N} c_j \frac{\partial \psi_j}{\partial \phi_i} \\ &\quad - \beta \sum_{j=1}^{2^N} \nabla \cdot \left( \frac{\nabla \psi_j}{|\nabla \psi_j|} \right) \frac{\partial \psi_j}{\partial \phi_i} + 2\lambda_i \phi_i. \end{aligned} \quad (28)$$

Using formulas (14), (15) and (19), it is easy to get  $\partial \psi_j / \partial \phi_i$ . If we replace the total variation norm of  $\psi_i$  by an approximation as in (22), then the term  $\left( \frac{\nabla \psi_j}{|\nabla \psi_j|} \right)$  should be replaced by  $\left( \frac{\nabla \psi_j}{\sqrt{|\nabla \psi_j|^2 + \epsilon}} \right)$ . We take  $\epsilon \approx 1e^{-3}$  in the numerical implementations.

Since  $u = \sum_i c_i \psi_i$ , and only the first term of  $F$  depends on  $u$ , the derivative with respect to  $c_i$  becomes

$$\frac{\partial L}{\partial c_i} = \int_{\Omega} \frac{\partial L}{\partial u} \frac{\partial u}{\partial c_i} dx = \int_{\Omega} \frac{\partial F}{\partial u} \frac{\partial u}{\partial c_i} dx = \int_{\Omega} (u - u_0) \psi_i dx. \quad (29)$$

The derivative of  $L$  with respect to  $\lambda_i$  essentially recovers the constraint:

$$\frac{\partial L}{\partial \lambda_i} = K_i = \phi_i^2 - 1. \quad (30)$$

All the derivatives (28), (29) and (30) must equal zero at a saddle point of  $L$ . To find the saddle point, we shall use an iterative algorithm. From initial guesses  $\phi^0, \mathbf{c}^0$  and  $\lambda^0$ , we iterate towards better approximations  $\phi^k, \mathbf{c}^k$  and  $\lambda^k$ . Since we want the three derivatives (28), (29) and (30) to equal zero, we increase  $k$  until none of  $\phi, \mathbf{c}$  or  $\lambda$  changes anymore. Then we have arrived at a saddle point. Using this approach, we need to choose three different schemes to get  $\phi^k, \mathbf{c}^k$  and  $\lambda^k$  from  $\phi^{k-1}, \mathbf{c}^{k-1}$  and  $\lambda^{k-1}$ .

First consider the minimization w.r.t.  $\phi$ , which is done by introducing an artificial time variable and finding a steady-state solution to the PDE

$$\phi_t = -\frac{\partial L}{\partial \phi}. \quad (31)$$

Note here that we have no theoretical foundation regarding existence and uniqueness of a solution of (31), but numerical experiments at least indicates existence of solutions. At steady state,  $\phi_t = 0$ , which means  $\frac{\partial L}{\partial \phi_i} = 0 \forall i$ . This is exactly what is needed for a saddle point of  $L$ . We discretize the time derivative using a forward Euler scheme

$$\phi_t \approx \frac{\phi^{new} - \phi^{old}}{\Delta t}. \quad (32)$$

Here  $\Delta t$  is a small positive time step. Combining (31) with (32), and rearranging the terms gives an updating scheme for  $\phi_i$

$$\phi_i^{new} = \phi_i^{old} - \Delta t \frac{\partial L}{\partial \phi_i}(\phi^{old}, \mathbf{c}^{k-1}, \lambda^{k-1}). \quad (33)$$

Observe that  $\phi_i$  is moved in the steepest descent direction, so this is essentially the gradient method. We use a fixed time step  $\Delta t$ , determined by a trial and error approach. The curvature term in  $\frac{\partial L}{\partial \phi}$  is the most restrictive term to the size of  $\Delta t$ . After a fixed number of iterations we let  $\phi^k = \phi^{new}$ . If an infinite number of iterations were done, i.e.  $t \rightarrow \infty$ , we would end up with the exact minimizer of  $L$  w.r.t.  $\phi$  with  $\mathbf{c}^{k-1}$  and  $\lambda^{k-1}$ .

Secondly, we consider the minimization of  $L$  w.r.t.  $\mathbf{c}$ , which is done by using (20).  $u$  is a linear combination of the basis functions, thus  $L$  is quadratic in  $\mathbf{c}$ . This means the minimization w.r.t.  $\mathbf{c}$  can be done by solving the  $2^N \times 2^N$  linear system  $A\mathbf{c} = \mathbf{b}$ , where  $A_{ij} = (\psi_i, \psi_j)_{L_2(\Omega)}$ , and  $\mathbf{b} = (u_0, \psi_i)_{L_2(\Omega)}$ .

$$\sum_{i=1}^{2^N} (\psi_i^k, \psi_j^k)_{L_2(\Omega)} c_i^k = (u_0, \psi_i^k)_{L_2(\Omega)}, \quad i = 1, 2, \dots, 2^N. \quad (34)$$

Last, an updating scheme for  $\lambda$  is constructed by combining (27) with (28) and (30). A saddle point of  $L$  must satisfy

$$0 = \frac{\partial L}{\partial \phi_i} = \frac{\partial F}{\partial \phi_i} + 2\lambda_i \phi_i. \quad (35)$$

By multiplying this equation with  $\phi_i$ , and noting that at a saddle point of  $L$  the constraint gives  $\phi_i^2 = 1$ , we can set this into (35) to get

$$\lambda_i = -\frac{1}{2} \phi_i \frac{\partial F}{\partial \phi_i}. \quad (36)$$

This is used as an updating scheme for  $\lambda$ :

$$\lambda_i^k = -\frac{1}{2} \phi_i^k \frac{\partial F}{\partial \phi_i}(\phi^k, \mathbf{c}^k). \quad (37)$$

Now the three updating formulas (33), (34) and (37) are combined to construct an algorithm using the Lagrangian approach. This scheme is essentially a projection Lagrangian algorithm.

---

#### Algorithm 1 (A Projection Lagrangian Method.)

---

**Initialize**  $\mathbf{c}^0, \phi^0, \lambda^0$ .

1. Update  $\phi^k$  by (33), to approximately solve

$$L(\mathbf{c}^{k-1}, \phi^k, \lambda^{k-1}) = \min_{\phi} L(\mathbf{c}^{k-1}, \phi, \lambda^{k-1}).$$

2. Construct  $u(\mathbf{c}^{k-1}, \phi^k)$  by

$$u = \sum_{i=1}^{2^N} c_i^{k-1} \psi_i^k.$$

3. Update  $\mathbf{c}^k$  by (34), to solve

$$L(\mathbf{c}^k, \phi^k, \lambda^{k-1}) = \min_{\mathbf{c}} L(\mathbf{c}, \phi^k, \lambda^{k-1}).$$

4. Update the multiplier by

$$\lambda_i^k = -\frac{1}{2} \phi_i^k \frac{\partial F}{\partial \phi_i}(\phi^k, \mathbf{c}^k), \quad \forall i = 1, 2, \dots, N.$$

5. Test convergence.

If necessary,  $k \leftarrow k + 1$ , repeat.

---

**Remark 1:** The minimization w.r.t.  $\mathbf{c}$  in step 3 should not be done too early in the process, e.g. not before  $|\phi_i| \approx 1$ , otherwise the matrix inversion in (34) becomes ill-conditioned. Minor perturbations of the level set functions will result in

large errors of the  $c_i$ -values. If  $|\phi_i|$  is far from 1, then  $\psi_i$  is far from orthogonal to  $\psi_j$ , and the inner product  $(\psi_i, \psi_j)$  in (34) will give contributions at points where it should not. This means that the matrix inversion in (34) does not give a good approximation to  $\mathbf{c}$  unless  $|\phi_i| \approx 1 \forall i$ .

**Remark 2:** The time step used in the gradient iteration in step 1 is influenced by the size of  $\nabla \cdot \left( \frac{\nabla \psi_i}{|\nabla \psi_i|} \right)$ . This term measures the curvature of the level curves  $\psi_i$ , i.e. essentially the second order derivatives of  $\psi_i$ . If the curvature becomes big, it might violate the CFL stability condition of the numerical scheme, unless the value of  $\beta$  is small. Thus larger  $\beta$  values require smaller time steps and vice versa. Since our numerical scheme is explicit, this behavior is to be expected. Using a semi-implicit or implicit scheme would make it possible to use larger time steps, but this is not in the scope of this paper.

**Remark 3:** The number  $n$  of gradient iterations performed in step 1 is usually set to a small number  $n \approx 10$ . This means that a gradient iteration is performed  $n$  times before the other steps in the algorithm are done. The minimization w.r.t.  $\phi$  is therefore not exact, but increasing  $n \rightarrow \infty$  would hopefully give an exact minimizer. We have observed that using ten gradient iterations usually gives a sufficiently good approximation to the exact minimizer before the other steps of the algorithm are performed.

### B. Augmented Lagrangian Algorithm

We can also solve the minimization problem by the augmented Lagrangian method. This is a combination of the multiplier method and the penalization method. Define the augmented Lagrangian functional as

$$L_\mu(\phi, \mathbf{c}, \boldsymbol{\lambda}) = F(\phi, \mathbf{c}) + \sum_{i=1}^N \int_{\Omega} \lambda_i K_i dx + \frac{1}{2} \mu \sum_{i=1}^N \int_{\Omega} K_i^2 dx. \quad (38)$$

Here  $\mu > 0$  is a penalization parameter, and the last term of (38) is called a penalization term. Similarly as in the Lagrangian approach, to minimize  $F(\phi, \mathbf{c})$ , we need to find a saddle point of (38). Thus we need updating schemes for  $\phi$ ,  $\mathbf{c}$  and  $\boldsymbol{\lambda}$ . Both  $\phi$  and  $\mathbf{c}$  are updated using the same techniques as in the Lagrangian approach. Hence we only need a new scheme for updating  $\boldsymbol{\lambda}$ , in addition to a scheme for updating the penalization parameter  $\mu$ . These two schemes are interconnected.

The original idea of a penalty method is to iteratively force the constraint to be fulfilled by increasing  $\mu$  to  $\infty$ . For the augmented Lagrangian method, due to the Lagrangian multipliers, the constraints are satisfied even if we use a fixed penalization parameter  $\mu$ . In practice, better convergence can be obtained if we increase the value of the penalization parameter. Let  $\boldsymbol{\lambda}^k$  denotes  $\boldsymbol{\lambda}$  at the  $k^{\text{th}}$  iteration. Following the approach in [31], [32], we use the following updating scheme

$$\boldsymbol{\lambda}^k = \boldsymbol{\lambda}^{k-1} + \mu \mathbf{K}(\phi^k). \quad (39)$$

Having determined  $\boldsymbol{\lambda}^{k-1}$ , we minimize  $L_\mu$  w.r.t.  $\phi$  by the gradient method updating scheme

$$\phi_i^{\text{new}} = \phi_i^{\text{old}} - \Delta t \frac{\partial L_\mu}{\partial \phi_i}(\phi^{\text{old}}, \mathbf{c}^{k-1}, \lambda_i^{k-1}), \quad (40)$$

where

$$\begin{aligned} \frac{\partial L_\mu}{\partial \phi_i} &= (u - u_0) \sum_{j=1}^{2^N} c_j \frac{\partial \psi_j}{\partial \phi_i} \\ &- \beta \sum_{j=1}^{2^N} \nabla \cdot \left( \frac{\nabla \psi_j}{|\nabla \psi_j|} \right) \frac{\partial \psi_j}{\partial \phi_i} \\ &+ 2\lambda_i \phi_i + 2\mu(\phi_i^2 - 1)\phi_i. \end{aligned} \quad (41)$$

Like in the first algorithm, after a few iterations we set  $\phi^k = \phi^{\text{new}}$ . The constraints  $K_i$  are independent of the constant values  $c_i$  and thus the updating for the  $c_i$  values will still be the same.

---

### Algorithm 2 (An augmented Lagrangian Method.)

---

**Initialize**  $\mathbf{c}^0, \phi^0, \boldsymbol{\lambda}^0, \mu$ .

1. Update  $\phi^k$  by (40), to approximately solve

$$L_\mu(\mathbf{c}^{k-1}, \phi^k, \boldsymbol{\lambda}^{k-1}) = \min_{\phi} L_\mu(\mathbf{c}^{k-1}, \phi, \boldsymbol{\lambda}^{k-1}).$$

2. Construct  $u(\mathbf{c}^{k-1}, \phi^k)$  by

$$u = \sum_{i=1}^{2^N} c_i^{k-1} \psi_i^k.$$

3. Update  $\mathbf{c}^k$  by (34), to solve

$$L_\mu(\mathbf{c}^k, \phi^k, \boldsymbol{\lambda}^{k-1}) = \min_{\mathbf{c}} L_\mu(\mathbf{c}, \phi^k, \boldsymbol{\lambda}^{k-1}).$$

4. Update the multiplier by

$$\boldsymbol{\lambda}^k = \boldsymbol{\lambda}^{k-1} + \mu \mathbf{K}(\phi^k).$$

5. Test convergence.

If necessary,  $k \leftarrow k + 1$ , iterate again.

---

This algorithm has a linear convergence and its convergence has been analyzed in Kunisch and Tai [33] under a slightly different context. This algorithm has also been used in Chan and Tai [4], [34] for a level set method for elliptic inverse problems.

**Remark 1:** In most of our simulations we have set  $\mu$  to be constant during the iterations. This is done to make the simulations as stable as possible. Better convergence behavior can be expected if  $\mu$  is increased during the iterations, but be aware of ill-conditioning if  $\mu$  is increased too quickly. This is a common approach when using the augmented Lagrangian method. See [31], [32] for details concerning the general algorithm.

**Remark 2:** As in the first algorithm,  $\mathbf{c}$  should not be updated too early in the process, to avoid ill-conditioning when inverting the matrix  $A$ . See Remark 1 of Algorithm 1.

**Remark 3:** In this algorithm,  $\Delta t$  in the gradient iteration depends on both  $\beta$  and  $\mu$ . A large  $\beta$  or  $\mu$  requires a small  $\Delta t$ . The constant  $\beta$  can be looked at as a parameter controlling the connectivities or oscillations of the different phases. A bigger  $\beta$  value will suppress oscillations, while a bigger  $\mu$  makes the level set functions  $\phi_i$  converge to  $\pm 1$  quicker. Choosing  $\mu$  too big will reduce the influence of the fitting term  $F(\phi, \mathbf{c})$  and thus may increase the iteration number needed to converge to the true solution. For practical problems, it is normally not too difficult to find an approximate range for these two parameters.

## V. NUMERICAL EXPERIMENTS

In this section numerical results are presented. We compare our method with the Chan-Vese CV model, qualitatively and quantitatively. For comparability with other methods, most of the images used in this section are standard images from articles in the literature of image processing [2], [13], [22].

Essentially the proposed model has two parameters,  $\beta$  and  $\mu$ . For each numerical example, we report the specific values used for  $\beta$  and  $\mu$ . In most of the simulations shown, we use  $\phi^0 \equiv 0$ , i.e. there is no need to initialize the level set functions. In examples where we compare with the CV model, the initial zero level curves are circles shown on top of  $u_0$ . Most of the images are imposed with noise, and we assume the noise is additive, i.e.

$$u_0 = u + \eta, \quad (42)$$

where  $\eta$  is Gaussian distributed noise. For each of the examples containing noise, we report the Signal to Noise Ratio (*SNR*):

$$SNR = \frac{\text{Variance of Data}}{\text{Variance of Noise}}. \quad (43)$$

If  $SNR \approx 1$ , the observation data is very noisy.

Even though the framework developed in this article is applicable for multiple level set functions, we only show numerical results using one or two level set functions. When using two functions, there is a need for an initial approximation of  $c$ . This is done by the following process: First a median filter is applied to the image, to produce a smoothed temporary image  $u_{tmp}$ , e.g.  $u_{tmp}(x_{ij})$  is taken to be the mean of a set of neighbor points of  $u_0(x_{ij})$ . Afterwards, a simple isodata approach is applied on  $u_{tmp}$  to find  $c^0$ , an approximation to the optimal  $c$ . We refer the reader to [35] for a discussion of the isodata algorithm which is based on a thresholding of the intensity values. When searching for only two phases, the initial value for  $c$  is not important, the algorithm converges to the same solution even if we start with an initial value far from the true one. This is due to the uniform convex nature of the objective functional in the two-subdomain-case. In the general case, the functional is only locally convex. All the examples shown are processed using the augmented Lagrangian Algorithm. For some experiments using the projection Lagrangian method, we refer to [36].

### A. One Level Set Function

**Example 1.** We start with an example where one level set function is used to detect two different subdomains. We want to test our method on a really challenging image with scattered data, i.e. a satellite image of Europe showing clusters of light. At every point in the image, the level set function must converge to  $\pm 1$ . Which point the level set function should equal 1 and which point it should equal  $-1$  is influenced by the regularization parameter  $\beta$ . A big  $\beta$  gives a “connected” result, while a small  $\beta$  gives a less “connected” result, see Fig. 2. No matter what kind of value we choose for  $\beta$ , the algorithms are able to get the level set function to converge to  $\pm 1$ . In Fig. 2 we show results using our method, and in Fig. 3 we show results using the CV method on the same image. For the CV model, the initial zero level of  $\phi$  is a circle with center

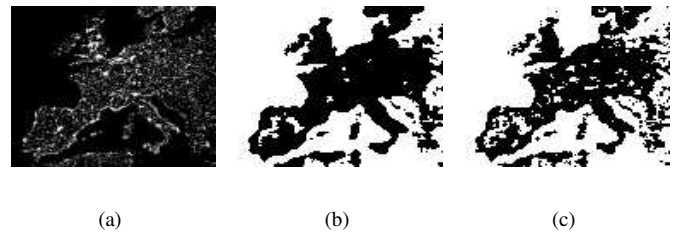


Fig. 2. Segmentation of a satellite image using our proposed augmented Lagrangian method. (a) The input image. (b) Processed image with  $\beta = 7 \cdot 10^{-4}$ . (c) Processed image with  $\beta = 3 \cdot 10^{-4}$ . Note that the topology of the resulting image depends on the choice of the regularization parameter.

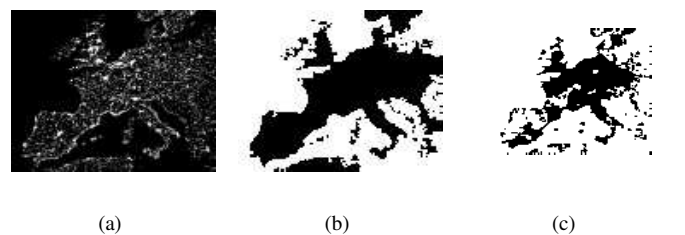


Fig. 3. Segmentation of the satellite image using the CV method. (a) The same input image as in Fig. 2(a). (b) Processed image without reinitialization. (c) Processed image with reinitialization every tenth iteration. Whether or not a reinitialization is done affects the topology of the resulting image.

in the middle of the image, and diameter equal to the height of the image. Both methods are terminated after 300 iterations on  $\phi$ . By choosing the  $\beta$  value properly, both methods are able to produce visually pleasing results. It was observed that the CV method produces different images with different initial values for the level set function. It is also true that the reinitialization process and how often doing the reinitialization for the level set functions alter the final results of the processed image. Using our method, we only need to choose the regularization parameter. For a comparison with other results, we refer the reader to [2], [10], [13], [23].

**Example 2.** In this example, we introduce a technique which can be used for accelerating the convergence of our algorithms. In some sense this is related to what was done in [14]. At convergence, the level set function should equal  $\pm 1$ . After a few gradient iterations, the level set functions could already have the correct sign, but it might take many iterations to get  $\phi$  exactly to  $\pm 1$ . This is a common behavior of the steepest descent method, due to its slow convergence rate. To accelerate the convergence, we start the algorithm, and perform a few iterations. Then we take the sign of the obtained  $\phi$  function as an initial value and start the algorithm again. In each iteration, the step size  $\Delta t$  is chosen using a line search algorithm. We show the results for a specific image in Fig. 4. Again we observe that the topological properties of the segmented image are related to the chosen value of  $\beta$ . We compare the result with the CV-method. See also the results in [13], [22].

As observed in this example, the number of iterations is dramatically decreased when the modified algorithm is used. In the case of two phase segmentation, the functional  $L(\phi, c, \lambda)$

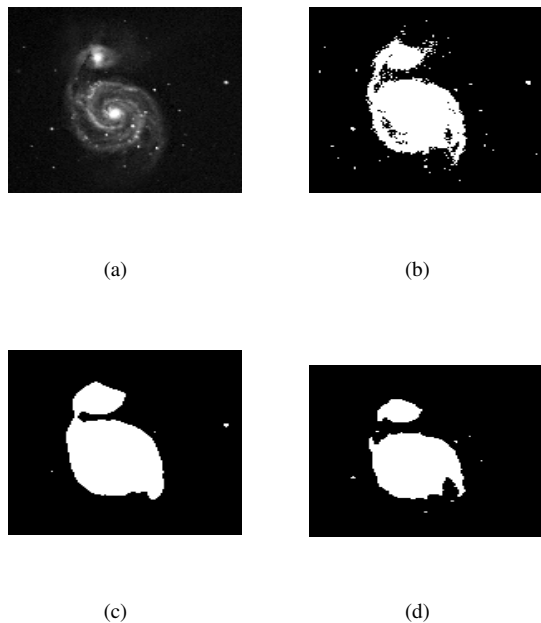


Fig. 4. An image of a galaxy is processed using the modified (quicker) version of the augmented Lagrangian algorithm. (a) The original image  $u_0$ . (b) A piecewise constant approximation to  $u_0$  after 20 iterations with  $\beta = 1 \cdot 10^{-6}$ . (c) Another piecewise constant approximation with  $\beta = 3 \cdot 10^{-4}$ . We compare with the result using CV in (d), using 300 iterations.

is convex w.r.t.  $c$  and  $\phi$ . For multiple phases,  $L(\phi, c, \lambda)$  is only locally convex w.r.t.  $c$  and  $\phi$ . The above technique can accelerate the convergence in the two-phase case. However, more careful tests are needed to draw some solid conclusions concerning the class of problems where the speedup-technique is applicable. Another and better way to accelerate the convergence of the algorithms would be using a Newton-type of iteration for  $\phi$ .

From the two numerical tests above, we see that our method is capable of computing segmentations of the same quality as the CV method. Both our method and CV require the user to supply an estimation of how much regularization to be performed. This is related to the noise level and to which kind of objects to be detected. In the above tests, we have set the parameter  $\epsilon$  controlling the approximation of the  $\delta$  and  $H$  functions to be 1 as in [2, p.272]. In addition,  $\nu$  controlling the area in (9) is set to 0. We have kept  $\mu$  in our model as a constant,  $\mu = 0.1$ . From Fig. 3 (b) and (c) we see that whether or not the reinitialization is performed can influence the resulting segmentation. Using our method, this is not an issue.

In the next section we will show a few numerical results using two functions  $\phi_1$  and  $\phi_2$  for detecting upto four different regions.

### B. Two Level Set Functions

**Example 3.** Using our method, we can start with continuous functions for the level set functions. In fact, in all but the last numerical experiments shown here, we initialize the level set functions as zero functions. At convergence, the level

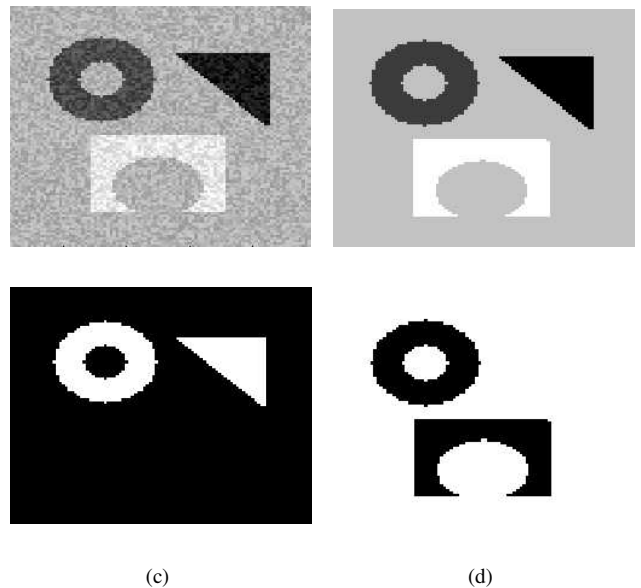


Fig. 5. Two level set functions are utilized for detecting four regions having distinct intensities. The values for the parameters are  $\mu = 1 \cdot 10^{-4}$  and  $\beta = 1 \cdot 10^{-4}$ .

set functions are discontinuous functions having values  $\pm 1$ . In some sense, we are not moving curves. For a given initial value for the level set functions, our algorithms determines a correction direction, and moves the level set function up or down according to this. This makes it easy to capture objects with arbitrary topology. This has also been observed in [8]. For other level set methods there is a need to start with curves inside the object, or like in [2], use approximations to  $H$ - and  $\delta$ -functions having global support to be able to identify inside "holes". Alternatively, topological derivatives can be used in order to identify complex geometries in level set methods [8], [37].

To demonstrate the capability of handling complicated geometry, we have tested the algorithms using two level set functions on the image depicted in the upper left corner of Fig. 5. The same image has been segmented using other methods in [22], [23] giving similar results. The image contains convex and concave shapes and a "hole". We have imposed the image with noise,  $SNR \approx 7.5$ . Our method is able find all the objects with rather good accuracy even under the moderate amount of noise. The sharp corners, concave shapes and "hole" presents no problems.

**Example 4.** As was pointed out in §III, and illustrated in Fig. 1, our regularization functional measures the length of edges accurately. In some applications it might be important that all edges are treated in a similar fashion, independently of other properties of the image, like the intensity-value. We will in this example illustrate a case where our regularizer treats all edges accurately, while the regularizer of CV does not. To do so, we chose the image  $u_0$  so that the difference between the two regularizers is clearly emphasized. An image containing long and thin (only 1 pixel wide) regions,  $\Omega_1$  and  $\Omega_2$ , having two distinct intensity values  $c_1$  and  $c_2$  in addition



to the background  $\Omega_3$  with intensity value  $c_3$  are used for this purpose. We choose  $c_1 \approx c_2$ , such that a misclassification of pixels between  $\Omega_1$  and  $\Omega_2$  will not count much in the fidelity terms of the functionals (23) and (9). Thus the regularization will have a great impact for the minimization problem. In this example, the exact intensity values  $c$  are assumed to be known prior to the segmentation. With no noise and no regularizer, both methods gave a perfect result. However, this is not the case if noise is present, and regularization is needed. The image  $u_0$  in Fig. 6 (b) is segmented using both the CV method (c) and our method (d). As can be observed in Fig. 6(d) and (c), our regularizer treats all edges in a similar manner, while the regularizer of CV does not. The boundaries  $\partial\Omega_1 \cap \partial\Omega_2$  and  $\partial\Omega_2 \setminus \partial\Omega_1 \cap \partial\Omega_2$  (measured once) in Fig. 6 (c) are more oscillatory than the boundary  $\partial\Omega_1 \setminus \partial\Omega_1 \cap \partial\Omega_2$  (measured twice). We emphasize that in most cases there will be little or no difference between the two regularizers. We are reporting a peculiar example in (a)-(d). As seen in Fig. 6 (e) and (f), where the intensity values are the same as in Fig. 6 (c) and (d), the areas covered by  $\Omega_1$ ,  $\Omega_2$ , and  $\Omega_3$  are the same, but the length of each phase is not the same. Both models manage to do a perfect segmentation in this case. In the example above, i.e. Fig. 6 (c) and (d), we maximized the difference between our model and the CV model by carefully choosing  $u_0$ . If the intensity values  $c_1$  and  $c_2$  are not as close as in this example, it is possible to find a  $\beta$  such that the regularizer in CV will produce the correct result.

**Example 5.** To conclude the numerical section we process one slice of a medical MR-image. We have picked out an image from the Brainweb database. This is an online database from where synthetic MR-images of the human brain can be obtained, [38]. The input image  $u_0$  in Fig. 7 contains 25% noise and is 20% inhomogeneous. The image is difficult to segment due to the fact that the interfaces between different tissues are topologically complicated and the intensity values are not homogeneous inside each phase. Another difficulty is the present noise,  $SNR \approx 15$ . When the noise level is low, both our method and the isodata-method [35] are able to produce rather good segmented images. The isodata-method may misclassify a number of points. However, the difference with our method is not so big. When the noise level is high, the isodata-method produces highly oscillating results, but our method can produce much better images. In order to accelerate the convergence for our method, we use the isodata-method [35] to construct initial guesses  $\{c_i\}_{i=1}^4$  and  $\{\phi_i\}_{i=1}^2$  for our method. By doing so, we can segment complex noisy images in a quick and reliable way.

The numerical result is shown in Fig. 7. In (b), we show the result of the isodata segmentation of  $u_0$ . We can here observe that the main structures are preserved, but also highly oscillating patterns occur. We use the results from the isodata-method to construct initial values for  $\phi_1$  and  $\phi_2$ , run our algorithm with these initial values, and end up with the image depicted in Fig. 7 (c). Observe that the main structures are still very well preserved, but most of the (unwanted) highly oscillating patterns are removed. By initializing the algorithm in this way, we both accelerate the convergence of our algorithm, and in addition more or less avoid the problem

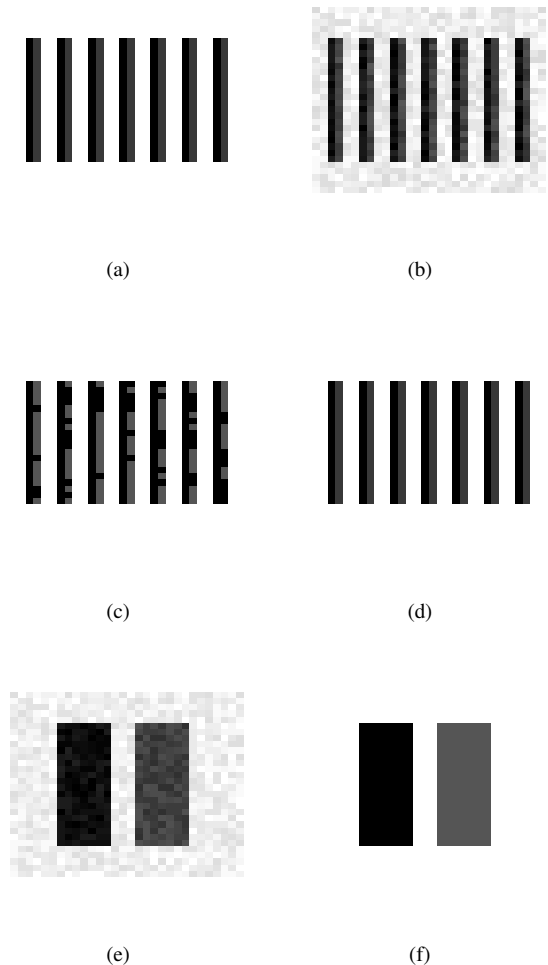


Fig. 6. With some special images containing long, thin (1 pixel wide) structures our regularizer is better at handling edges than the regularizer used in the CV model. A little amount of noise is added to the image (a) to produce (b). This image is segmented using both CV and our method. In (c) we show the result of CV and in (d) we show the result of our method. The image in (e) is very well segmented using CV (f).

of local minimizers. In this example, we used the parameters  $\beta = 5 \cdot 10^{-3}$ ,  $\mu = 5$ , and 1000 iterations.

## VI. CONCLUSIONS

In this work and also in [23], we have proposed some piecewise constant level set methods for capturing interfaces. These methods are related both to the phase-field methods and the level set methods. Numerical experiments indicate that these methods are able to trace interfaces with complicated geometries and sharp corners. The level set functions are discontinuous at convergence, but the minimization functionals are smooth and at least locally convex. In this work, we have only tested the methods for image segmentation, and we have used simple gradient methods for the iterative algorithms. Due to the fact that the functionals are smooth and the method is truly variational, it is possible to design fast iterative algorithms for solving the minimization problems, i.e. by using Newton type of iterations instead of the simple gradient methods. The numerical results indicate that our methods

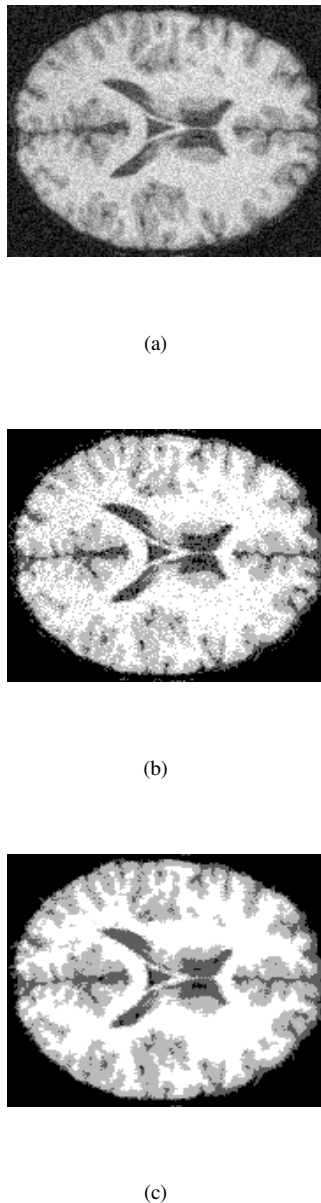


Fig. 7. In this example, the image (a) is first segmented using the isodata method (b). Then the result is further processed using our method, and the final result is shown in (c).

give as good results as methods using continuous level set functions. The methods proposed in this article and in [23] are not superior, but are interesting alternatives to other methods used for interface problems. Our methods are not moving the interfaces during the iterative procedure, and thus have some advantages in treating geometries, for example in situations where inside "holes" need to be identified. Using our approach, we have removed the reinitialization procedure sometimes needed in traditional level set methods. We have proposed and demonstrated the validity of an alternative approach for interface identification, in particular for image segmentation.

## ACKNOWLEDGMENTS

We would like to thank the unknown referees for their suggestions, tips and for pointing out references.

## REFERENCES

- [1] S. Osher and J. A. Sethian, "Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations," *J. Comput. Phys.*, vol. 79, no. 1, pp. 12–49, 1988.
- [2] T. F. Chan and L. A. Vese, "Active contours without edges," *IEEE Trans. Image Processing*, vol. 10, no. 2, pp. 266–277, 2001.
- [3] M. Sussman, P. Smereka, and S. Osher, "A level set approach for computing solutions to incompressible two phase flow," *J. Comput. Phys.*, vol. 114, pp. 146–159, 1994.
- [4] T. F. Chan and X.-C. Tai, "Level set and total variation regularization for elliptic inverse problems with discontinuous coefficients," *J. Comput. Phys.*, vol. 193, pp. 40–66, 2003.
- [5] R. P. Fedkiw, G. Sapiro, and C.-W. Shu, "Shock capturing, level sets, and PDE based methods in computer vision and image processing: a review of Osher's contributions," *J. Comput. Phys.*, vol. 185, no. 2, pp. 309–341, 2003.
- [6] M. Burger, "A framework for the construction of level set methods for shape optimization and reconstruction," *Interfaces Free Bound.*, vol. 5, no. 3, pp. 301–329, 2003.
- [7] M. Hintermüller and W. Ring, "A second order shape optimization approach for image segmentation," *SIAM J. Appl. Math.*, vol. 64, no. 2, pp. 442–467 (electronic), 2003/04.
- [8] M. Burger, B. Hackl, and W. Ring, "Incorporating topological derivatives into level set methods," *J. Comput. Phys.*, vol. 194, no. 1, pp. 344–362, 2004.
- [9] S. Osher and R. Fedkiw, "Level set methods: An overview and some recent results," *J. Comput. Phys.*, vol. 169, no. 2, pp. 463–502, 2001.
- [10] —, *Level Set Methods and Dynamic Implicit Surfaces*, ser. Applied Mathematical Sciences. Springer, 2003, vol. 153.
- [11] J. A. Sethian, *Level set methods and fast marching methods*, 2nd ed., ser. Cambridge Monographs on Applied and Computational Mathematics. Cambridge: Cambridge University Press, 1999, vol. 3, evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science.
- [12] X.-C. Tai and T. F. Chan, "A survey on multiple level set methods with applications for identifying piecewise constant functions," *International J. Numerical Analysis and Modelling*, vol. 1, no. 1, pp. 25–48, 2004.
- [13] F. Gibou and R. Fedkiw, "A fast hybrid k-means level set algorithm for segmentation," *Stanford Technical Report*, 2002.
- [14] B. Song and T. Chan, "A fast algorithm for level set based optimization," *CAM-UCLA*, no. 68, 2002, under revision for publication in *SIAM J. Sci. Comput.*
- [15] S. Baldo, "Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids," *Ann. Inst. H. Poincaré Anal. Non Linéaire*, vol. 7, no. 2, pp. 67–90, 1990.
- [16] B. Bourdin and A. Chambolle, "Design-dependent loads in topology optimization," *ESAIM Control Optim. Calc. Var.*, vol. 9, pp. 19–48 (electronic), 2003.
- [17] J. Müller and M. Grant, "Model of surface instabilities induced by stress," *PHYSICAL REVIEW LETTERS*, vol. 82, no. 8, 1999.
- [18] C. Samson, L. Blanc-Féraud, G. Aubert, and J. Zerubia, "A variational model for image classification and restoration," *TPAMI*, vol. 22, no. 5, pp. 460–472, may 2000.
- [19] D. Mumford and J. Shah, "Optimal approximation by piecewise smooth functions and associated variational problems," *Comm. Pure Appl. Math.*, vol. 42, p. 577685, 1989.
- [20] A. Chambolle, "Image segmentation by variational methods: Mumford and Shah functional and the discrete approximations," *SIAM J. Appl. Math.*, vol. 55, no. 3, pp. 827–863, 1995.
- [21] —, "Finite-differences discretizations of the Mumford-Shah functional," *M2AN Math. Model. Numer. Anal.*, vol. 33, no. 2, pp. 261–288, 1999.
- [22] L. A. Vese and T. F. Chan, "A multiphase level set framework for image segmentation using the mumford and shah model," *International Journal of Computer Vision*, vol. 50, no. 3, pp. 271–293, 2002.
- [23] J. Lie, M. Lysaker, and X.-C. Tai, "A variant of the levelset method and applications to image segmentation," *CAM-UCLA*, 2003.
- [24] J. T. Chung and L. A. Vese, "Image segmentation using a multilayer level-set approach," *UCLA-CAM 03-53*, 2003.

- [25] Y. R. Tsai, "Rapid and accurate computation of the distance function using grids." *J. Comput. Phys.*, vol. 178, pp. 175–195, 2002.
- [26] Y.-H. R. Tsai, L.-T. Cheng, S. Osher, and H.-K. Zhao, "Fast sweeping algorithms for a class of Hamilton-Jacobi equations," *SIAM J. Numer. Anal.*, vol. 41, no. 2, pp. 673–694 (electronic), 2003.
- [27] J. A. Sethian, "A review of level set and fast marching methods for image processing," in *Modern methods in scientific computing and applications (Montréal, QC, 2001)*, ser. NATO Sci. Ser. II Math. Phys. Chem. Dordrecht: Kluwer Acad. Publ., 2002, vol. 75, pp. 365–396.
- [28] M. Kass, A. Witkin, and D. Terzopoulos, "Snakes: Active contour models," *Int. J. Comput. Vision*, vol. 1, pp. 321–331, 1988.
- [29] V. Caselles, F. Catte, T. Coll, and F. Dibos, "A geometric model for active contours in image processing," *Numerische Mathematik*, vol. 66, pp. 1–31, 1993.
- [30] W. P. Ziemer, *Weakly differentiable functions*, ser. Graduate Texts in Mathematics. New York: Springer-Verlag, 1989, vol. 120, sobolev spaces and functions of bounded variation.
- [31] D. P. Bertsekas, *Constrained optimization and Lagrange multiplier methods*, ser. Computer Science and Applied Mathematics. New York: Academic Press Inc., 1982.
- [32] J. Nocedal and S. J. Wright, *Numerical optimization*, ser. Springer Series in Operations Research. New York: Springer-Verlag, 1999.
- [33] K. Kunisch and X.-C. Tai, "Sequential and parallel splitting methods for bilinear control problems in Hilbert spaces," *SIAM J. Numer. Anal.*, vol. 34, pp. 91–118, 1997.
- [34] T. F. Chan and X.-C. Tai, "Identification of discontinuous coefficients in elliptic problems using total variation regularization," *SIAM J. Sci. Comput.*, vol. 25, pp. 881–904, 2003.
- [35] F. R. Dias Velasco, "Thresholding using the ISODATA clustering algorithm," *IEEE Trans. Systems Man Cybernet.*, vol. 10, no. 11, pp. 771–774, 1980.
- [36] J. Lie, M. Lysaker, and X.-C. Tai, "A binary level set model and some applications to image processing," *UCLA, CAM 04-31*, 2004.
- [37] J. Sokołowski and A. Żochowski, "On the topological derivative in shape optimization," *SIAM J. Control Optim.*, vol. 37, no. 4, pp. 1251–1272 (electronic), 1999.
- [38] "Brainweb: Simulated brain database." [Online]. Available: [www.bic.mni.mcgill.ca/brainweb](http://www.bic.mni.mcgill.ca/brainweb)