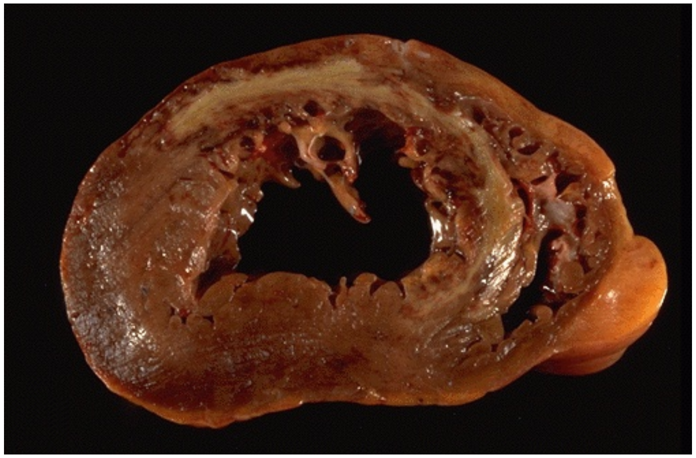
A level set framework for infarction modeling; an inverse problem



by
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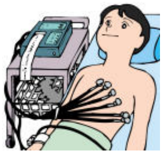
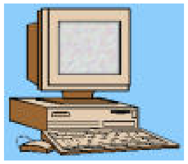
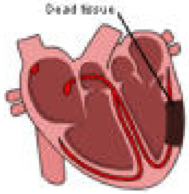
&
Bjørn Fredrik Nielsen

Is it possible to use mathematical models and computer simulation to estimate the size, the shape and the location of an infarction?

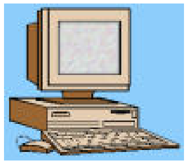






- ECG measurements
- Mathematical models and computer simulation
- Estimate size, shape and location of an infarction
- Guide surgery

Is it possible to use mathematical models and computer simulation to estimate the size, the shape and the location of an infarction?



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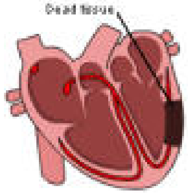

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The Bidoman model versus the Monodomain model:

$$\chi C v_t + \chi I_{\text{ion}}(v) = \nabla\cdot(M_i\nabla v) + \nabla\cdot(M_i\nabla u_e) \quad \text{in } \Omega \quad (1)$$

$$\nabla\cdot(M_i\nabla v) + \nabla\cdot((M_i+M_e)\nabla u_e) = 0 \quad \text{in } \Omega \quad (2)$$

$$(Mi\nabla v + M_i\nabla u_e)\cdot n = 0 \quad \text{on } \partial\Omega \quad (3)$$

$$v(x,0) = v_0(x) \quad \text{in } \Omega. \quad (4)$$

Assume that $M_e = \lambda M_i$, then (2) gives

$$\nabla\cdot(M_i\nabla v) + \nabla\cdot((M_i+\lambda M_i)\nabla u_e) = 0$$

$$\Rightarrow \nabla\cdot(M_i\nabla v) + (1+\lambda)\nabla\cdot(M_i\nabla u_e) = 0$$

$$\Rightarrow \nabla\cdot(M_i\nabla u_e) = \frac{-1}{1+\lambda}\nabla\cdot(M_i\nabla v). \quad (5)$$

Use (5) in (1) and (3) to get

$$\chi C v_t + \chi I_{\text{ion}}(v) = \frac{\lambda}{1+\lambda}\nabla\cdot(M_i\nabla v) \quad \text{in } \Omega \tag{6}$$

$$\frac{\lambda}{1+\lambda}(M_i\nabla v)\cdot n = 0 \quad \text{on } \partial\Omega. \tag{7}$$

Let $I(v) = \frac{1}{C}I_{\text{ion}}(v)$, $k = \frac{\lambda}{\chi C(1+\lambda)}M_i$, and use (6)-(7) to find the following model:

$$v_t + I(v) = \nabla\cdot[k\nabla v] \quad \text{in } \Omega \tag{8}$$

$$(k\nabla v)\cdot n = 0 \quad \text{on } \partial\Omega \tag{9}$$

$$v(x,0) = v_0(x) \quad \text{in } \Omega. \tag{10}$$

$$I(v) = -A^2(v+v_{\text{rest}})(v+v_{\text{th}})(v-v_{\text{peak}}). \tag{11}$$

This simplified Monodomain equation can be used to describe the electrical activity in a healthy heart

$$v_t + I(v) = \nabla\cdot[k\nabla v] \quad \text{in } \Omega, \tag{12}$$

$$(k\nabla v)\cdot n = 0 \quad \text{on } \partial\Omega, \tag{13}$$

$$v(x,0) = v_0(x) \quad \text{in } \Omega. \tag{14}$$

How should we modify (12)-(14) in order to model the effect of infarctions?

1) Block or reduce the ion transport $I(v)$ in infarcted areas.

2) The conductivity function k should depend on whether or not infarctions are present.

To do so, introduce the *infarction parameters* $p_1, p_2, \ldots, p_M$.

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1) Block or reduce the ion transport in infarcted areas, i.e. replace $I(v)$ with $gI(v)$;

$$g(x; p_1, p_2, \ldots, p_M) = \begin{cases} 0 & \text{if } x \text{ in } D \\ 1 & \text{if } x \text{ in } \Omega \setminus D \end{cases} \tag{15}$$

2) The conductivity function k should depend on whether or not infarctions are present, i.e. let k be given as

$$k(x; p_1, p_2, \ldots, p_M) = \begin{cases} k_1 & \text{if } x \text{ in } D \\ k_2 & \text{if } x \text{ in } \Omega \setminus D \end{cases} \tag{16}$$

D denotes the infarcted area of the heart, i.e.

$$D = D(p_1, p_2 \ldots, p_M).$$

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Model for the electrical potential in the heart Ω with an infarcted region D:

$$v_t + gI(v) = \nabla\cdot[k\nabla v] \quad \text{in } \Omega$$

$$k\nabla v\cdot n = 0 \quad \text{on } \partial\Omega$$

$$v(x,0) = v_0(x) \quad \text{in } \Omega$$

$$g(x; p_1, p_2, \ldots, p_M) = \begin{cases} 0 & \text{if } x \text{ in } D \\ 1 & \text{if } x \text{ in } \Omega \setminus D \end{cases} \tag{17}$$

$$k(x; p_1, p_2, \ldots, p_M) = \begin{cases} k_1 & \text{if } x \text{ in } D \\ k_2 & \text{if } x \text{ in } \Omega \setminus D \end{cases} \tag{18}$$

$$D = D(p_1, p_2 \ldots, p_M)$$

movie1 movie2

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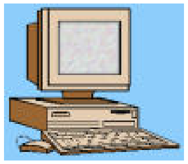

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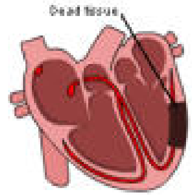

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What do we consider as important factors regarding the geometrical representation?

$$J(p_1, p_2, \ldots, p_M) = \frac{1}{2}\int_0^{t^*}\int_{\partial\Omega} \left[d(x,t) - v(x,t;p_1,p_2\ldots,p_M)\right]^2 d$$

$$\min_{p_1,p_2,\ldots,p_M} J(p_1,p_2,\ldots,p_M).$$

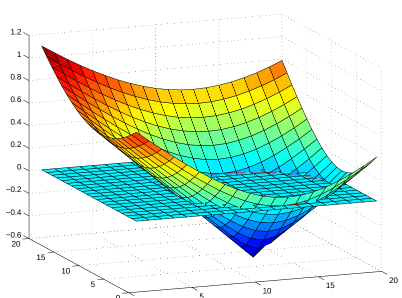
1. Number of free parameters
2. Flexibility
3. Is it possible to observe a small change of the infarction parameters (shape, size and location) in the boundary measurements?
4. Implementation aspects

Let Γ be a closed curve in $\mathbb{R}^2$. If ϕ is a function such that

$$\begin{cases} \text{if} \quad \phi(x) < 0 & \Rightarrow \quad \text{x is inside} \quad \Gamma, \\ \text{if} \quad \phi(x) = 0 & \Rightarrow \quad \text{x is at } \Gamma, \\ \text{if} \quad \phi(x) > 0 & \Rightarrow \quad \text{x is outside } \Gamma, \end{cases} \tag{19}$$

then Γ is implicitly represented by ϕ, in the sense that

$$\Gamma = \{x;\ \phi(x) = 0\}.$$



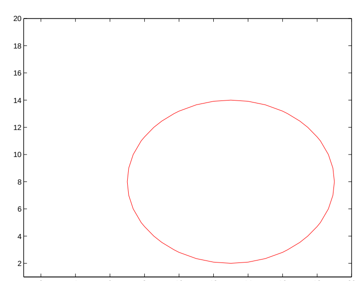


Figure 1: A level set function ϕ and its zero level set $\Gamma = \{x;\ \phi(x) = 0\}$

How do we describe the infarcted region D in terms of the parameters $p_1, p_2 \ldots, p_M$?

Assume we have a Finite Element mesh, and introduce the level set function

$$\phi = \phi(x; p_1, p_2 \ldots, p_M) = \sum_{i=1}^{M} p_i N_i(x),$$

where $\{N_i(x)\}_{i=1}^M$ denotes the basis functions. Note that ϕ is usually defined by the following formula

$$\begin{cases} \phi(x) = -\operatorname{dist}(\Gamma, \mathrm{x}) & \text{if } x \text{ is in } D \\ \phi(x) = \quad 0 & \text{if } x \text{ is at } \partial D \\ \phi(x) = \quad \operatorname{dist}(\Gamma, \mathrm{x}) & \text{if } x \text{ is outside } D. \end{cases}$$

$$p_1, p_2 \ldots, p_M \;\rightarrow\; \phi \;\rightarrow\; D \;\rightarrow\; \text{Infarcted regions}$$

Let H denote the Heaviside function:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$g(x; p_1, p_2 \ldots, p_M) = H(\phi) = \begin{cases} 0 & \text{if } \phi < 0 \\ 1 & \text{if } \phi \geq 0 \end{cases}$$

and

$$k(x; p_1, p_2 \ldots, p_M) = k_1(1 - H(\phi)) + k_2 H(\phi) = \begin{cases} k_1 & \text{if } \phi < 0 \\ k_2 & \text{if } \phi \geq 0 \end{cases}$$

$$H_\alpha(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x}{\alpha}\right)$$

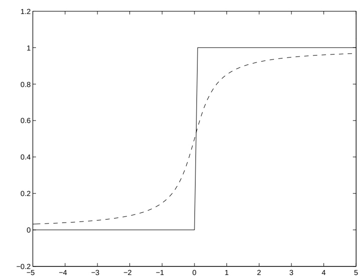
$$\delta_\alpha(x) = H'_\alpha(x) = \frac{1}{\pi}\left(\frac{\alpha}{\alpha^2 + x^2}\right)$$

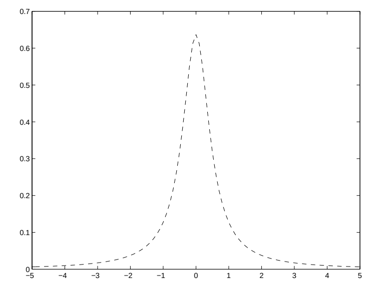
A realistic model should include a smooth boarder-zone between the healthy and the infarcted tissue

$$g_\alpha(\phi) = H_\alpha(\phi) \approx \begin{cases} 0 & \text{if } x \text{ in } D, \\ 1 & \text{if } x \text{ in } \Omega \setminus D, \end{cases}$$

and

$$k_\alpha(\phi) = k_1(1 - H_\alpha(\phi)) + k_2 H_\alpha(\phi) \approx \begin{cases} k_1 & \text{if } x \text{ in } D, \\ k_2 & \text{if } x \text{ in } \Omega \setminus D. \end{cases}$$


(a) Heaviside functions


(b) Delta function

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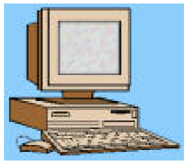
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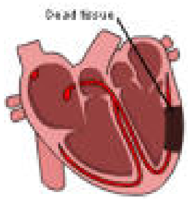

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The complete form of this minimization problem can now be written as

$$\min_{p_1, p_2, \ldots, p_M} J(p_1, p_2, \ldots, p_M) =$$

$$\min_{p_1, p_2, \ldots, p_M} \left(\frac{1}{2} \int_0^{t^*} \int_{\partial\Omega} \left[d(x,t) - v(x,t; p_1, p_2 \ldots, p_M) \right]^2 dx\, dt$$

subject to the constraint that v solves

$$v_t + g_\alpha(\phi) I(v) = \nabla \cdot [k_\alpha(\phi) \nabla v] \quad \text{in } \Omega, \quad (20)$$

$$k_\alpha(\phi) \nabla v \cdot n = 0 \quad \text{along } \partial\Omega, \quad (21)$$

$$v(x, 0) = v_0(x) \quad \text{in } \Omega, \quad (22)$$

where

$$\phi = \sum_{i=1}^{M} p_i N_i(x), \quad g_\alpha(\phi) = H_\alpha(\phi) \text{ and } k_\alpha(\phi) = k_1(1 - H_\alpha(\phi)) + k_2 H$$

Different techniques can be used to solve this minimization problem, let us focus on a gradient-type of method, i.e. find

$$\frac{\partial J}{\partial p_1}, \frac{\partial J}{\partial p_2}, \ldots, \frac{\partial J}{\partial p_M},$$

and for all old parameter values, we will get new values by

$$\sum_{i=1}^{M} p_i^{\text{new}} N_i(x) = \sum_{i=1}^{M} p_i^{\text{old}} N_i(x) - \beta \sum_{i=1}^{M} \frac{\partial J}{\partial p_i}(p_1^{\text{old}}, p_2^{\text{old}}, \ldots, p_M^{\text{old}}) N_i(x).$$

Remember that

$$\phi = \phi(x; p_1, p_2 \ldots, p_M) = \sum_{i=1}^{M} p_i N_i(x),$$

so actually we update the level set function, whereupon D is changed, meaning that the infarcted region is modified!

Algorithm 1

1. *Choose an initial value* v_0, *and a constant value for* β
2. *Choose* $\mathbf{p}^0$, *where* $\mathbf{p}^0 = (p_1^0, p_2^0, \ldots, p_M^0)^T$
3. *For* $n = 0, 1, \ldots$ *until convergence do:*
   - (a) *Solve* **Forward Problem** *for* $v^n = v(\mathbf{p}^n)$
   - (b) *Solve* **Adjoint Problem** *for* $w^n = w(\mathbf{p}^n)$
   - (c) *For* $i = 1, 2, \ldots, M$ *compute*

$$\frac{\partial J}{\partial p_i}(\mathbf{p}^n) = -\int_0^{t^*} \int_\Omega \Big(I(v^n)) w^n + (k_2 - k_1) \nabla v^n \cdot \nabla w^n \Big) \delta_\alpha(\phi^n) \phi$$

   - (d) *Update the infarction parameters by*

$$\mathbf{p}^{n+1} = \mathbf{p}^n - \beta \nabla J(\mathbf{p}^n),$$

where $\nabla J(\mathbf{p}^n) =$
$[\partial J/\partial p_1(\mathbf{p}^n), \partial J/\partial p_2(\mathbf{p}^n), \ldots, \partial J/\partial p_M(\mathbf{p}^n)]^T$

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Due to the ill-posed nature of this inverse problem, the process of locate infarctions suffers from noise artifacts

$$|\partial D| = \int_\Omega |\nabla H_\alpha(\phi)| \, dx = \int_\Omega \delta_\alpha(\phi) |\nabla \phi| \, dx \qquad (23)$$

$$|D| = \int_\Omega (1 - H_\alpha(\phi)) \, dx \qquad (24)$$

$$J\epsilon(p_1, p_2, \ldots, p_M) = J(p_1, p_2, \ldots, p_M) + \epsilon \int_0^{t^*} \int_\Omega (1 - H_\alpha(\phi)) \, dx \, dt$$

$$\frac{\partial J_\epsilon}{\partial p_i} = \frac{\partial J}{\partial p_i} - \epsilon \int_0^{t^*} \int_\Omega \delta_\alpha(\phi) \phi_{p_i} \, dx \, dt$$
$$= -\int_0^{t^*} \int_\Omega \Big(I(v) w + (k_2 - k_1) \nabla v \cdot \nabla w + \epsilon \Big) \delta_\alpha(\phi) \phi_{p_i} \, dx \, dt$$

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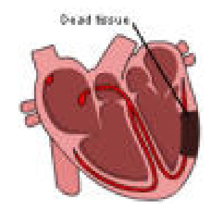

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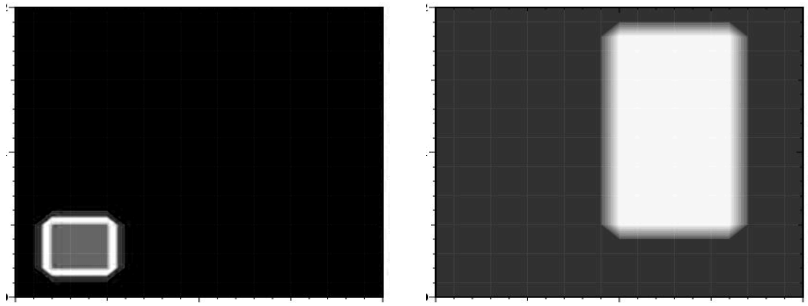
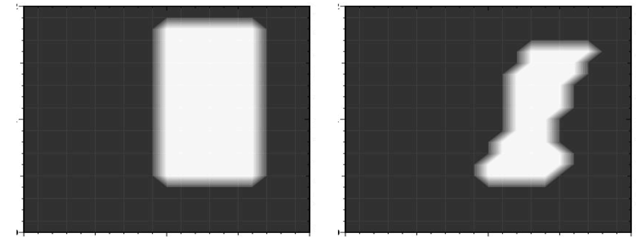


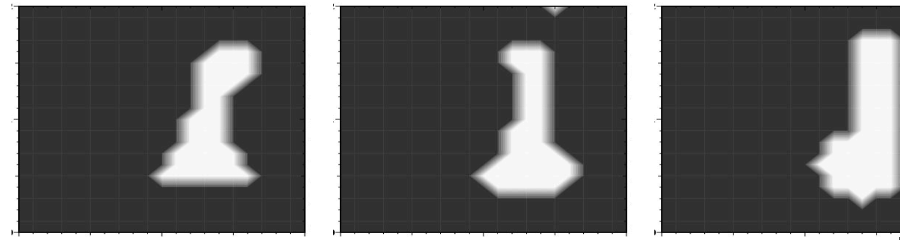
(a) (b)

Let $\tilde{d}(x_s, t_l) = d(x_s, t_l) + r(x_s, t_l)$ denote the corrupted measurements recorded at the boundary, in position x_s at time t_l. The relative error in the measurements is given by

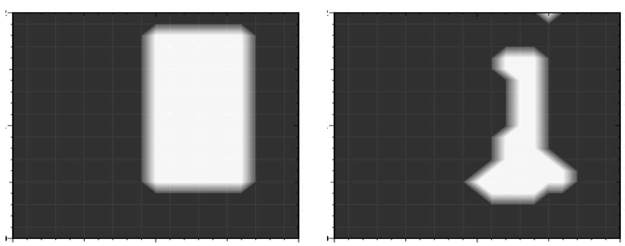
$$e = \frac{\sum_{s=1}^{S} \sum_{l=1}^{L} |r(x_s, t_l)|}{\sum_{s=1}^{S} \sum_{l=1}^{L} |d(x_s, t_l)|}. \tag{25}$$



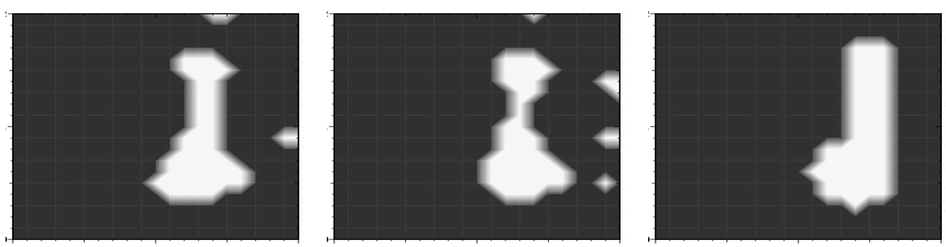
(c) initial guess (d) 4 iterations



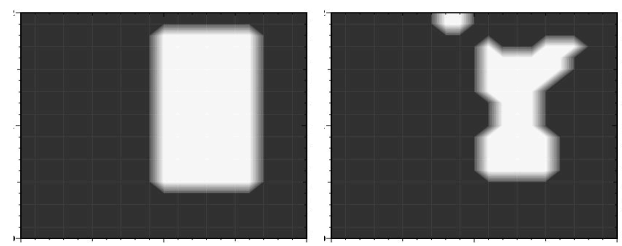
(e) 7 iterations (f) 200 iterations (g) true infarctio



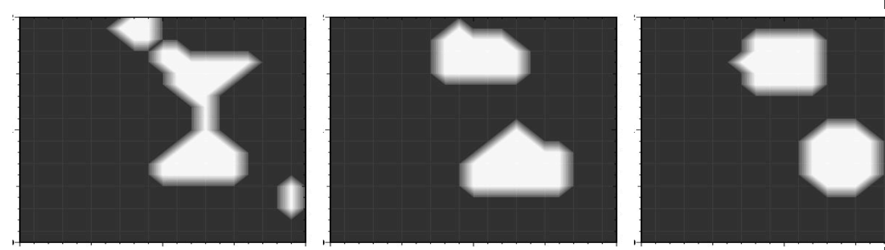
(a) initial guess (b) $e = 0.01$



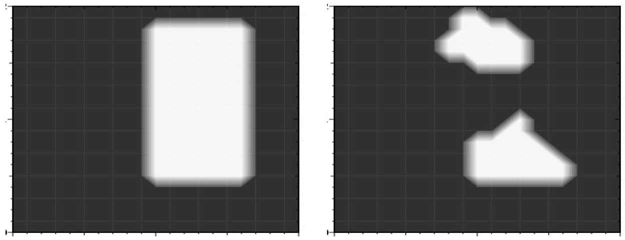
(c) $e = 0.05$ (d) $e = 0.10$ (e) true infarction



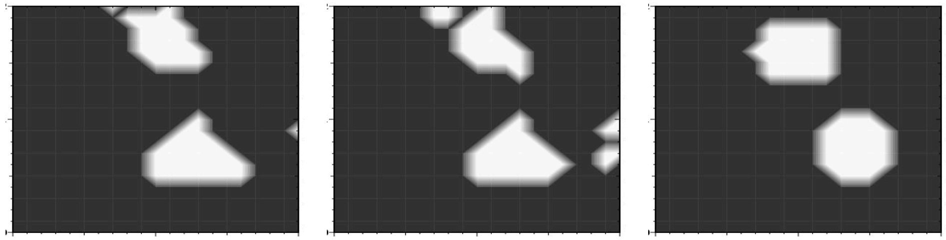
(a) initial guess (b) 10 iterations



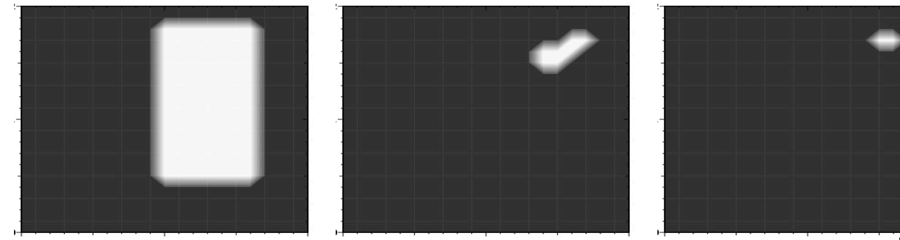
(c) 20 iterations (d) 200 iterations (e) true infarctio



(a) initial guess (b) $e = 0.01$



(c) $e = 0.05$ (d) $e = 0.10$ (e) true infarction



(a) initial guess (b) 100 iterations (c) true infarctio

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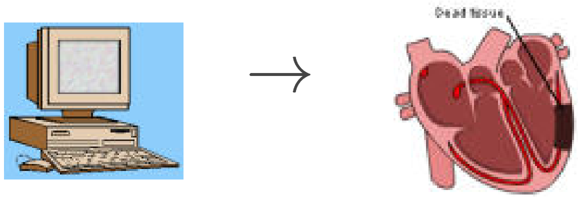

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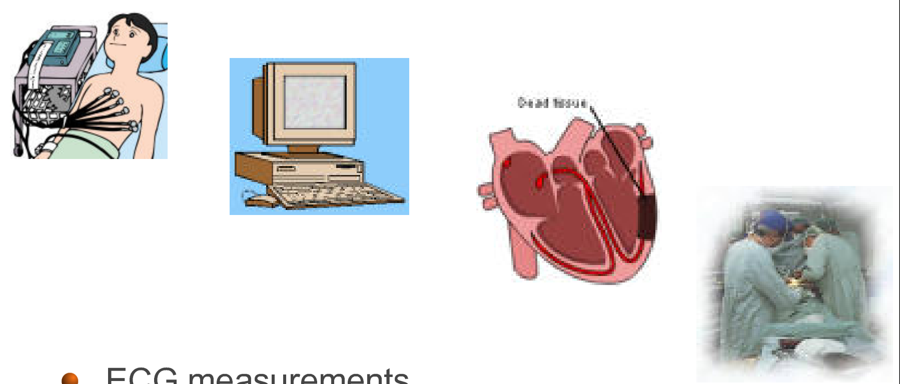

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