# Level Set Methods For Inverse Obstacle Problems

Martin Burger

UCLA University of California Los Angeles

University Linz, Austria

## Outline





## Introduction

Many applications deal with the reconstruction and optimization of geometries (shapes, topologies);

e.g.:

- Identification of piecewise constant parameters
- Inverse obstacle scattering
- Inclusion detection
- Structural optimization
- Optimal design of photonic bandgap structures
- .....



## Introduction

In such applications, there is no natural a-priori information on shapes or topological structures of the solution (number of connected components, star-shapedness, convexity, ...)

→ flexible representations of the shapes needed!

## Level Set Methods

Osher & Sethian, JCP 1987
Osher & Fedkiw, Springer, 2002

Basic idea: implicit shape representation

$$\Omega = \{ \ x \mid \Phi(x) < \ 0 \ \}$$
$$\partial\Omega \ = \{ \ x \mid \Phi(x) = \ 0 \ \}$$

with continuous level-set function

$$\Phi \ : \mathbb{R}^d \to \ \mathbb{R}$$



## Level Set Methods

Evolution of a curve $\Gamma \ = \partial\Omega$

with velocity $\mathbf{V}$; $\Gamma(t) = \{ \ \mathbf{x}(S,t) \mid S \in \Sigma\}$

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{V}(\mathbf{x},t) \ ; \ \mathbf{x} \in \Gamma(t)$$

Implicit representation

$$\Gamma(t) = \{ \ \mathbf{x} \mid \Phi(\mathbf{x},t) = 0\}$$
$$\Rightarrow 0 = \frac{d}{dt}\Phi(\mathbf{x}(S,t),t) = \frac{\partial \Phi}{\partial t} + \nabla\Phi . \frac{\partial \mathbf{x}}{\partial t}$$
$$\Rightarrow \frac{\partial \Phi}{\partial t} + \mathbf{V}(\mathbf{x},t) . \nabla\Phi = 0$$



## Geometric Motion

Tangential velocity $\mathbf{V}.\tau$ corresponds to change of parametrization only, i.e.

$$\{ \ \Phi_1 < \ 0\} \ = \{ \ \Phi_2 < \ 0\}$$
$$\text{if} \qquad \frac{\partial \Phi_j}{\partial t} + \mathbf{V_j}.\nabla\Phi_j = 0$$
$$\text{and} \qquad \mathbf{V_1}.\mathbf{n} = \mathbf{V_2}.\mathbf{n}$$

→ Restriction to normal velocities is natural: $\mathbf{V} = v\mathbf{n}$



## Geometric Motion

Normal can be computed from level set function $\Phi$ :

$$\frac{\partial}{\partial S}\Phi(\mathbf{x}(S,t),t) = 0$$
$$\Rightarrow \quad \nabla\Phi . \frac{\partial \mathbf{x}}{\partial S} = 0$$
$$\frac{\partial \mathbf{x}}{\partial S} = \lambda\tau \Rightarrow \nabla\Phi \text{ is normal direction}$$
$$\Rightarrow \mathbf{n}(S,t) = \frac{\nabla\Phi}{|\nabla\Phi|}(\mathbf{x}(S,t))$$

## Geometric Motion

Evolution becomes nonlinear transport equation: $\frac{\partial \Phi}{\partial t} + v \mid \nabla \Phi \mid = 0$

In general, normal velocity $v$ may depend on the geometric properties of $\Gamma$, e.g.

$v = v(x, t; \mathbf{n}, \kappa)$

$\mathbf{n} = \frac{\nabla \Phi}{|\nabla \Phi|}$ normal vector

$\kappa = \mathrm{div}(\mathbf{n}) = \mathrm{div}(\frac{\nabla \Phi}{|\nabla \Phi|})$ mean curvature



## Geometric Motion

$\Rightarrow \quad v = \tilde{v}(x, t; \nabla \Phi, D^2 \Phi)$

$\tilde{v}$ is homogeneous extension.

Fully nonlinear parabolic equation

$$\frac{\partial \Phi}{\partial t} + \tilde{v}(x, t; \ \nabla \Phi, D^2 \ \Phi) = 0$$



## Geometric Motion

Classical geometric motions:

Eikonal equation

$$\frac{\partial \Phi}{\partial t} + v(x) \mid \nabla \Phi \mid = 0, \quad v(x) \geq 0$$

computes minimal arrival times

$$T(x) = \min_{\Phi(x,t)=0} t$$

in a velocity field v



## Geometric Motion

Mean curvature flow

$$v = -\kappa$$

$$\frac{\partial \Phi}{\partial t} = \mid \nabla \Phi \mid \mathrm{div} \left( \frac{\nabla \Phi}{|\nabla \Phi|} \right)$$

## Viscosity Solutions

In general, nonlinear parabolic and Hamilton-Jacobi equations do not have classical solutions.

Standard notion of weak solutions are viscosity solutions.

First-order Hamilton-Jacobi

$$\frac{\partial \Phi}{\partial t} + H(x,t,\nabla\Phi) = 0$$

(Crandall-Lions)



## Viscosity Solutions

Viscosity subsolution

$\frac{\partial \Psi}{\partial t}(\bar{x},\bar{t}) + H(\bar{x},\bar{t},\nabla\Psi) \leq 0$
if $(\bar{x},\bar{t})$ is local maximum of $\Phi - \Psi$

Viscosity supersolution

$\frac{\partial \Psi}{\partial t}(\bar{x},\bar{t}) + H(\bar{x},\bar{t},\nabla\Psi) \geq 0$
if $(\bar{x},\bar{t})$ is local minimum of $\Phi - \Psi$

Viscosity solution = subsolution + supersolution



## Viscosity Solutions

Typical type of regularity

$$\Phi \in BUC(\mathbb{R}^d \times [0,T])$$

$$\text{If } \Phi_0 \in C^{(0,1)}(\mathbb{R}^d) \text{ then}$$

$$\Phi \in C^{(0,1)}(\mathbb{R}^d \times [0,T])$$



## Viscosity Solutions

Comparison Theorems

$$\Phi_0 \geq \tilde{\Phi}_0 \quad \text{in } \mathbb{R}^d$$

$$\Phi \text{ viscosity supersolution}$$

$$\tilde{\Phi} \text{ viscosity subsolution}$$

$$\Rightarrow \Phi \geq \tilde{\Phi} \quad \text{in } \mathbb{R}^d \times [0,T]$$

## Properties of Level Sets

Level sets are independent of chosen initial value:

$\{\Phi_0 < \ 0\} = \{\tilde{\Phi}_0 < \ 0\}$

$\Phi_t + v \mid \nabla\Phi \mid = 0$

$\tilde{\Phi}_t + v \mid \nabla\tilde{\Phi} \mid = 0$

$\Rightarrow \{\Phi < \ 0\} = \{\tilde{\Phi} < \ 0\}$ for all $t \in \mathbb{R}^+$



## Properties of Level Sets

Comparison:

$\{\Phi_0 < \ 0\} \subset \{\tilde{\Phi}_0 < \ 0\}$

$\Rightarrow \{\Phi < \ 0\} \subset \{\tilde{\Phi} < \ 0\}$ for all $t$

In particular

$\{\Phi_0 < \ p\} \subset \{\Phi_0 < \ q\}$

$\Rightarrow \{\Phi < \ p\} \subset \{\Phi < \ q\}$ for all $t$



## Higher-Order Evolutions

Comparison results still hold for second order evolutions like mean curvature.

No comparison results for higher order evolutions, e.g. surface diffusion

$v = -\Delta_S \kappa$

(4th order)



## Higher-Order Evolutions

Mullins-Sekerka: $v = [\frac{\partial u}{\partial n}]$

$\Delta u = 0$

$u = \kappa \ \text{ on } \Gamma$

(3rd order)

➡ No global level set method!

## Computing Viscosity Solutions

First-order equations

$$\Phi_t + v \mid \nabla\Phi \mid = 0$$

Explicit time discretization

$$\Phi^{k+1} = F(\Phi^k)$$

Stability bound

$$\Delta t. \max_x \mid V(x,t) \mid \leq \Delta x$$

„CFL-condition"



## Computing Viscosity Solutions

As in numerical schemes for conservation laws, first-order Hamilton-Jacobi equations

$$\Phi_t + H(\Phi_x, \Phi_y) = 0$$

are solved by a scheme of the form

$$\frac{\Phi^{k+1} - \Phi^k}{\Delta t} = \tilde{H}(D^x_+\Phi^k, D^x_-\Phi^k, D^y_+\Phi^k, D^y_-\Phi^k)$$

with approximate numerical flux $\tilde{H}$ - analogous to conservation laws (Godunov, Lax-Friedrichs, ENO, WENO)



## Computing Viscosity Solutions

Mean curvature type equation

$$\Phi_t - |\nabla\Phi| \left(\mathrm{div}\, \frac{\nabla\Phi}{|\nabla\Phi|} - v\right) = 0$$

Set $Q = \sqrt{|\nabla\Phi|^2 + \epsilon}\,, \quad \epsilon > 0$

$$\frac{\Phi_t}{Q} - \mathrm{div}\left(\frac{\nabla Q}{Q}\right) = -v$$



## Computing Viscosity Solutions

Discretization with linear finite elements ($Q$, $\nabla\Phi$ piecewise constant)

$$\sum_T \frac{1}{Q_T}\left(\int_T \Phi_t\ v\ dx + |T|\ (\nabla\Phi)_T(\nabla\Psi)_T\right) = = -\sum_T \int_T v\ \Psi\ dx$$

Convergence to viscosity solution as $\epsilon \to 0$ (Deckelnick, Dziuk, 2002)

## Redistancing

In order to prevent fattening

$(\nabla\Phi \approx 0 \text{ around } \{\Phi = 0\})$

and for initial values, $\Phi$ should be close to signed distance function $b$ .

$b$ is limit of $\Psi$ solving

$$\frac{\partial\Psi}{\partial s} + \text{sign}(\Phi)(|\ \nabla\Psi\ | - 1) = 0$$
$$\Psi(s=0) = \Phi$$

as $s \to \infty$ (Osher, Sussman, Smereka, 1994)



## Redistancing

Upwind scheme, first order

$$\Phi_{ij}(t+\Delta t) = \Phi_{ij}(t) - \Delta t\ S(\Phi^0_{ij})\ G(\Phi)_{ij}$$

$$S(\Phi) = \frac{\Phi}{\sqrt{\Phi^2 + \Delta x^2}}$$



## Redistancing

$$G(\Phi)_{ij} = \begin{cases} \sqrt{\max(a_+^2, b_-^2) + \max(c_+^2, d_-^2)} - 1 \\ \sqrt{\max(a_-^2, b_+^2) + \max(c_-^2, d_+^2)} - 1 \end{cases}$$

$$a = D_x^- \Phi_{ij} = \frac{\Phi_{ij} - \Phi_{i-1j}}{\Delta x}$$

$$b = D_x^+ \Phi_{ij} = \frac{\Phi_{i+1j} - \Phi_{ij}}{\Delta x}$$

$$c = D_y^- \Phi_{ij} = \frac{\Phi_{ij} - \Phi_{ij-1}}{\Delta x}$$

$$d = D_y^+ \Phi_{ij} = \frac{\Phi_{ij+1} - \Phi_{ij}}{\Delta x}$$



## Velocity Extension

In many cases, natural velocity is given on the interface $\Gamma$ only.

Simplest extension is constant in normal direction:

$$\nabla v.n = \nabla v.\nabla\Phi = 0, \text{ on } \mathbb{R}^d - \Gamma$$

Extension velocity is the limit of the linear transport equation

$$\frac{\partial v}{\partial S} + \text{sign}(\Phi)(\nabla v.\nabla\Phi) = 0 \qquad \text{as } S \to \infty$$

## Velocity Extension

Upwind scheme, first order

$$v_{ij}(t+\Delta t) = v_{ij} - \Delta t\ G(v,\Phi)_{ij}$$

$$S(\Phi) = \frac{\Phi}{\sqrt{\Phi^2+\Delta x^2}}$$

$$(D_x\Phi)_{ij} = \frac{\Phi_{i+1j}-\Phi_{i-1j}}{2\Delta x}$$

$$(D_y\Phi)_{ij} = \frac{\Phi_{ij+1}-\Phi_{ij-1}}{2\Delta x}$$

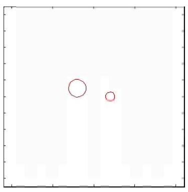


## Velocity Extension

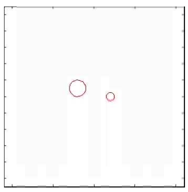
$$\begin{aligned} G(v,\Phi)_{ij} &= \max(S(\Phi_{ij})(D_x\Phi)_{ij},0).\left(\frac{v_{ij}-v_{i-1j}}{\Delta x}\right) \\ &+ \min(S(\Phi_{ij})(D_x\Phi)_{ij},0).\left(\frac{v_{i+1j}-v_{ij}}{\Delta x}\right) \\ &+ \max(S(\Phi_{ij})(D_y\Phi)_{ij},0).\left(\frac{v_{ij}-v_{ij-1}}{\Delta x}\right) \\ &+ \min(S(\Phi_{ij})(D_y\Phi)_{ij},0).\left(\frac{v_{ij+1}-v_{ij}}{\Delta x}\right) \end{aligned}$$



## Example: Eikonal Equation



$v(x) \equiv 1$



$v(x) = 1 - ax_2$



## Optimal Geometries

# Optimal Geometries

Classical problem for optimal geometry:

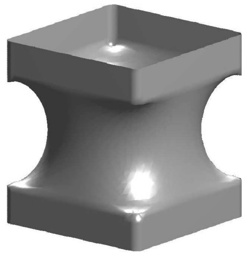
PLATEAU PROBLEM
(MINIMAL SURFACE PROBLEM)

Minimize area of surface between fixed boundary curves.



# Optimal Geometries

Minimal surface (L.T.Cheng, PhD 2002)





# Optimal Geometries

Wulff-Shapes: crystals tend to minimize energy at fixed volume.

Pure surface energy:

$J(\Omega) = \int_{\partial\Omega} \varphi(v) ds \to \min$

subject to $\int_\Omega 1 dx = V_0$

$v$ is the normal on $\partial\Omega$

$\varphi : S^{d-1} \to \mathbb{R}$

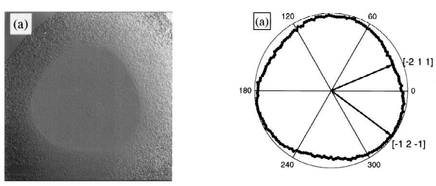
given anisotropic surface tension



# Optimal Geometries

Wulff-Shapes: Pb[111] in Cu[111]

Surnev et al, J.Vacuum Sci. Tech. A, 1998



## Optimal Geometries

Isotropic case: $\varphi(v) = | v | = 1$

Minimization of perimeter, yields ball $\Omega$



## Optimal Geometries

Free discontinuity problems:

find the set of discontinuity $\Gamma$ from a noisy observation $\tilde{u}$ of a function.

Mumford-Shah functional

$$J(u,\Gamma) = \lambda \int_D | u - \tilde{u} |^2 \, dx + \int_{D\setminus\Gamma} | \nabla u |^2 \, dx + \\ + \alpha \text{ Per } (\Gamma)$$

Again, $u$ solves partial differential equation with interface condition on $\Gamma$ .



## Optimal Geometries

Structural topology optimization

$J(\Omega) = \tilde{J}(\mathbf{u}_\Omega, \Omega)$
subject to
$\int_\Omega 1 dx = V_0$
$- \operatorname{div}(\sigma(\mathbf{u}_\Omega)) = 0 \text{ in } \Omega$
$+ \text{ boundary conditions on } \partial\Omega$

Design of Photonic Crystals, Semiconductor Design, Electromagnetic Design, ...



## Optimal Geometries

Inverse Obstacle Problems

E.g., inclusion detection

$\int_{\Gamma_n} | u - f |^2 \, ds \to \min$
$- \Delta u = 0 \text{ in } D\setminus\Omega$
$\frac{\partial u}{\partial n} = g \text{ on } \Gamma_n \subset \partial D$
$u = 0 \text{ on } \Gamma_d \subset \partial D$
$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$

Inverse Obstacle Scattering, Impedance Tomography, Identification of Discontinuities in PDE Coefficients, ...

## Gradient Flows

Physical Processes tend to minimize energy $E$ by a gradient flow:

$$\frac{\partial u}{\partial t} = -E'(u)$$

E.g., heat diffusion, thermal energy

$$E(u) = \int_\Omega |\nabla u|^2 \, dx$$

$$\frac{\partial u}{\partial t} = -E'(u) = \Delta u$$



## Gradient Flows

Gradient flow can be obtained as limit of variational problems

$$E(u(t \to \Delta t)) + \frac{1}{2\Delta t}\|u(t+\Delta t) - u(t)\|^2 \to \min_{u(t+\Delta t)}$$

$$\text{for } \Delta t \to 0$$

(Fife 1978; „Minimizing movements“, De Giorgi 1974)

Scales of gradient flows are obtained by changing the norm.



## Geometric Gradient Flows

For geometric motion, there is no natural Hilbert space setting, generalized notion of gradient flow needed.

Natural velocity replacing $\frac{\partial u}{\partial t}$ is normal velocity $v$ on $\partial\Omega$ .

$$E(\Omega(t+\Delta t)) + \frac{\Delta t}{2}\|v\|^2_{\mathcal{H}(t)} \to \min_{v \in \mathcal{H}(t)}$$

Where $\Omega(t+\Delta t)$ is the shape obtained by the motion of $\Omega(t)$ with normal velocity $v$ (Almgren-Taylor 1994)



## Geometric Gradient Flows

Scale of geometric gradient flows obtained, in the limit $\Delta t \to 0$ by using different Hilbert spaces $\mathcal{H}(t)$ for the velocity $v$ .

Variational form for $\Delta t \to 0$ :

$$E'(\Omega)w + < v, w >_{\mathcal{H}(t)} = 0, \quad \forall w \in \mathcal{H}(t)$$

where $E'(\Omega)$ is the shape derivative

$$E'(\Omega) = \frac{d}{dt} E(\Omega(t))$$

## Geometric Gradient Flows

$E(\Omega) = - \mid \Omega \mid$

$E'(\Omega)\ w = - \int_{\partial\Omega} w\ ds$

$\mathcal{H}(t) = L^2(\partial\Omega(t))$

$\rightarrow \int_{\partial\Omega(t)} (-1 + v)\ w\ ds = 0 \quad \forall w$

$\rightarrow v = 1$ , Eikonal Equation



## Geometric Gradient Flows

$E(\Omega) = \int_{\partial\Omega} 1\ ds = \mathcal{H}^{d-1}(\partial\Omega)$

$E'(\Omega) = - \int_{\partial\Omega} \kappa\ v\ ds$

$\kappa$ ... mean curvature $= \text{div}\ (n)$

$\mathcal{H}(t) = L^2(\partial\Omega(t))$

$\rightarrow v = \kappa$ , mean-curvature flow



## Geometric Gradient Flows

$\mathcal{H}(t) = \{w \in L^2(\partial\Omega(t)) \mid \int_{\partial\Omega(t)} w\ ds = 0\}$

$\rightarrow v = \kappa + \lambda, \quad \lambda = - \int_{\partial\Omega(t)} \kappa\ ds$

volume-conserving mean curvature flow



## Geometric Gradient Flows

$E(\Omega) = \int_{\partial\Omega} 1\ ds$
$E'(\Omega) = - \int_{\partial\Omega} \kappa\ v\ ds$
$\mathcal{H}(t) = H^{-1}(\partial\Omega(t))$

$$\|w\|^2_{\mathcal{H}(t)} = \int_{\partial\Omega(t)} \mid \nabla_s \Psi \mid^2 \ ds$$

$$-\Delta_s \Psi = w$$

$$\int_{\partial\Omega(t)} \Psi\ ds = 0$$

$\rightarrow v = -\Delta_s\ \kappa$ surface diffusion

## Geometric Gradient Flows

$E(\Omega) = \int_{\partial\Omega} 1 \, ds$
$E'(\Omega) = -\int_{\partial\Omega} \kappa \, v \, ds$
$\mathcal{H}(t) = H^{-\frac{1}{2}}(\partial\Omega(t))$

$\|w\|^2_{\mathcal{H}(t)} = \int_D | \nabla\Psi |^2 \ dx$

$-\Delta\Psi = 0 \quad \text{in } D \supset \Omega(t)$

$\Psi = w \quad \text{on } \partial\Omega(t)$

$\rightarrow v = -[\frac{\partial\Psi}{\partial n}]_{\partial\Omega(t)} \ , \ w = \kappa$

„Mullins-Sekerka Problem", Bulk diffusion



## Geometric Gradient Flows

$E(\Omega) = \int_{\partial\Omega} 1 \, ds$

$E'(\Omega) = -\int_{\partial\Omega} \kappa \, v \, ds$

$\mathcal{H}(t) = H^1(\partial\Omega(t))$

$\rightarrow -\Delta_s v + v = \kappa$



## Geometric Gradient Flows

$\mathcal{H}(t) = H^{\frac{1}{2}}(\partial\Omega(t))$
$\|w\|^2_{\mathcal{H}(t)} = \int_D | \nabla\Psi |^2 \ dx$

$-\Delta\Psi = 0 \quad \text{in } D \supset \Omega(t)$

$-[\frac{\partial\Psi}{\partial n}] = w \quad \text{on } \partial\Omega(t)$

$\rightarrow v = \Psi_{\partial\Omega(t)} \ , \ w = \kappa$



# Inverse Obstacle Problems & Shape Optimization

## Inverse Obstacle Problem

$\mathcal{F} : \mathcal{D} \to X$

$\mathcal{D}$ ... Set of shapes

$X$ ... Hilbert space

$\mathcal{F}$ ... Nonlinear operator

Given noisy measurement $z^\delta$ for $z := \mathcal{F}(\hat{\Omega})$ find a shape $\Omega$ approximating $\hat{\Omega}$.

$\Omega = \Omega(\delta) \to \hat{\Omega} \quad \text{as } z^\delta \to z$

Associated Least-Squares Problem

$$J_0(\Omega) := \|\mathcal{F}(\Omega) - z^\delta\|^2 \to \min_\Omega$$



## Inverse Obstacle Problem

In general, minimization of $J_0(\Omega)$ is ill-posed:

- $J_0(\Omega_k) \to 0$ without convergence of a subsequence $\Omega_k$ possible.

No stable dependence of minimizer (if existing) on the data $z^\delta$.



## Inverse Obstacle Problem

Ill-posedness causes need for Regularization.

(i) **Tikhonov-type regularization**

$$\frac{1}{\alpha} J_0(\Omega) + R(\Omega) \to \min_\Omega$$

with regularization functional

$$R : \mathcal{D} \to \mathbb{R}$$



## Inverse Obstacle Problem

(ii) **Iterative regularization concept**

Apply iterative (level set) method directly to $J_0$, use appropriate stopping criterium, e.g. stop at the first iteration where residual is less than $\tau *$ (noise level), $\tau > 1$.

## Inverse Obstacle Problem

Regularization functional $R : \mathcal{D} \to \mathbb{R}$ must be defined on general class of shapes (multiply connected, no fixed parametrization with respect to reference shape, ...).

Popular choice: Perimeter

$$R(\Omega) = \mathcal{H}^{d-1}(\partial\Omega) = TV(\chi^\Omega)$$

$\chi^\Omega$ ... indicator function of $\Omega$



## Inverse Obstacle Problem

Other possibilities for regularization functionals:

based on distance function $d_\Omega$ .

Reference (starting) shape $\Omega_0$ :

$$R(\Omega) = \int_{\Omega\Delta\Omega_0} | d_{\Omega_0} |^p \ dx, \quad p \geq 1$$
$$R(\Omega) = \int_{\Omega\Delta\Omega_0} | d_{\Omega} |^p \ dx, \quad p \geq 1$$
$$R(\Omega) = \int_{\partial\Omega_0} | d_{\Omega} |^p \ ds, \quad p \geq 1$$
$$R(\Omega) = \int_{\partial\Omega} | d_{\Omega_0} |^p \ ds, \quad p \geq 1$$



## Shape Optimization

$$J(\Omega) \to \min$$
$$e(\Omega) = 0$$
$$c(\Omega) \leq 0$$

$J : \mathcal{D} \to \mathbb{R}$ shape functional

$e : \mathcal{D} \to X$ equality constraints in Banach space

$c : \mathcal{D} \to Y$ inequality constraints in ordered Banach space



## Shape Optimization

In general, existence of minimizer not guaranteed (except simple 2D cases, e.g. Chambolle 2001)

→ Perimeter constraint $P(\Omega) \leq p^*$ or penalization by perimeter

$$J(\Omega) + \alpha \ P(\Omega) \to \min_\Omega$$

## Metrics on Shapes

For analysis of inverse obstacle and shape optimization problems, metrics on classes of shapes are needed!

- Transformation-metrics
- $L^1$-metric
- Hausdorff-metric



## Transformation Metrics

Transformation metrics are based on cost of transformation of shapes:

$$d_T\,(\Omega_0, \Omega_1) := \inf_{v \in \tilde{\mathcal{D}}} \int_0^1 \|v(.\,,t)\|^2 dt$$

subject to $\Phi_t + v|\nabla\Phi| = 0$

$$\Omega_0 = \{\Phi(.\,,0) < 0\}$$

$$\Omega_1 = \{\Phi(.\,,1) < 0\}$$



## Transformation Metrics

- ➡ use any appropriate Hilbert space norm
- ➡ used e.g. for conformal mappings
- ➡ restricts class of admissible shapes



## $L^1$-metric

$L^1$ -metric measures distance of shapes via their indicator functions:

$$d_1(\Omega_0, \Omega_1) = \|\chi_{\Omega_1} - \chi_{\Omega_2}\|_{L^1}$$

Many shape functionals are lower semicontinuous with respect to $L^1$-metric, typically if

$$J(\Omega) = \tilde{J}(\chi_\Omega)$$

## $L^1$-metric

Perimeter is weakly lower semicontinuous with respect to $L^1$-metric

$\{\Omega : P(\Omega) \leq c\}$ is pre-compact with respect to $L^1$ -metric



## Hausdorff Metric

Natural metric of shapes (?)

$$d_{\mathcal{H}}(\Omega_0, \Omega_1) := \{ \sup_{x \in \partial\Omega_0} d_{\Omega_1}(x) \ \sup_{x \in \partial\Omega_1} d_{\Omega_0}(x) \}$$

Perimeter is lower-semicontinuous with respect to $d_{\mathcal{H}}$ on the class of compact sets in $\mathbb{R}^2$ with finite number of connected components (Golab's Theorem)



## Hausdorff Metric

Neumann-Problems for elliptic partial differential equations are lower semicontinuous with respect to $d_{\mathcal{H}}$ on the class of compact sets in $\mathbb{R}^2$ with finite number of connected components (Chambolle 2002, DalMaso-Toader 2002)



## Regularization by Perimeter

Assumptions: $\|z - z^\delta\| \leq \delta$

$\mathcal{F}(\Omega) = \tilde{\mathcal{F}}(\chi_\Omega)$

$\mathcal{F}$ continuous on $L^1$

$\alpha \to 0, \ \frac{\delta^2}{\alpha}, \ \text{as } \delta \to 0$

Let $\Omega_\alpha^\delta$ be minimizer of

$$\frac{1}{\alpha} \|\mathcal{F}(\Omega) - z^\delta\|^2 + P(\Omega) \to \min_\Omega$$

## Regularization by Perimeter

Respectively

$\frac{1}{\alpha}\|\tilde{\mathcal{F}}(\chi_\Omega) - z^\delta\|^2 + TV(\chi_\Omega) \to \min$

$\chi_\Omega \in BV(D;\{0,1\})$

Then there exists supsequence $(\delta_k, \alpha_k)$ such that $\Omega_{\alpha_k}^{\delta_k} \to \hat{\Omega} \quad \text{as } \delta_k \to 0$

where $\hat{\Omega}$ solves $P(\hat{\Omega}) \to \min$

s.t. $\mathcal{F}(\hat{\Omega}) = z$



## Regularization by Perimeter

Uniqueness of the limit problem implies

$\Omega_\alpha^\delta \to \hat{\Omega}$

as $\delta \to 0$ .



# Shape Sensitivity Analysis



## Sensitivity Analysis

As usual for optimization problems sensitivities (derivatives) are needed.

Gateâux-Derivatives in Banach spaces

$$J'(u)v = \lim_{\epsilon \downarrow 0} \frac{J(u+\epsilon v) - J(u)}{\epsilon}$$

## Sensitivity Analysis

Alternative definition:

$$J'(u)v = \frac{d}{dt} J(u(t))|_{t=0}$$

$$\frac{du}{dt}(t) = v, \; t \in (0, \epsilon)$$

$$u(0) = u$$

„Speed Method“



## Speed Method

Analogous to Gateâux-Derivative define „Shape Derivative“ of $\Omega$

$$J'(\Omega)v = \frac{d}{dt} J(\Omega(t))|_{t=0}$$

$$\Omega(t) = \{\Phi( \; . \; , \; t) < 0\}$$

$$\frac{\partial \Phi}{\partial t} + v \; |\nabla \Phi| = 0$$



## Speed Method

Classical definition for smooth shapes $(C^1)$ and velocities $(C^2)$

→ extension via level set method



## Volume Functionals

Let $J(\Omega) := \int_\Omega g \; dx$

Level-set formulation, $D \supset \Omega$

$$\tilde{J}(\Phi) = \int_D H(\Phi) \; g \; dx$$

with Heaviside function $H$.

## Volume Functionals

Formal derivative:

$$\frac{d}{dt}\tilde{J}(\Phi(\ .\ ,t)) = \int_D \delta(\Phi)\frac{\partial\Phi}{\partial t}\ g\ dx$$

$$= -\int_D \delta(\Phi)\ |\ \nabla\Phi\ |\ v\ g\ dx$$

with Dirac $\delta$ -distribution

$$\delta = H'$$



## Surface Functionals

Let $J(\Omega) = \int_\Gamma g\ ds$
$\Gamma = \partial\Omega$

Level-set formulation, $D \supset \Omega$

$\tilde{J}(\Phi) = \int_D \delta(\Phi)\ |\ \nabla\Phi\ |\ g\ dx$

Formal derivative

$$\frac{d}{dt}\tilde{J}(\Phi(\ .\ ,t)) = \int_D (\delta'(\Phi)\ |\ \nabla\Phi\ |\ \frac{\partial\Phi}{\partial t}$$

$$+\delta(\Phi)\frac{\nabla\Phi}{|\ \nabla\Phi\ |}\nabla\left(\frac{\partial\Phi}{\partial t}\right))\ g\ dx$$



## Surface Functionals

$$\frac{d}{dt}\tilde{J}(\Phi(\ .\ ,t)) = -\int_D (\delta'(\Phi)\nabla\Phi.\nabla\Phi v$$

$$+\delta(\Phi)\frac{\nabla\Phi}{|\nabla\Phi|}\nabla(v|\nabla\Phi|))\ g\ dx$$

Extension of $v$, arbitrary on $D\backslash\Gamma$

➡ use constant normal extension

$$\nabla v\ .\ \nabla\Phi = \nabla v\ .\ \mathbf{n}\ |\nabla\Phi| = 0$$



## Surface Functionals

Use $\delta'(\Phi)\nabla\Phi = \nabla(\delta(\Phi))$ and

Gauss-Theorem:

$$\begin{aligned}\frac{d}{dt}\tilde{J}(\Phi(\ .\ ,t)) &= \int_D \delta(\Phi)(\mathrm{div}(g\ v\ \nabla\Phi) \\ &\quad -\frac{\nabla\Phi}{|\nabla\Phi|}v\ g\ \nabla(|\nabla\Phi|)\ )\ dx \\ &= \int_D \delta(\Phi)\ |\nabla\Phi|\ v\ (g\ \mathrm{div}\left(\frac{\nabla\Phi}{|\nabla\Phi|}\right) + \nabla g\frac{\nabla\Phi}{|\nabla\Phi|})\ dx \\ &= \int_\Gamma v\ (g\ \kappa + \frac{\partial g}{\partial n})\ ds\end{aligned}$$

# Level Set Methods Based on Gradient Flows



## Gradient Flows

In the above framework of gradient flows, we can derive equations for velocity $v$ by minimizing

$$J(\Omega(t+\Delta t)) + \frac{\Delta t}{2} \|v\|^2_{\mathcal{H}(t)}$$

with respect to $v \in \mathcal{H}(t)$
and $\Delta t \to 0$



## Gradient Methods

Variational equation for $v$

$$< v, w >_{\mathcal{H}(t)} = -J'(\Omega(t))\ w$$

yields continuous time evolution for $v$.

Classical gradient method is explicit time discretization of gradient flow.



## Gradient Methods

Set $t=0, k=0$ , initial value $\Phi_0$
Loop:

- Set $\Omega_k := \{\Phi(\ .\ , t_k) < 0\}$
- Compute $v_k$ from variational equation at time $t_k$
- Select time step $\Delta t_k,\ t_{k+1} = t_k + \Delta t_k$
- Solve $\frac{\partial \Phi}{\partial t} + v_k\ |\nabla \Phi| = 0$ in $(t_k,\ t_{k+1})$
- $k = k+1$

## Gradient Methods

Lemma: $v_k$ is descent direction, i.e.

$$J(\Omega_{k+1}) < J(\Omega_k)$$

If $v_k \neq 0$ , $\Delta t_k$ sufficiently small.

Proof:

$$J(\Omega_{k+1}) - J(\Omega_k) = \Delta t_k \; J'(\Omega_k)v_k + o(\Delta t_k)$$

$$J'(\Omega_k)v_k = -\|v_k\|^2_{\mathcal{H}(t)} < 0$$



## Example 1

$$J(\Omega) = \frac{1}{2}\int_M |u-f|^2 dx$$
$$-\Delta u = \chi_\Omega = H(\Phi) \quad \text{in } D$$
$$u = 0 \quad \text{on } \partial D$$

---

$\Omega \subset D$ and $M \subset D$

$$J'(\Omega)v = \int_M (u-f)\; u' \; dx$$
$$-\Delta u' = -\delta(\Phi)\; |\nabla\Phi| \; v \quad \text{in } D$$
$$u' = 0 \quad \text{on } \partial D$$



## Example 1

Define adjoint state $u^*$ via

$$-\Delta u^* = \chi_M.(u-f) \quad \text{in } D$$
$$u^* = 0 \quad \text{on } \partial D$$



## Example 1

$$\begin{aligned} J'(\Omega)\; v &= \int_M (u-f)\; u' \; dx \\ &= -\int_D \Delta u^* \; u' \; dx \\ &= -\int_D u^* \; \Delta u' \; dx \\ &= -\int_{\partial\Omega} u^* \; v \; ds \end{aligned}$$

$J'(\Omega)\; v$ independent of $u'$

„Adjoint Method“

# Example 1

$\mathcal{H}(t) = L^2(\partial\Omega(t))$

$\int_{\partial\Omega(t)} v\ w\ ds = \int_{\partial\Omega(t)} u^*\ w\ ds$

$\Rightarrow v = u^*$

$\mathcal{H}(t) = H^{-\frac{1}{2}}(\partial\Omega(t))$

$$\begin{aligned} -\Delta\Psi &= 0 \quad \text{in } D\backslash\Gamma \\ \Psi &= u^* \quad \text{on } \Gamma \\ \Psi &= 0 \quad \text{on } \partial D \\ v &= -\left[\frac{\partial\Psi}{\partial n}\right] \end{aligned}$$



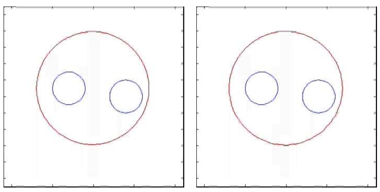
# Example 1

Alternative: $v = \left[\frac{\partial u^*}{\partial n}\right]$

$$\begin{aligned} \rightarrow J'(\Omega)\ v &= -\int_{\partial\Omega} u^* \left[\frac{\partial u^*}{\partial n}\right] ds \\ &= -\int_{\Omega} |\nabla u^*|^2\ dx \end{aligned}$$



# Example 1

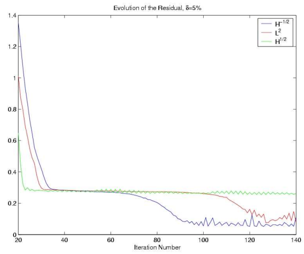


$L^2$ - Norm $\qquad H^{-\frac{1}{2}}$ - Norm



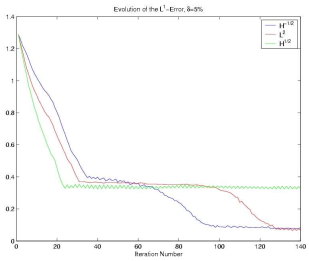
# Example 1

Residual



## Example 1

$L^1$ - error






## Example 2

$J(\Omega) = \frac{1}{2}\int_M |u-f|^2 \; ds$

$$
\begin{aligned}
-\Delta u &= 0 \quad \text{in } D\backslash\partial\Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega \\
\frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma_n \subset \partial D \\
u &= 0 \quad \text{on } \Gamma_d = \partial D\backslash\Gamma_n
\end{aligned}
$$

$\Omega \subset D, \;\; M \subset \Gamma_d$



## Example 2

$J'(\Omega)\; v = \int (u-f) \; u' \; ds$

$$
\begin{aligned}
-\Delta u' &= 0 \quad \text{in } D\backslash\partial\Omega \\
\frac{\partial u'}{\partial n} &= -\,\text{div}\,(\nabla u \,.\, n\; v) \quad \text{on } \partial\Omega \\
\frac{\partial u'}{\partial n} &= 0 \quad \text{on } \Gamma_n \\
u' &= 0 \quad \text{on } \Gamma_d
\end{aligned}
$$



## Example 2

Adjoint state $u^*$ defined by

$$
\begin{aligned}
-\Delta u^* &= 0 \quad \text{in } D\backslash\partial\Omega \\
\frac{\partial u^*}{\partial n} &= 0 \quad \text{on } \partial\Omega \\
\frac{\partial u^*}{\partial n} &= \chi_M(u-f) \quad \text{on } \Gamma_d \\
u^* &= 0 \quad \text{on } \Gamma_n
\end{aligned}
$$

$\Rightarrow$ $J'(\Omega)\; v = \int_{\partial\Omega} v[\nabla u \,.\, \nabla u^*] \; ds$

## Example 2

$\mathcal{H} = L^2(\partial\Omega(t))$

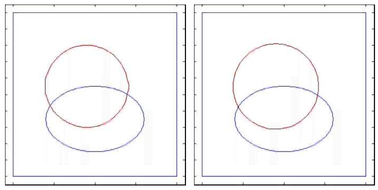
$$v = -\left[\nabla u \,.\, \nabla u^*\right] \quad \text{on } \partial\Omega$$

$\mathcal{H} = H^{\frac{1}{2}}(\partial\Omega(t))$

$$\begin{aligned} -\Delta\Psi &= 0 \quad \text{in } D \\ \left[\frac{\partial\Psi}{\partial n}\right] &= -\left[\nabla u \,.\, \nabla u^*\right] \quad \text{on } \partial\Omega \\ \Psi &= 0 \quad \text{on } \partial D \\ v &= -\Psi \end{aligned}$$



## Example 2

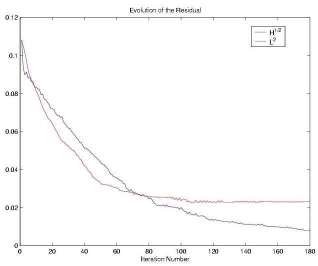


$L^2$ - Norm $\quad H^{\frac{1}{2}}$ - Norm



## Example 2

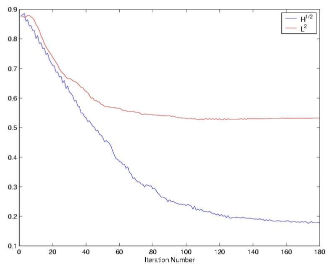
Residual






## Example 2

$L^1$ - error



## Levenberg-Marquardt

$$J(\Omega) = \|\mathcal{F}(\Omega) - z\|^2$$

Levenberg-Marquardt obtained from first-order expansion of $\mathcal{F}$ together with penalty on velocity

$$\|\mathcal{F}(\Omega) + \mathcal{F}'(\Omega)\ v - z\|^2 + \beta\|v\|^2_{\mathcal{H}(t)} \rightarrow \min_v$$

Variational equation

$$< \mathcal{F}'(\Omega)\ v,\ \mathcal{F}'(\Omega)\ w > + \beta < v, w >_{\mathcal{H}(t)} =$$
$$= - < \mathcal{F}(\Omega) - z,\ \mathcal{F}'(\Omega)\ w > \quad \forall w \in \mathcal{H}(t)$$



## Levenberg-Marquardt

Set $t_0 = 0,\ k = 0$, initial value $\Phi_0$

Loop:

- Set $\Omega_k := \{\Phi(\ .\ ,\ t_k) < 0\}$
- Compute $v_k$ from variational equation at time $t_k$
- Select time step $\Delta t_k,\ t_{k+1} = t_k + \Delta t_k$
- Solve $\frac{\partial \Phi}{\partial t} + v_k|\nabla\Phi| = 0$ in $(t_k,\ t_{k+1})$
- $k = k + 1$



## Example 1

$$\mathcal{F} : \mathcal{D} \rightarrow H^1_0(D),\ \Omega \mapsto u_{/M}$$

$$\begin{aligned} -\Delta u &= \chi_\Omega \\ \mathcal{F}'(\Omega)\ v &= u'_{v/M} \\ -\Delta u'_v &= \delta_\Gamma\ v,\ \Gamma = \partial\Omega \end{aligned}$$



## Example 1

Levenberg-Marquardt update

$$< \mathcal{F}'(\Omega_k)\ v,\ \mathcal{F}'(\Omega_k)\ w > + \beta\ < v, w > =$$
$$= - < \mathcal{F}'(\Omega_k)\ w,\ \mathcal{F}(\Omega_k) - z >$$

becomes

$$\int_M u'_v\ u'_w\ dx + \beta\ < v, w >_{\mathcal{H}(t)} =$$
$$= - \int_M u'_v(u - z)\ dx$$

## Example 1

Define Lagrange parameter $\lambda$

$$\int_M u'_v \; \Psi \; dx + \int_D \nabla\lambda \; \nabla\Psi \; dx =$$
$$= - \int_M u'_v \; (u - z) \; dx \quad \forall \Psi \in H^1_0(D)$$

$$\Psi = u'_v$$
$$\Rightarrow \int_D \nabla\lambda \; \nabla u'_w \; dx = - \int_\Gamma w \; \lambda \; ds$$



## Example 1

Primal-Dual formulation $(u' = u'_v)$

$$A(u', v \; ; \Psi, w) = \int_M u' \; \Psi \; dx + \beta \; < v, w >_{\mathcal{H}(t)}$$

$$B(u', v \; ; \lambda) = \int_D \nabla u' \; \nabla\lambda \; dx - \int_\Gamma v \; \lambda \; ds$$

$$< \mathcal{F}, \Psi > = - \int_M (u - z) \; \Psi \; dx$$



## Example 1

$$A(u', v \; ; \Psi, w) + B(\Psi, w \; ; \lambda) \;=\; < \mathcal{F}, \Psi >$$

$$B(u', v \; ; \mu) \;=\; 0$$

$$\forall \; \Psi \in H^1_0(D), \;\; v \in \mathcal{H}(t), \;\; \mu \in H^1_0(D)$$

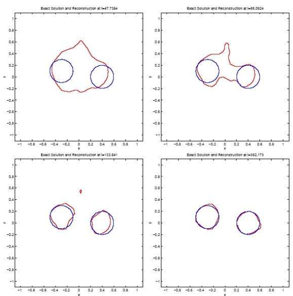


## Example 1

Noise level 1%

10 15

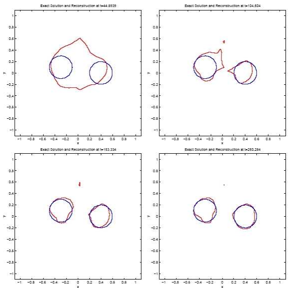
20 25



# Example 1

Noise level 4%

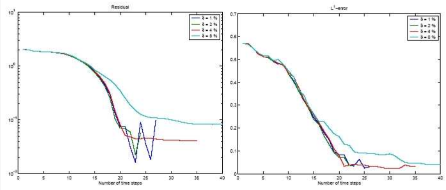
10 20



30 40



# Example 1






# Example 2

$$
\begin{aligned}
\mathcal{F}(\Omega) &= u|_M \\
-\Delta u &= 0 \quad D\backslash\Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \partial\Omega \\
\frac{\partial u}{\partial n} &= g \quad \Gamma_n \\
u &= 0 \quad \Gamma_d \\
\mathcal{F}'(\Omega)\ v &= u'_v|_M
\end{aligned}
$$

$$\int_{D\backslash\Omega} \nabla u'_v \nabla\Psi \ dx = \int_{\partial\Omega} v\ (\nabla u\ .\ \nabla\Psi)\ ds$$

$$\forall\ \Psi \in H^1_{0,d}(D\backslash\Omega)$$



# Example 2

Primal-Dual formulation

$$
\begin{aligned}
A(u', v\ ; \Psi, w) &= \int_M u'\ \Psi\ ds + \beta\ < v, w >_{\mathcal{H}(t)} \\
B(u', v\ ; \lambda) &= \int_{D\backslash\Omega} \nabla u'\ \nabla\lambda\ dx - \\
&\quad - \int_{\partial\Gamma} v\ (\nabla u\ .\ \nabla\lambda)\ ds \\
< \mathcal{F}, \Psi > &= -\int_M (u - z)\ \Psi\ ds
\end{aligned}
$$

## Example 2

Multiple State Equations

$$-\Delta u_k = 0, \ \frac{\partial u_k}{\partial n} = g_k, \ k = 1, ..., l$$

$$\mathcal{F}(\Omega) = (u_k|_M)_{k=1,...,l}$$

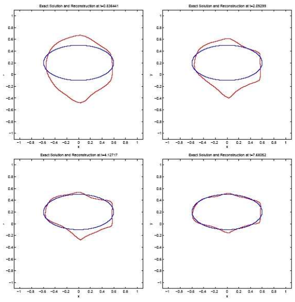
$$\sum_k A(u_k', v\ ; \Psi_k, w) + \sum_k B(\Psi_k, w\ ; \lambda_k) = \sum_k < \mathcal{F}_k, \Psi >$$

$$\sum_k B(u_k', v\ ; \mu_k) = 0$$



## Example 2

2 4



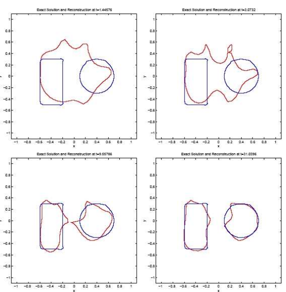
6 8



## Example 2

Noise level 0.1 %

5 10

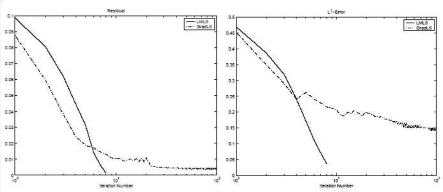


15 20

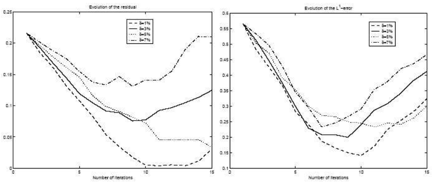


## Example 2

Comparison with Gradient method



# Example 2





# Newton-Type Methods

Basic structure: compute second derivative

$$J''(\Omega)(v,w) = \frac{d}{dt}(J'(\Omega(t))\ v)$$

$$\text{for } \Omega(t) = \{\Phi(\ .\ ,t) < 0\}$$

$$\frac{\partial \Phi}{\partial t} + w|\nabla \Phi| = 0$$



# Newton-Type Methods

Compute velocity $v_k$ by solving

$$J''(\Omega_k)(v_k,w) = -J'(\Omega_k)\ w, \quad \forall\ w \in \mathcal{H}(t)$$

$$J(\Omega) = \int_{\partial\Omega} 1\ ds$$

$$J'(\Omega) = \int_{\partial\Omega} \kappa\ v\ ds$$

$$J''(\Omega)(v,w) = \int_{\partial\Omega} \left[vw\left(\kappa^2 + \frac{\partial \kappa}{\partial n}\right) + \nabla v . \nabla w\right] ds$$

Hintermüller, Ring, SIAP 2003



# Numerical Methods

## Numerical Methods

Besides computational techniques for level set evolution, (Hamilton-Jacobi solver, redistancing, velocity extension), we need numerical methods to solve partial differential equations with/on interfaces (state/adjoint equation, Newton equation, ...).



## Numerical Methods

Possibilities for elliptic PDEs with interfaces:

1. Resolve interface by mesh (e.g. finite elements) accurately
   ⇒ Remeshing at each iteration step is needed. Expensive in particular in 3D, difficult.



## Numerical Methods

2. Use adaptive refinement of basic mesh (fixed during iteration).
3. Moving meshes
   ⇒ Problems with too strong change of obstacle.



## Numerical Methods

4. Immersed interface method:
   finite difference discretization on fixed grid with local connections to system matrix, around interface.
5. Partion of Unity FE/Extended Finite Element.
   FE analogous to immersed interface method, fixed triangular grid + discontinuous elements around interface.

## Numerical Methods

6. Fictitious Domain Methods: extend problem to larger domain, use Lagrange parameter for correction.

So far, all methods require construction of the zero level set

$$\Gamma = \{\Phi = 0\}$$

Expensive in 3D!



## Numerical Methods

7. Averaged fictitious domain methods: use weighted average over several level sets.



## Numerical Methods

Let $w_\epsilon \to \delta$ as $\epsilon \to 0$

$\int_{\{\Phi<0\}} g\ dx = \int_D H(-\Phi)\ g\ dx$

$\to \int w_\epsilon(p) \int_D H(p - \Phi)\ g\ dx\ dp =$
$= \int_D H_\epsilon(-\Phi)\ g\ dx$

$H_\epsilon(t) = \int w_\epsilon(p) H(p + t)\ dp$

$\int_{\{\Phi=0\}} g\ ds \to \int w_\epsilon(p) \int_{\{\Phi=p\}} g\ ds\ dp$
$= \int_D w_\epsilon(\Phi) |\nabla\Phi|\ g\ dx$



## Example 1

State equation, weak form

$\int_D \nabla u\ \nabla\Psi\ dx = \int_D H(-\Phi)\ \Psi\ dx$
$\to \int_D \nabla u\ \nabla\Psi\ dx = \int_D H_\epsilon(-\Phi)\ \Psi\ dx$

Linearized state equation, weak form

$\int_D \nabla u'\ \nabla\Psi\ dx = \int_{\{\Phi=0\}} \Psi\ v\ ds$
$\to \int_D \nabla u'\ \nabla\Psi\ dx = \int_D w_\epsilon(\Phi)\ \Psi\ v\ |\nabla\Phi|\ dx$

Use adaptize finite element method on fixed grid on $D$.

# Example 2

State equation, weak form

$$\int_{\{\Phi>0\}} \nabla u \ \nabla \Psi \ dx = \int_{\Gamma_n} g \ \Psi \ ds$$

„Ersatz material“, stiffness

$$A^\epsilon = \epsilon + (1-\epsilon) H(\Phi)$$

$$\int w_\epsilon(p) \int_D A^\epsilon(\Phi) \nabla u \ \nabla \Psi \ dx \ dp = \int_{\Gamma_n} g \ \Psi \ ds$$

$$\int_D \tilde{A}^\epsilon(\Phi) \nabla u \ \nabla \Psi \ dx = \int_{\Gamma_n} g \ \Psi \ ds$$



# Example 2

Linearized state equation

$$\int_D \tilde{A}^\epsilon(\Phi) \nabla u' \ \nabla \Psi \ dx =$$
$$= \int_D \frac{\partial \tilde{A}^\epsilon}{\partial \Phi}(\Phi) \ v \ \nabla u \ \nabla \Psi \ |\nabla \Phi| dx$$