Adaptive Variational Multiscale Method: Basic A Posteriori Error Estimation Framework

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Goal

- We want to find computational methods for solving multiscale problems in a Galerkin finite element setting.
- We need an a posteriori estimation framework to measure the reliability of our solution.
- We also want to use the error bounds for adaptivity.
- We start with two scales in two dimensions.

Outline

- Model Problem
- Variational Multiscale Method
- Choice of Coarse and fine Spaces
- The Basic Idea of our Method
- Error Estimates
- Adaptive Strategy
- Numerical Examples
- Future Work

Model Problem

Poisson Equation. Find $u \in H_0^1(\Omega)$ such that

$$-\nabla \cdot a\nabla u = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

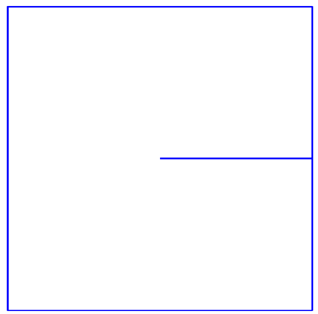
where $f \in H^{-1}(\Omega)$, $a > 0$ bounded, and Ω is a domain in $\mathbf{R}^d$, $d = 1, 2, 3$.

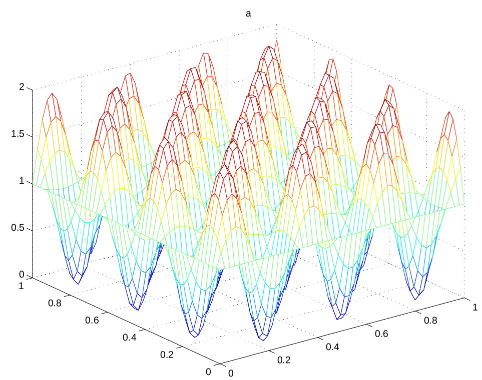
Weak form. Find $u \in H_0^1(\Omega)$ such that

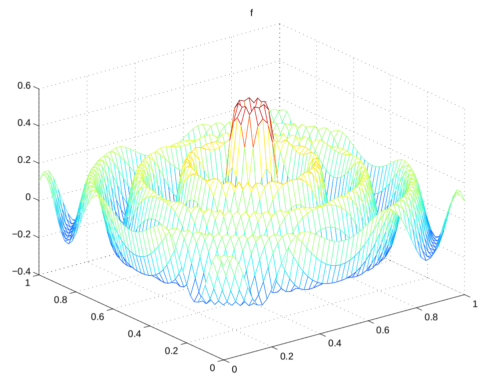
$$(a\nabla u, \nabla v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega).$$

Multiscale Problems

Below are three examples of multiscale problems.







The £rst one represents dif£culties in the domain (cracks, holes, ...) the second one oscillations in a and the third one oscillations in f.

Variational Multiscale Method

- See for instance T.J.R. Hughes (1995).
- $H_0^1 = V_c \oplus V_f$, $u = u_c + u_f$, and $v = v_c + v_f$.

Find $u_c \in V_c$ and $u_f \in V_f$ such that

$$
\begin{aligned}
(a\nabla u_c, \nabla v_c) + (a\nabla u_f, \nabla v_c) &= (f, v_c) \quad \text{for all } v_c \in V_c, \\
(a\nabla u_f, \nabla v_f) &= (f, v_f) - (a\nabla u_c, \nabla v_f) \\
&:= (R(u_c), v_f) \quad \text{for all } v_f \in V_f.
\end{aligned}
$$

Variational Multiscale Method

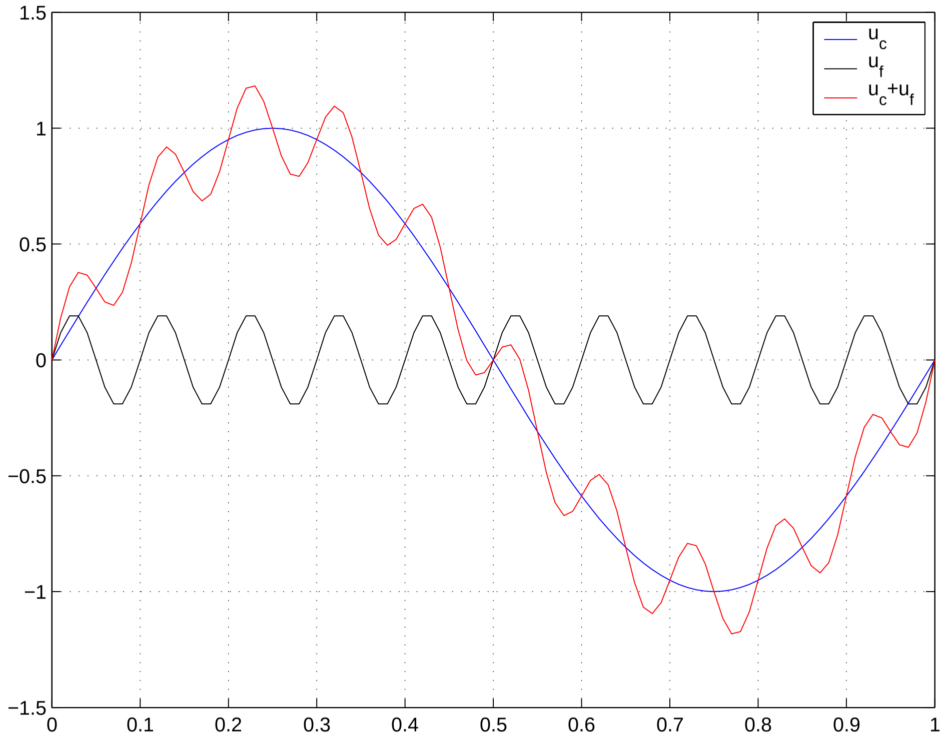


Figure 1: u_c, u_f, and $u_c + u_f$.

Variational Multiscale Method

- The fine scale is driven by the coarse scale residual.
- Approximation to fine scale solution solved on each element analytically (Green's functions).
- fine scale information is then used to modify the coarse scale equation.

$$(a\nabla u_c, \nabla v_c) + (a\nabla \hat{A}_f^{-1} R(U_c), \nabla v_c) = (f, v_c) \;\; \forall v_c \in V_c.$$

Choice of V_c and V_f

We use the splits proposed by Vassilevski-Wang (1998) and also used by Aksoylu-Holst (2004).

- Hierarchical basis, HB.
- Wavelet modified hierarchical basis, WHB.

The aim with WHB is to make V_f more $L^2(\Omega)$ orthogonal to V_c than in ordinary HB.

$$(Q_c^a v, w) = (v, w), \quad \text{for all } w \in V_c.$$

$$\varphi_{WHB} = (I - Q_c^a)\varphi_{HB}.$$

Choice of V_c and V_f

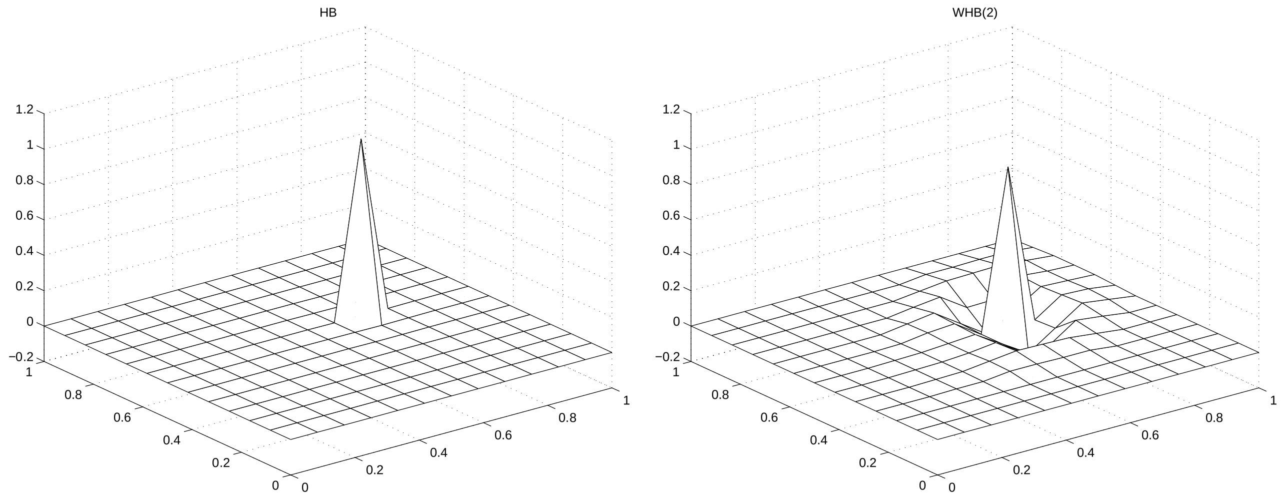


Figure 2: HB-function and WHB-function with two Jacobi iterations.

Basic Idea

- Discretization of V_f by (W)HB-functions (V_f^h).
- Solve localized fine scale problems for each coarse node (or some coarse nodes).
- Possibility to do this in parallel.
- A posteriori error estimation framework.
- Adaptive strategy for this setting.

Decouple fine Scale Equations

Remember the fine scale equations:

$$(a\nabla U_f, \nabla v_f) = (R(U_c), v_f), \quad \text{for all } v_f \in V_f^h.$$

Include a partition of unity,

$$(a\nabla U_f, \nabla v_f) = (R(U_c), v_f) \;= \sum_{i=1}^{n} (R(U_c), \varphi_i v_f),$$

let $U_f = \sum_i^n U_{f,i}$ where
$(a\nabla U_{f,i}, \nabla v_f) = (R(U_c), \varphi_i v_f)$.

Approximate Solution

Find $U_c \in V_c$ and $U_f = \sum_i^n U_{f,i}$ where $U_{f,i} \in V_f^h(\omega_i)$ such that

$$(a\nabla U_c, \nabla v_c) + (a\nabla U_f, \nabla v_c) = (f, v_c) \quad \text{for all } v_c \in V_c,$$
$$(a\nabla U_{f,i}, \nabla v_f) = (R(U_c), \varphi_i v_f) \quad \text{for all } v_f \in V_f^h(\omega_i).$$

- Since φ_i has support on a star S_i^1 in node i we solve the fine scale equations approximately on ω_i with $U_{f,i} = 0$ on $\partial\omega_i$.

Refinement and Layers

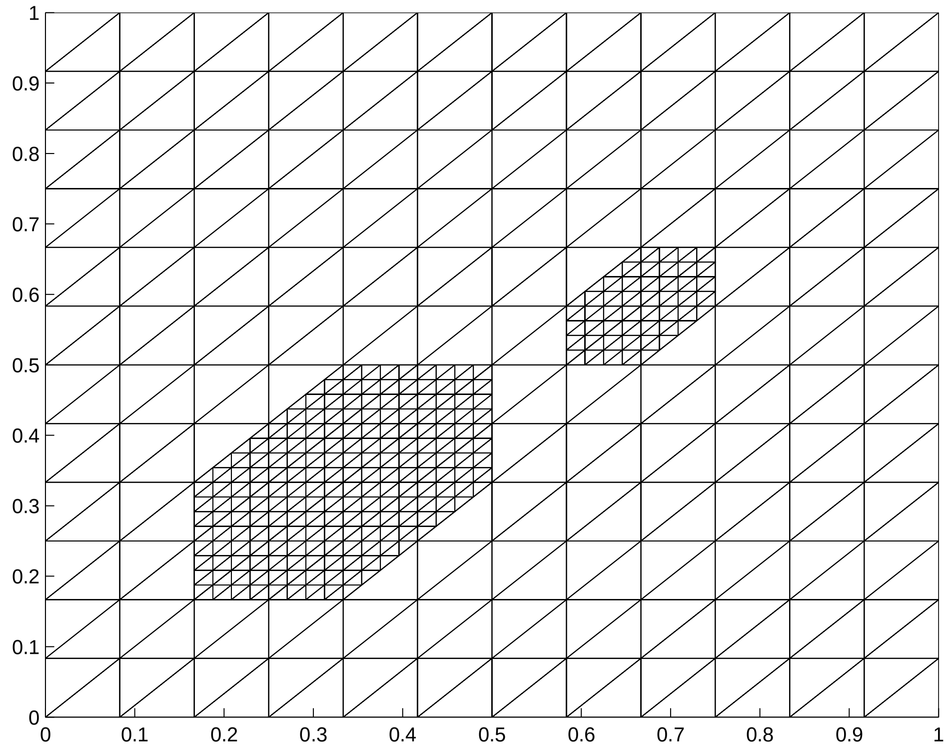


Figure 3: One, S_i^1, and two, S_i^2, layer stars.

Iterative or Direct

Iterative $U_{f,i}^0 = 0$,

$$
\begin{aligned}
(a\nabla U_c^k, \nabla v_c) &= (f, v_c) - (a\nabla U_f^{k-1}, \nabla v_c), \\
(a\nabla U_{f,i}^k, \nabla v_f) &= (R(U_c^k), \varphi_i v_f),
\end{aligned}
$$

or in matrix form,

$$
\begin{aligned}
A_c U_c^k &= b_c(U_f^{k-1}) \\
\hat{A}_f U_{f,i}^k &= b_f(U_c^k)
\end{aligned}
$$

Iterative or Direct

Direct

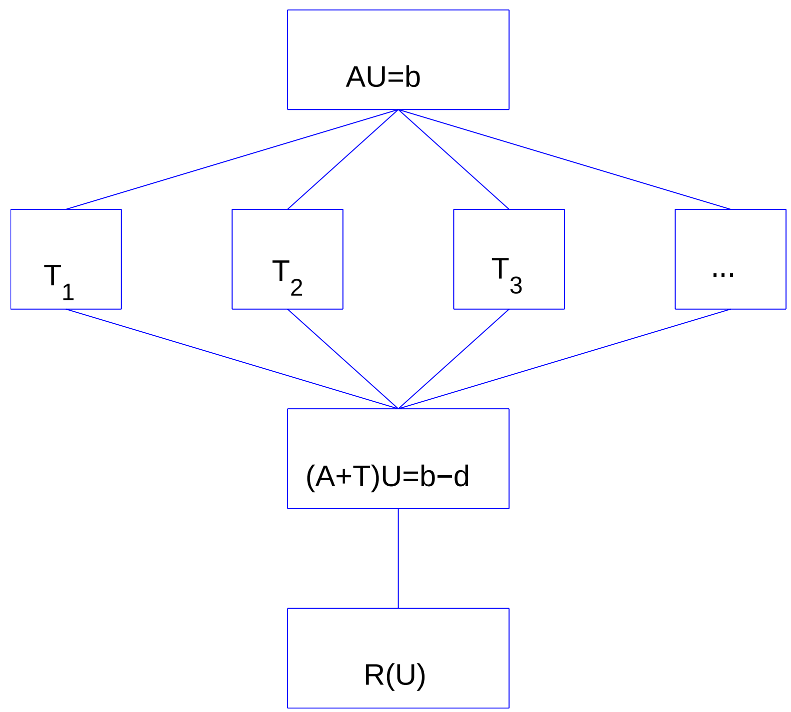
$$(a\nabla U_c, \nabla v_c) + (\nabla \hat{A}_f^{-1} R(U_c), \nabla v_c) = (f, v_c)$$

or in matrix form,

$$(A_c + T)U_c = b - d,$$

where $b_j = (f, \varphi_j)$,
$T_{ij}\varphi_j + d_i = (\nabla \hat{A}_f^{-1}(R(\varphi_i)), \nabla \varphi_j)$.

Algorithm



Error Estimation

We let $e = u - U = u_c + \sum_{i=1}^n u_{f,i} - U_c - \sum_{i=1}^n U_{f,i}$ denote the error. We further let $e_c = u_c - U_c$ and $e_{f,i} = u_{f,i} - U_{f,i}$.

- Energy norm error estimate for primal solution, $\|\nabla e\|$, in the case when a is a constant.
- Linear functional error estimate for the case when a is a constant.
- Application on the dual problem.

Energy norm estimate

We now focus on the case when $a = 1$. Remember the weak form for the exact solution, Find $u_c \in V_c$ and $u_f \in V_f$ such that

$$\begin{aligned} (\nabla u_c, \nabla v_c) + (\nabla u_f, \nabla v_c) &= (f, v_c) \quad \text{for all } v_c \in V_c, \\ (\nabla u_f, \nabla v_f) &= (f, v_f) - (\nabla u_c, \nabla v_f) \quad \text{for all } v_f \in V_f. \end{aligned}$$

Since the first equation also holds for the approximate solution we have

$$(\nabla e_c, \nabla v_c) + (\nabla e_f, \nabla v_c) = 0.$$

Energy norm estimate

$$\|\nabla e\|^2 = (\nabla e, \nabla e) = (\nabla e, \nabla e_f) = (\nabla e, \nabla e_f - P_f^h e) + (\nabla e, \nabla P_f^h e),$$

where P_f^h is the L^2 projection onto V_f^h.

$$(\nabla e, \nabla P_f^h e) = \sum_{i=1}^{n} (\nabla e_c, \nabla \varphi_i P_f^h e) + \sum_{\text{fine}} (\nabla e_{f,i}, \nabla P_f^h e) + \sum_{\text{coarse}} (\nabla u_{f,i}, \nabla P_f^h e)$$

Energy norm estimate

$$
\begin{aligned}
(\nabla e, \nabla P_f^h e) &= \sum_{i=1}^{n} (\nabla e_c, \nabla \varphi_i P_f^h e) + \sum_{\text{fine}} (\nabla e_{f,i}, \nabla P_f^h e) \\
&\quad + \sum_{\text{coarse}} (\nabla u_{f,i}, \nabla P_f^h e) \\
&= \sum_{\text{fine}} (R(U_c), \varphi_i P_f^h e) - (\nabla U_{f,i}, \nabla P_f^h e) \\
&\quad + \sum_{\text{coarse}} (R(U_c), \varphi_i P_f^h e)
\end{aligned}
$$

Energy norm estimate

$$\|\nabla e\|^2 = (\nabla e, \nabla e - P_f^h e) + \sum_{\text{coarse}} (R(U_c), \varphi_i P_f^h e)$$
$$+ \sum_{\text{fine}} (R(U_c), \varphi_i P_f^h e) - (\nabla U_{f,i}, \nabla P_f^h e)$$
$$= I + II + III$$

Energy norm estimate

I

$$(\nabla e, \nabla e - P_f^h e) \leq \|hR(U_c + U_f)\| \|\nabla e\|$$

II

$$\sum_{\text{coarse}} (R(U_c), \varphi_i P_f^h e) \leq C \left(\sum_{\text{coarse}} \|HR(U_c)\|_{S_i^1} \right) \|\frac{1}{H} P_f^h e\|$$

$$\leq C \left(\sum_{\text{coarse}} \|HR(U_c)\|_{S_i^1} \right) \|\nabla e\|$$

Energy Norm Estimate

III

On the black board...

Energy Norm Estimate

$$\|\nabla e\| \leq C\|hR(U_c + U_f)\| + C \sum_{\text{coarse}} \|HR(U_c)\|_{S_i^1}$$
$$+ C\sqrt{H} \sum_{\text{fine}} \|\Sigma_i\|_{\partial\omega_i}$$

- The £rst term is referred to as the truth mesh error (reference).
- The third term is the normal derivative of the fine scale solutions on $\partial\omega_i$.

Dual Problem

The standard approach to get a bound of a linear functional of the error is to introduce a dual problem: £nd $\phi \in H_0^1$ such that

$$-\triangle \phi = \psi.$$

We when get for $\pi\phi \in V_h$,

$$(e, \psi) = (e, -\triangle \phi) = (\nabla e, \nabla \phi) = (\nabla e, \nabla \phi - \pi\phi).$$

And after integration by parts we get

$$(e, \psi) = (R(U), \phi - \pi\phi).$$

Dual Problem

- The dual solution ϕ need to be approximated but not in V.
- Regular refinement or higher order method allocate lots of memory.

Instead we solve the dual problem by local problems in each coarse node,

$$(e,\psi) = \sum_{i=1}^{n} (R(U), \Phi_{f,i}) + (R(U), \phi_f - \Phi_f).$$

Dual Problem

The second term can be estimated in the following way,

$$\begin{aligned}(\nabla e, \nabla(\phi_f - \Phi_f)) &\leq \|\nabla e\| \, \|\nabla(\phi_f - \Phi_f)\| \\ &\leq \|\nabla e\| \, \|\nabla(\phi - (\Phi_c + \Phi_f))\|.\end{aligned}$$

And we get the energy norm of the error in the dual solution which can be estimated.

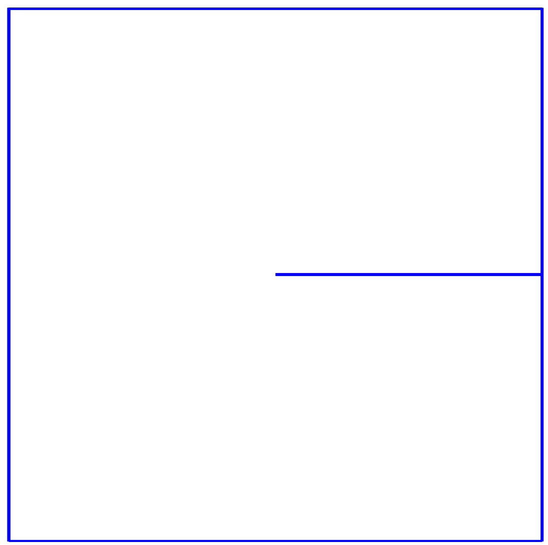
Adaptive Strategy

$$\|\nabla e\| \leq C\|hR(U_c + U_f)\| + C \sum_{\text{coarse}} \|HR(U_c)\|_{S_i^1}$$
$$+ C\sqrt{H} \sum_{\text{fine}} \|\Sigma_i\|_{\partial\omega_i}$$

- We focus on the last two terms.
- We calculate these for each $i \in \{\text{coarse fine}\}$.
- Big values $i \in$ coarse $\longrightarrow$ more local problems.
- Big values $i \in$ fine $\longrightarrow$ more layers.

Numerical Examples

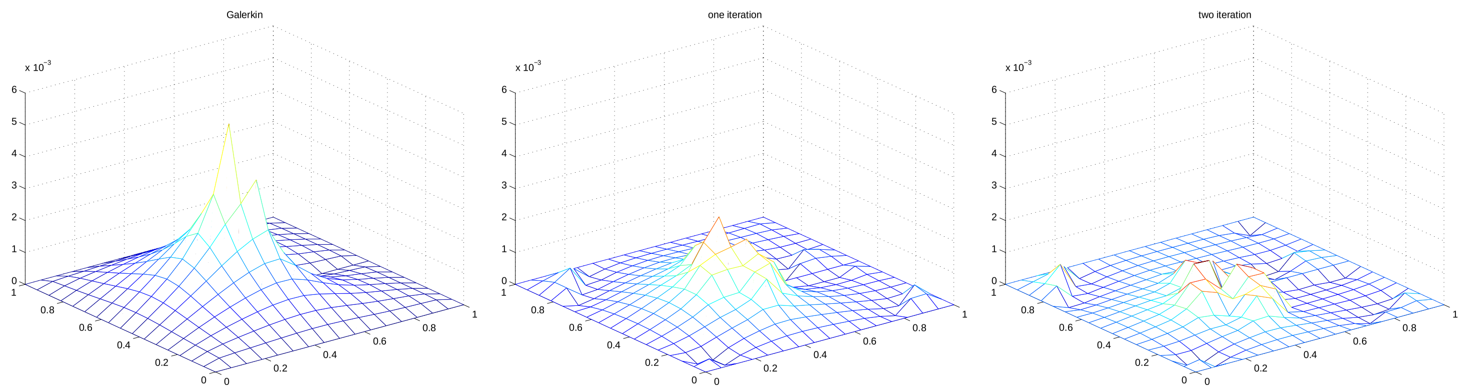
We start with a unit square containing a crack.



We let the coef£cient $a = 1$ and solve, $-\triangle u = f$ with $u = 0$ on the boundary including the crack.

Numerical Examples

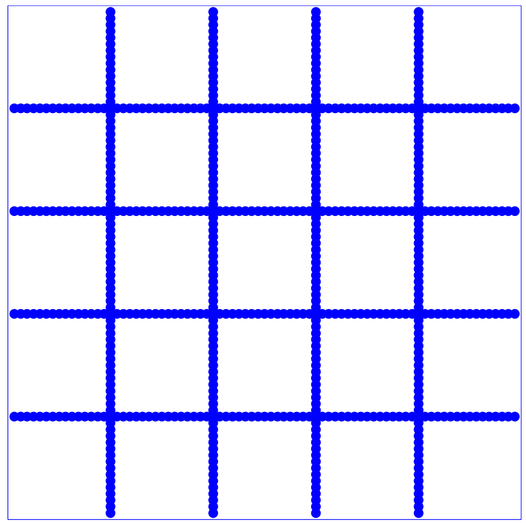
We solve the problem by using the adaptive algorithm with a refinement level of 10 % each iteration.



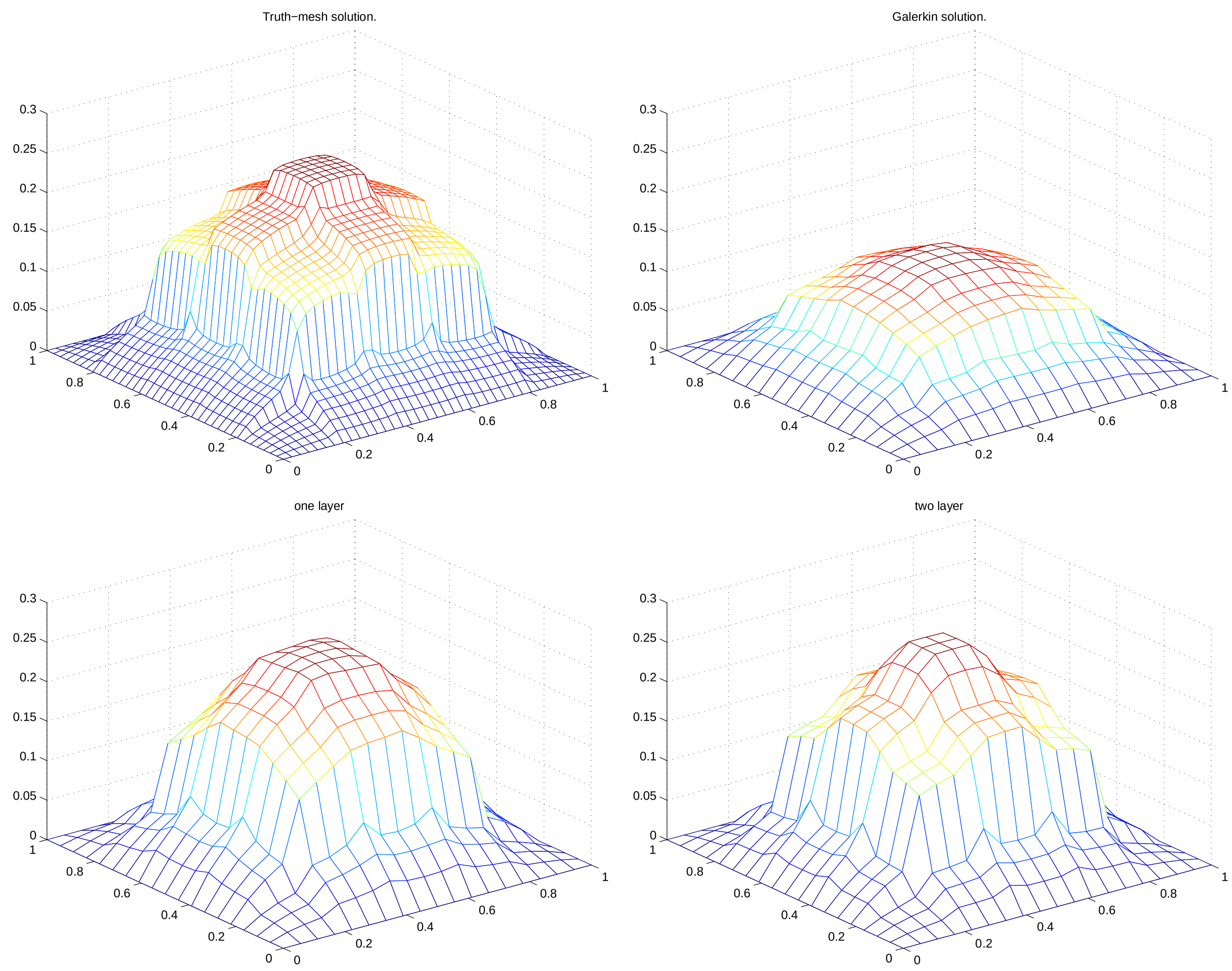
We plot the difference between our solution and a reference solution.

Numerical Examples

In this example we study a discontinuous coef£cient a in $-\nabla \cdot a \nabla u = f$. $a = 1$ (white) and $a = 0.05$ (blue).



Numerical Examples



Future Work

- Error estimates in the case when $a \neq 1$.
- Extended numerical tests in both 2D and 3D.
- More scales.
- Other equations (convection-diffusion, ...).
- Comparing results with classical Homogenization theory.

References

References

[1] B. Aksoylu and M. Holst *An odyssey into local refinement and multilevel preconditioning II: stabilizing hierarchical basis methods*, SIAM J. Numer. Anal. in review

[2] T. J.R. Hughes, *Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation* ... Comput. Methods Appl. Mech. Engrg. 127 (1995) 387-401.