Finite Elements - A Crash Course

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Outline

- Function Spaces
- Function Approximation
- Finite Elements
- Time Dependent Problems
- Stabilized Methods
- Mixed Methods
- Systems of PDE

Linear Functions

Let $P_1(I)$ be the space of linear functions on the interval $I = [a, b]$.

A natural basis for $P_1(I)$ is $\{\lambda_1, \lambda_2\}$, where

$$\lambda_1(x) = \frac{b - x}{b - a}, \qquad \lambda_2(x) = \frac{x - a}{b - a},$$

because any $v(x) \in P_1(I)$ can be written:

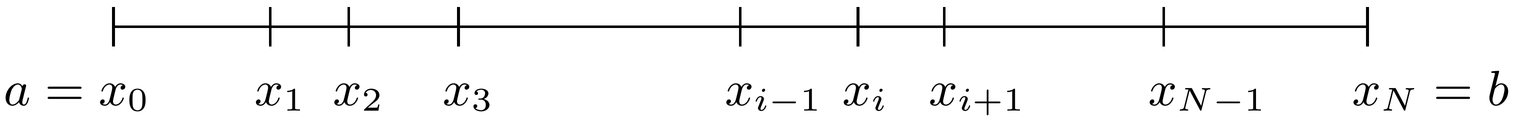
$$\boxed{v(x) = v(a)\lambda_1(x) + v(b)\lambda_2(x)}$$

Interval Partition

For a given interval $I = [a, b]$ let

$$a = x_0 < x_1 < x_2 < \ldots < x_N = b,$$

be a partition of I into intervals $I_i = (x_{i-1}, x_i)$ of length $h_i = x_i - x_{i-1}$.

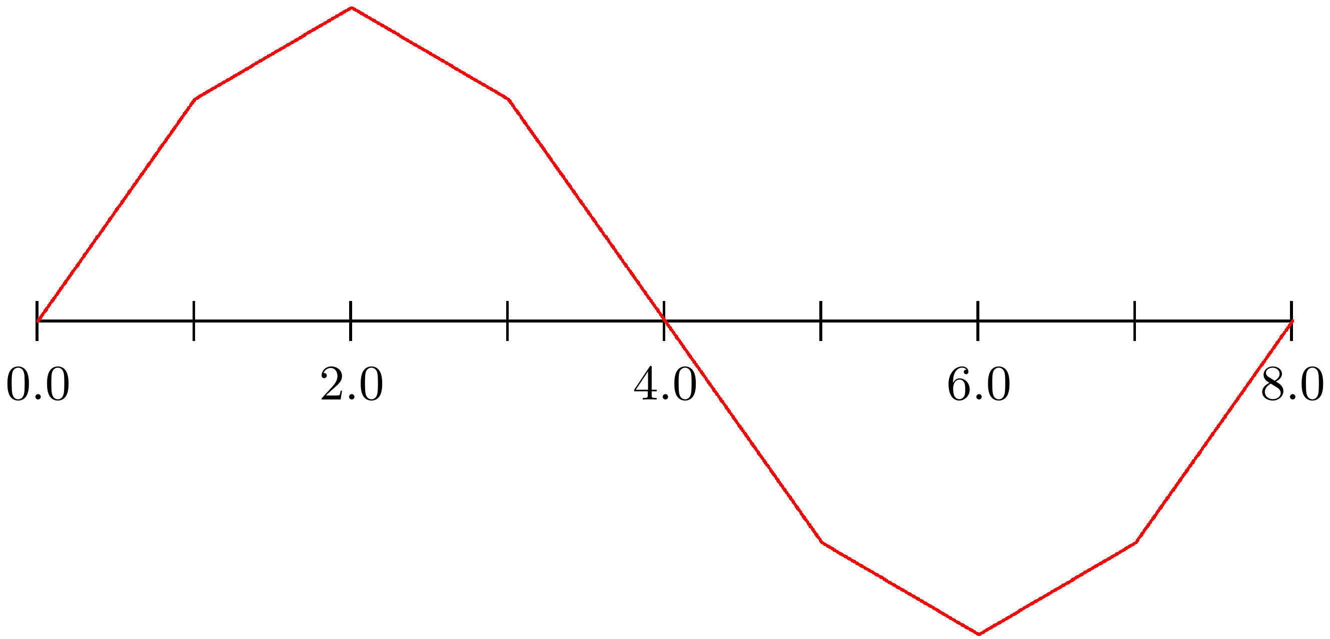


Continuous Piecewise Linears

Define V_h as the space of all continuous piecewise linear functions on I

$$V_h = \{v \in C(I) : v|_{I_j} \in P_1(I_j)\}$$

Example: *Piecewise linear continuous function*

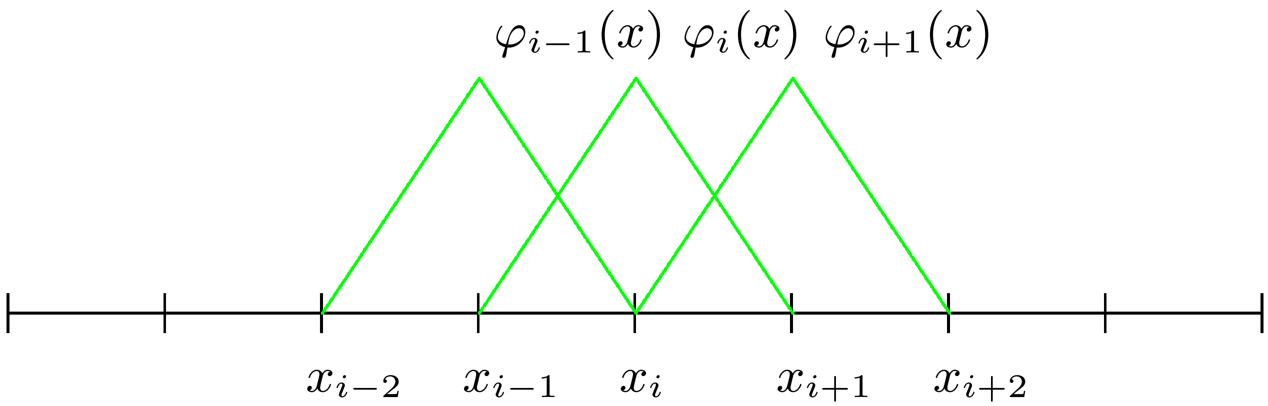


Nodal Basis

A basis for V_h is defined by

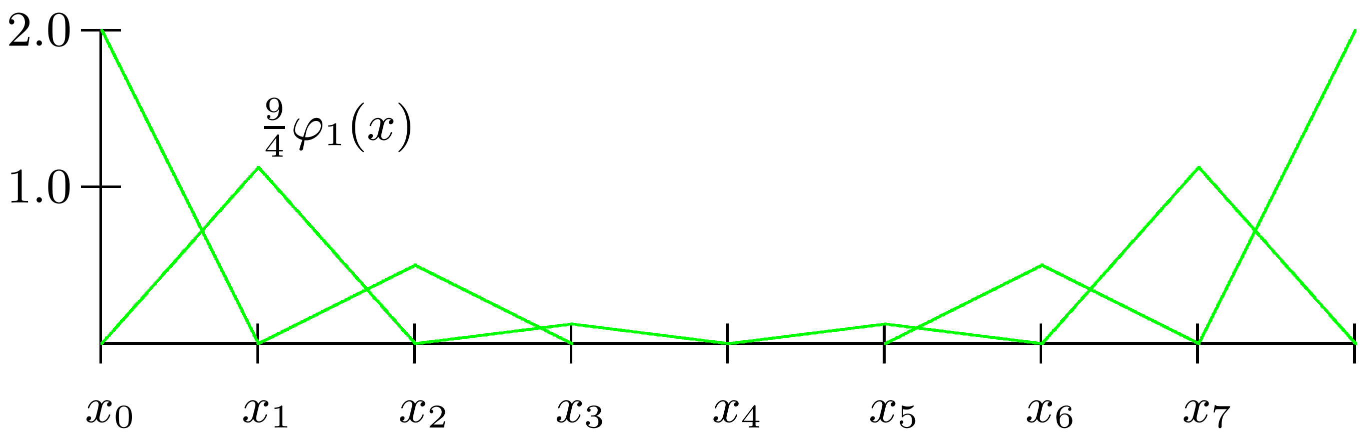
$$\varphi_i(x_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Basis functions (*hat functions*) are locally supported



Function Construction

Scale and add basis functions together

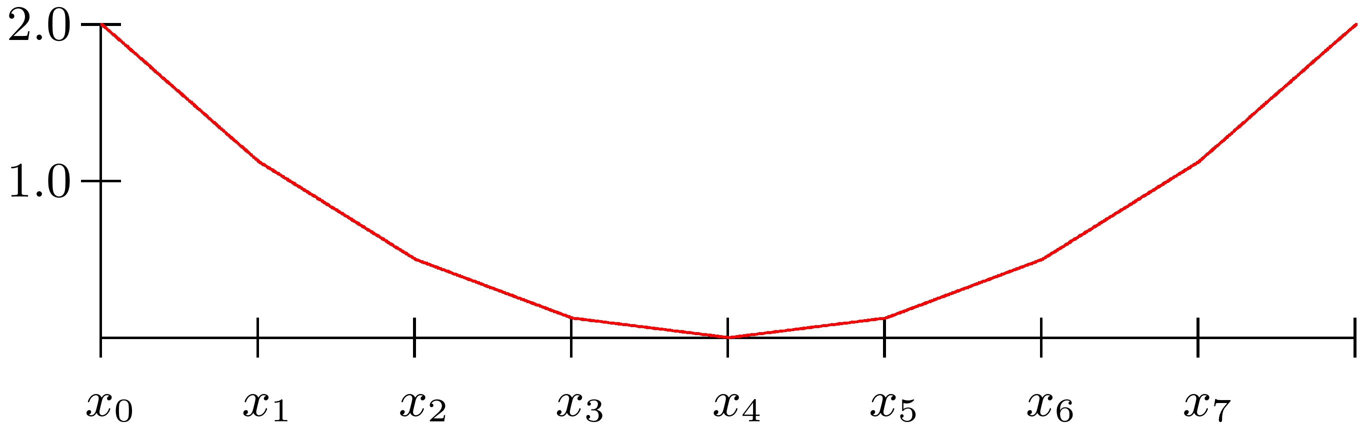


Linear combination of basis functions

$$v(x) = \frac{9}{4}(\varphi_0 + \varphi_9) + \frac{13}{8}(\varphi_1 + \varphi_8) + \varphi_2 + \varphi_7 + \frac{5}{8}(\varphi_4 + \varphi_6)$$

Function Construction, cnt

Resulting function $v(x)$ continuous piecewise linear

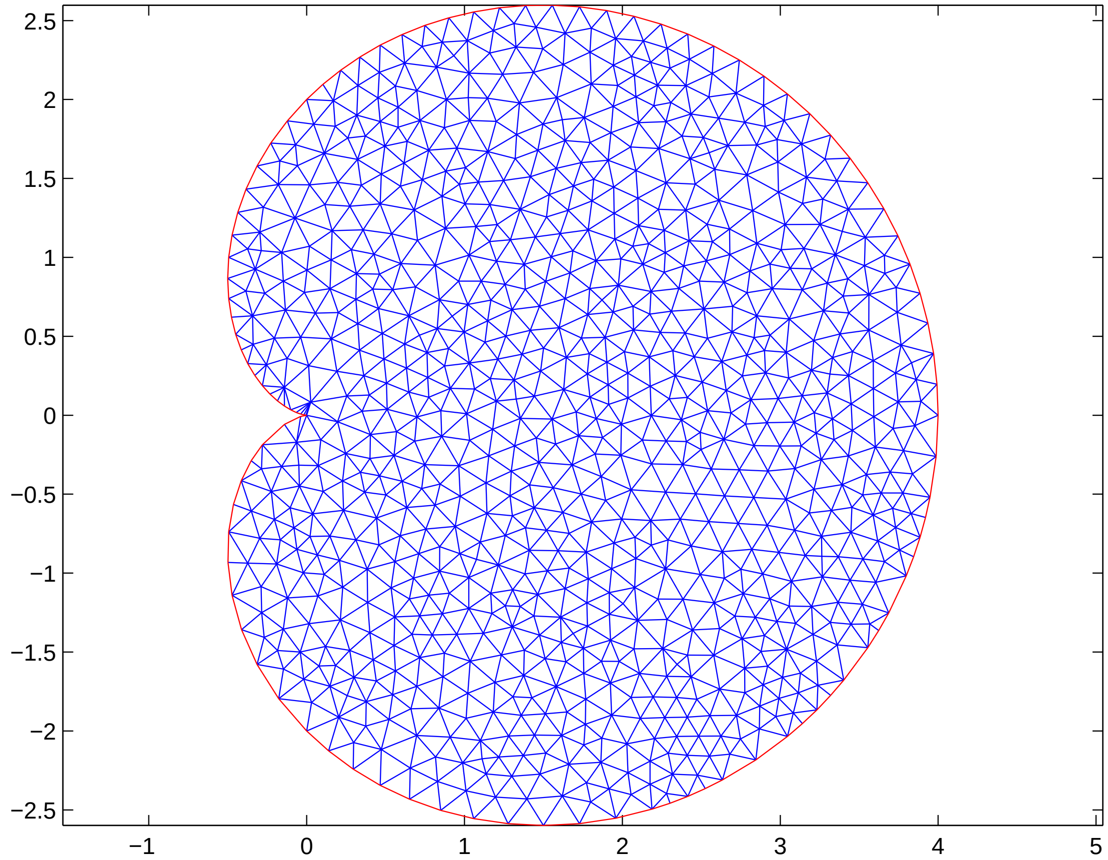


Every function $v \in V_h$ can be written:

$$v(x) = \sum_{i=0}^{N} v(x_i)\varphi_i(x)$$

Generalizing to 2 Dimensions

Given a domain $\Omega \subset \mathbb{R}^2$ we construct partition into triangles, viz.



Triangulations

Basic data structures of a triangulation

1. A set of nodes $\mathcal{P} = \{p_i\}$ (the triangle vertices)

Node i defined by its coordinates $p_i = (x_i, y_i)$

```
MESH dimension 2 ElemType Triangle Nnode 3

Coordinates
0.000000 0.000000
0.250000 0.000000
0.250000 0.250000
0.500000 0.250000
0.500000 0.500000
```

Triangulations, cnt

2. A set of elements $\mathcal{K} = \{K_i\}$ (triangles)

Triangle corner indices stored (connectivity)

```
Elements
1 2 3
1 3 4
4 3 5
4 5 6
7 9 8
```

Local mesh size defined by $h_K = diam(K)$.

Smallest angle of K denoted α_K. Assume $\alpha_K \geq \alpha_0 > 0$ for some constant α_0.

Linear functions on a triangle

Let

$$P_1(K) = \{v(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2\}$$

be the space of linear functions on triangle K.

Using nodal basis defined by

$$\lambda_i(p_j) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases}$$

any function $v(x_1, x_2) \in P_1(K)$ can be written

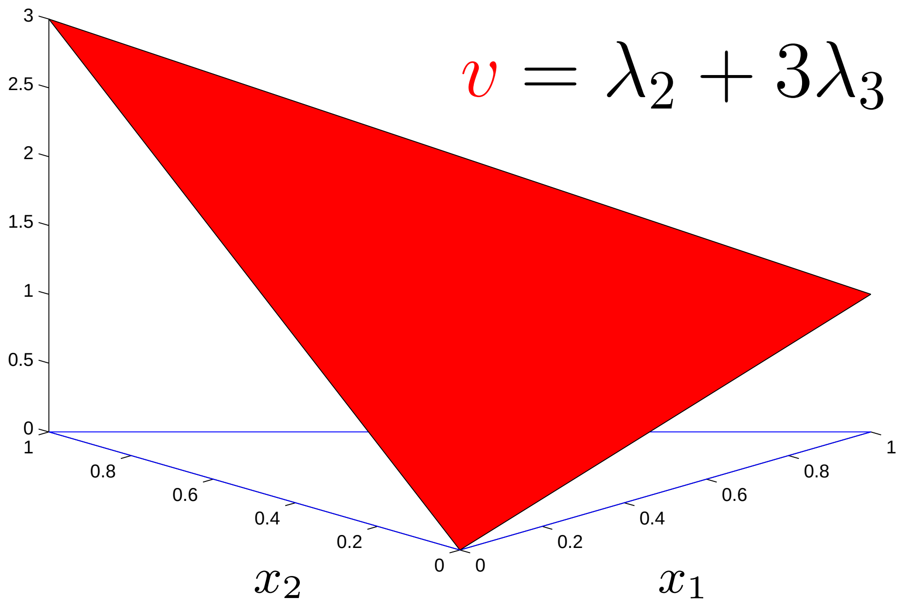
$$\boxed{v(x_1, x_2) = \sum_{i=1}^{3} v(p_i)\lambda_i(x_1, x_2)}$$

Linear functions, cnt

Example: *If* $p_1 = (0,0)$, $p_2 = (1,0)$ *and* $p_3 = (0,1)$ *then*

$$\lambda_1 = 1 - x_1 - x_2, \quad \lambda_2 = x_1, \quad \lambda_3 = x_2.$$

Function $v = x_1 + 3x_2$ is linear combination of bases



Piecewise Continuous Linear Functions on a Triangulation

On a given triangulation let

$$V_h = \{v \in C(\Omega) : v(x)|_{K \in \mathcal{K}} \in P_1(K)\}$$

be the space of piecewise linear continous functions.

Defining a set of basis functions for V_h by

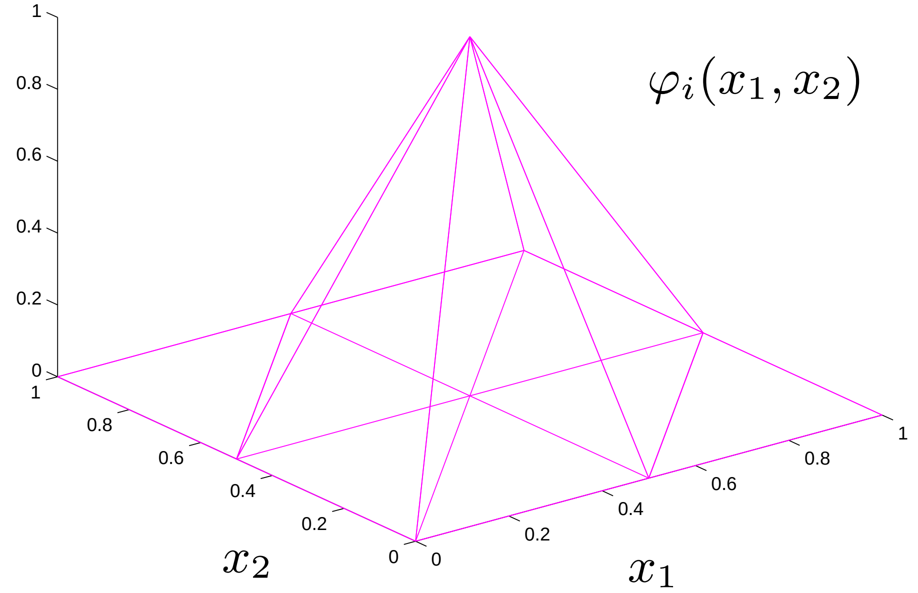
$$\varphi_i(p_j) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases}$$

any $v \in V_h$ can be expressed as:

$$\boxed{v(x) = \sum_{i=0}^{N} v(x_i)\varphi_i(x)}$$

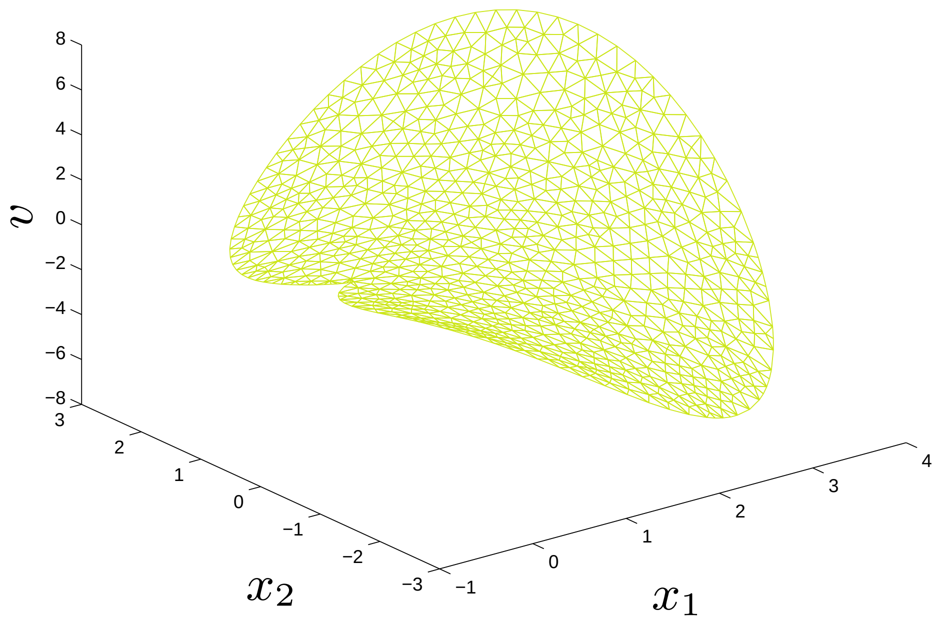
Basis Function

A basis function of V_h (*tent function*)



Functions on a Triangulation

Example: *Continuous piecewise linear function on a triangulation.*



At nodes p_j *function* $v(x_1, x_2) = x_1 x_2$.

Bilinear Elements

Triangulation can also be a quadrilateral mesh.

Let $B_1(K)$ be the space of bilinear functions, i.e.

$$B_1(K) = \{v(x) = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2\},$$

on a quadrilateral element K.

Basis defined by

$$\lambda_i(p_j) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}$$

Bilinear Elements, cnt

On the reference element $K = [-1, 1] \times [-1, 1]$:

$$\lambda_1 = \frac{1}{4}(1 + x_1)(1 + x_2)$$
$$\lambda_2 = \frac{1}{4}(1 - x_1)(1 + x_2)$$
$$\lambda_3 = \frac{1}{4}(1 + x_1)(1 - x_2)$$
$$\lambda_4 = \frac{1}{4}(1 - x_1)(1 - x_2)$$

Other Elements

- *Discontinuous elements* (piecewise constants)
- *Higher order polynomials* (quadratics, cubics)
- $\nabla\times$ *conforming elements* (electromagnetics)
- *Beam-plate-shell elements* (solid mechanics)

Function approximations

Given a function f, find approximation to f in V_h.

Possible approximation methods:

- *Interpolation - minimize error pointwise*
- *Projection - minimize error norm over subspace.*

Interpolation

Defined by interpolation operator π

$$\pi : C(\Omega) \to V_h,$$

such that the interpolant πv of $v(x)$ satisfies

$$\pi v(x) = \sum_{i=0}^{N} v(x_i)\varphi_i(x)$$

where $\{x_i\}_0^N$ is a set of nodes and $\{\varphi_i\}_1^N$ a basis of V_h.

Interpolation Error Estimate

Interpolation error satisfies

$$\|v - \pi v\|_t \leq C h^{s-t} \|v\|_s,$$

where h is meshsize and C a constant.

L_2-projection

Given a function f we seek its projection Pf onto V_h.

Error $e = f - Pf$ should be orthogonal to all $v \in V_h$,

$$(f - Pf, v) = 0,$$

for all $v \in V_h$. Here $(v, w) = \int_\Omega vw \, dx$.

Find projection $Pf \in V_h$ to f such that

$$(f, v) = (Pf, v)$$

for all $v \in V_h$.

Question: *How can we compute* Pf*?*

L_2-projection, cnt

Note that $(f, v) = (Pf, v)$ is equivalent to

$$(f, \varphi_i) = (Pf, \varphi_i)$$

for all φ_i, $i = 1, 2, \ldots, N$.

Recall also that

$$Pf(x) = \sum_{j=1}^{N} \xi_j \varphi_j(x)$$

with unknown coefficients ξ_j.

L_2-projection, cnt

We obtain

$$b_i = (f, \varphi_i) = \sum_{j=1}^{N} \xi_j (\varphi_j, \varphi_i)$$

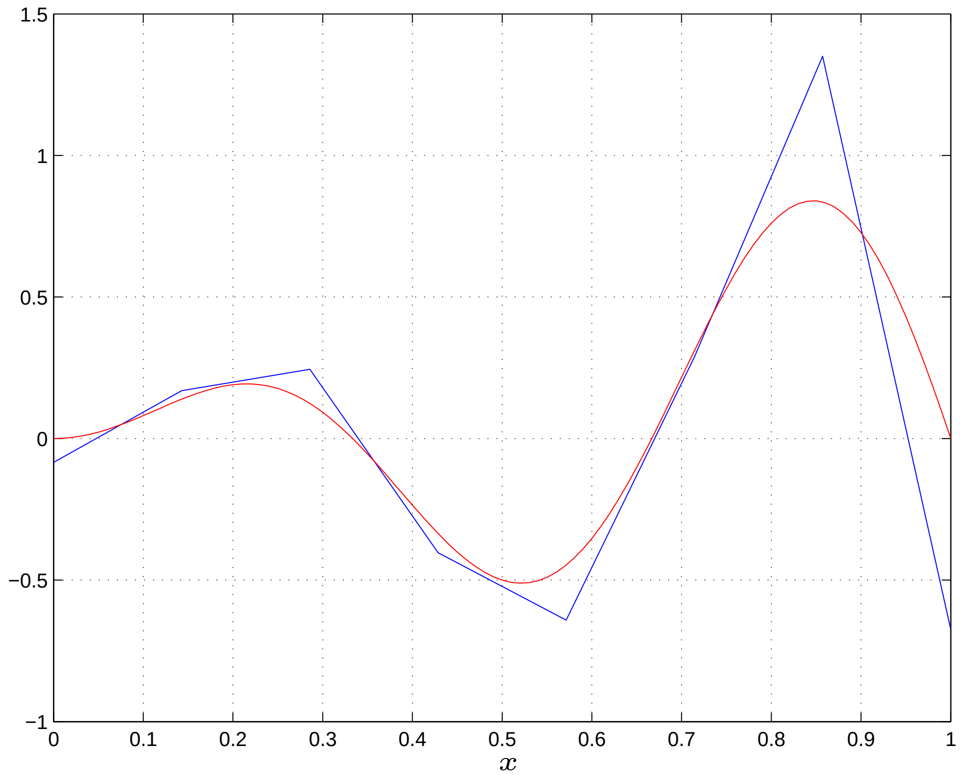
$$= \sum_{j=1}^{N} m_{ij} \xi_j, \qquad i = 1, 2, \ldots, N,$$

which is just a linear system of equations:

$$M\xi = b$$

L_2-projection, cnt

Example: *L_2-approximation of* $x \sin 3\pi x$



Basic Error Estimates

Projection Pf is best approximation to f over V_h.

$$\|f - Pf\| \leq \|f - v\|,$$

for all $v \in V_h$.

Proof:

$$\begin{aligned} \|f - Pf\|^2 &= (f - Pf, f - Pf) \\ &\leq (f - Pf, f - v) + \underbrace{(f - Pf, v - Pf)}_{0 \ orthogonality} \\ &\leq \|f - Pf\| \|f - v\| \end{aligned}$$

where $\|u\|^2 = (u, u)$.

Basic Error Estimates, cnt

Choose $v = \pi f$ to get error estimate

$$\|f - Pf\| \leq \|f - \pi f\| \leq \|h^s f\|_s.$$

Model Problem (Poisson)

Find u such that

$$\begin{aligned} -\Delta u &= f, \qquad && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \end{aligned}$$

where f is a given function, and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Variational statement

Let H_0^1 be the Hilbert space defined by

$$H_0^1(\Omega) = \{v : \|v\|^2 + \|\nabla v\|^2 < \infty,\ v\,|_{\partial\Omega} = 0\}.$$

Multiply $-\Delta u = f$ by function $v \in H_0^1$ and use that

$$\int_\Omega fv = \int_\Omega -\Delta uv = \int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \partial_n uv,$$

i.e. integration by parts.

Variational statement

Using that $v = 0$ on $\partial\Omega$, we get variational form.

Variational Statement: Find $u \in H_0^1$ such that

$$\int_\Omega fv = \int_\Omega \nabla u \cdot \nabla v,$$

for all $v \in H_0^1$.

Finite Elements

Finite elements are obtained by replacing H_0^1 by V_h.

Finite Element Method: Find $U \in V_h$ such that

$$\int_\Omega fv = \int_\Omega \nabla U \cdot \nabla v,$$

for all $v \in V_h$.

Question: *How can we compute U?*

Finite Elements, cnt

First note that the problem is equivalent to

$$(\nabla U, \nabla \varphi_i) = (f, \varphi_i) \qquad i = 1, \ldots, N,$$

and that

$$U = \sum_{j=1}^{N} \xi_j \varphi_j(x)$$

with unknown coefficients ξ_j .

Finite Elements, cnt

This gives the problem

$$b_i = (f, \varphi_i) = \sum_{j=1}^{N} \xi_j (\nabla \varphi_j, \nabla \varphi_i)$$

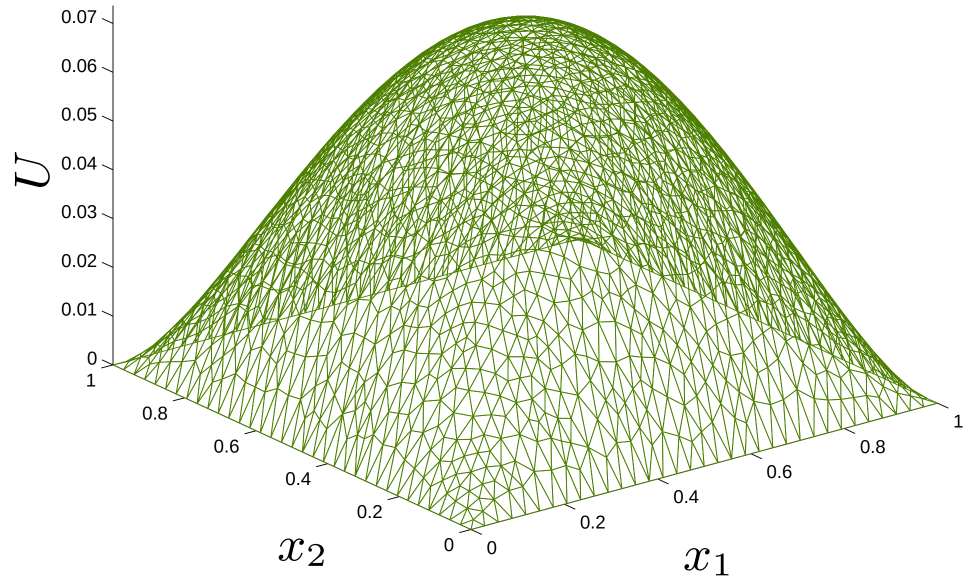
$$= \sum_{j=1}^{N} a_{ij} \xi_j, \qquad i = 1, 2, \ldots, N,$$

i.e. a linear system of equations

$$b = A\xi$$

Finite Elements, cnt

Example: *Solution of Poisson problem on* $\Omega = [0,1] \times [0,1]$.



Here $f = 1$ *with boundary condition* $u = 0$ *on* $\partial\Omega$.

Galerkin Orthogonality

Note that we have

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v,$$

$$\int_\Omega \nabla U \cdot \nabla v = \int_\Omega f v,$$

so

$$\int_\Omega \nabla (u - U) \cdot \nabla v = 0,$$

for all $v \in V_h$. Error $e = u - U$ is orthogonal to V_h.

Energy Error Estimate

Mechanical analogy: Energy norm defined by

$$\|u\|_E^2 = \|\nabla u\|^2 = \int_\Omega |\nabla u|^2.$$

Convienient measure of error $e = u - U$.

Basic error estimate

$$\|\nabla(u - U)\| \leq \|\nabla(u - v)\|,$$

for all $v \in V_h$.

Energy Error Estimate, cnt

Proof:

$$\begin{aligned}\|\nabla(u-U)\|^2 &= \int_\Omega \nabla(u-U)\cdot\nabla(u-U)\\ &= \int_\Omega \nabla(u-U)\cdot\nabla(u-v+v-U)\\ &= \int_\Omega \nabla(u-U)\cdot\nabla(u-v)\\ &\leq \|\nabla(u-U)\|\,\|\nabla(u-v)\|,\end{aligned}$$

thus

$$\|\nabla(u-U)\| \leq \|\nabla(u-v)\|.$$

L_2-Error Estimate

Based on *dual problem*

$$
\begin{aligned}
-\Delta\phi &= e, && \text{in } \Omega,\\
\phi &= 0, && \text{on } \partial\Omega.
\end{aligned}
$$

Dual solution ϕ gives error estimate

$$
\begin{aligned}
\|\nabla e\|^2 &= (\nabla e, \nabla e)\\
&= (\nabla e, \nabla(\phi - \pi\phi))\\
&\leq \|\nabla e\| \|\nabla(\phi - \pi\phi)\|\\
&\leq Ch\|\nabla e\|\\
&\leq Ch^2|u|_2.
\end{aligned}
$$

Robin and Neumann BC

Consider problem of finding u such that

$$
\begin{aligned}
-\Delta u &= f, && \text{in } \Omega, \\
u &= 0, && \text{on } \Gamma_1, \\
\partial_n u + a u &= g, && \text{on } \Gamma_2,
\end{aligned}
$$

where f, a and g are given data and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$.

Robin and Neumann BC, cnt

Integration by parts gives variational statement

$$\int_\Omega fv = -\int_\Omega \Delta uv$$

$$= \int_\Omega \nabla u \cdot \nabla v - \int_{\Gamma_2} \partial_n uv$$

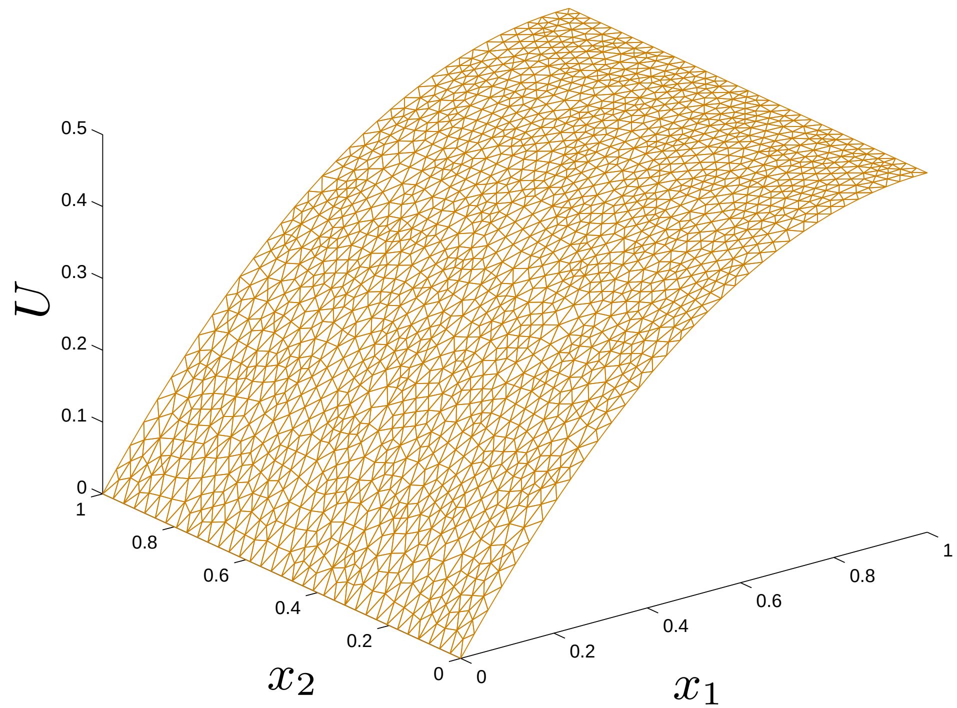
$$= \int_\Omega \nabla u \cdot \nabla v - \int_{\Gamma_2} (g - au)v.$$

Find u such that

$$\int_\Omega \nabla u \cdot \nabla v + \int_{\Gamma_2} auv = \int_\Omega fv + \int_{\Gamma_2} gv.$$

Robin and Neumann BC, cnt

Example: *Solution of Poisson problem on* $\Omega = [0,1] \times [0,1]$.



Here $f = 1$ *with boundary conditions* $u = 0$ *on* Γ_1, *the* x_2 *axis, and* $\partial_n u = 0$ *on* Γ_2, *the rest of* $\partial\Omega$.

Abstract Setting

Let V be a Hilbert space with norm $\| \cdot \|_V$.

Consider problem of finding u such that

$$a(u, v) = l(v)$$

for all $v \in V$, where $a(\cdot, \cdot)$ is *bilinear form* satisfying

- $m\|v\|_V \leq a(v, v)$ (coercivity)
- $a(v, w) \leq M\|v\|_V\|w\|_V$ (continuity)

and $l(v)$ *linear functional* satisfying

- $|l(v)| \leq C\|v\|_V$.

Abstract Setting, cnt

Example: *Poisson model problem*

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v, \quad l(v) = \int_\Omega f v.$$

Let $V_h \subset V$ be a finite dimensional subspace of V.

FEM: Find $u \in V_h$ such that

$$a(u,v) = l(v)$$

for all $v \in V_h$.

Galerkin Orthogonality

Using abstract notations $a(\cdot, \cdot)$ and $l(\cdot)$ yields

$$a(u, v) = l(v) \quad v \in V,$$
$$a(U, v) = l(v) \quad v \in V_h,$$

so

$$a(u - U, v) = 0,$$

for all $v \in V_h$.

Equivalent Minimization

Problem of finding u such that

$$a(u, v) = l(v)$$

and minimization problem

$$\min_{v \in V_h} F(v) = \min_{v \in V_h} \frac{1}{2} a(v, v) - l(v)$$

have the same solution (*Lax Milgram*).

Error Estimate

Error depending on constants m and M of $a(\cdot,\cdot)$

$$\|u - U\|_V \leq \frac{M}{m}\|u - v\|_V \text{ for all } v \in V_h$$

Proof:

$$\begin{aligned} m\|u - U\|^2 &\leq a(u - U, u - U) \\ &= a(u - U, u - v) + \underbrace{a(u - U, v - U)}_{0,\ v-U\in V_h} \\ &\leq M\|u - U\|\|u - v\| \end{aligned}$$

Time Dependent Problems

Ordinary Differential Equations

Find $u : [0, T] \to \mathbb{R}^n$ such that

$$\begin{aligned} \dot{u} + Au &= f, \quad 0 < t \leq T, \\ u(0) &= u_0. \end{aligned}$$

Here $A = A(t) \in \mathbb{R}^{n \times n}$ and $f(t) \in \mathbb{R}^n$ given function.

Time Dependent Problems, cnt

Partition $0 \leq t \leq T$ into time intervals

$$0 = t_0 < t_1 < t_2 < \ldots < t_N,$$

of length $k_N = t_N - t_{N-1}$.

As before define a function space on the partition.

V_q^c - space of continuous piecewise polynomials of degree q.

V_q^d - space of discontinuous piecewise polynomials of degree q.

Galerkin Method

Multiply by test function $v \in V_{q-1}^d$ and integrate to get

$$\int_0^T fv = \int_0^T \dot{u}v + \int_0^T Auv,$$

for all $v \in V_{q-1}^d$.

Continuous piecewise linear solution approximation.

cG(1): Find $U \in V_1^c$ such that

$$\int_0^T fv = \int_0^T \dot{u}v + \int_0^T Auv,$$

for all $v \in V_0^d$.

Galerkin Method, cnt

Solution basis functions on $I_N = [t_{N-1}, t_N]$

$$\psi_{N-1}(t) = \frac{t_N - t}{k_N}, \qquad \psi_N(t) = \frac{t - t_{N-1}}{k_N}.$$

So $U = U_N \psi_N(t) + U_{N-1}\psi_{N-1}(t)$ on I_N.

Evaluate weak form on I_N, i.e.

$$\int_{I_N} fv = \int_{I_N} \dot{U} v + \int_{I_N} AUv,$$

to get iteration scheme.

Galerkin Method, cnt

Example: *Assume* $A(t) = 1$ *and* $f = 0$.

$$
\begin{aligned}
0 &= \int_{I_N} \dot{U} v + \int_{I_N} U v \\
&= \int_{I_N} \frac{U_N - U_{N-1}}{k_N} v + \int_{I_N} (U_N \psi_N + U_{N-1}\psi_{N-1})\, v \\
&= U_N - U_{N-1} + \frac{k_N}{2} U_N + \frac{k_N}{2} U_{N-1},
\end{aligned}
$$

since $v = 1$. *Crank-Nicholson* iteration form

$$
\left(1 + \frac{k_N}{2}\right) U_N = \left(1 - \frac{k_N}{2}\right) U_{N-1},
$$

Heat equation

Combination of time and space discretization.

Consider the *Heat Equation*

$$\begin{aligned} \dot{u} - \Delta u &= f, \quad &&\text{in } \Omega \times [0, T], \\ u(0, x) &= u_0, \quad &&\text{on } \Omega, \\ u(t, .) &= 0, \quad &&\text{on } \partial\Omega. \end{aligned}$$

Multiplying by function $v \in \mathcal{W}$ with

$$\mathcal{W} = L^2([0, T]) \bigotimes H^1(\Omega)$$

and integrating over $\Omega \times [0, T]$ yields (next slide)

Heat Equation, cnt

$$\int_0^T (\dot{u}, v) - (\Delta u, v)\, dt = \int_0^T (f, v)\, dt.$$

Integration by parts and BC gives variational form.

Find $u \in \mathcal{W}$ such that

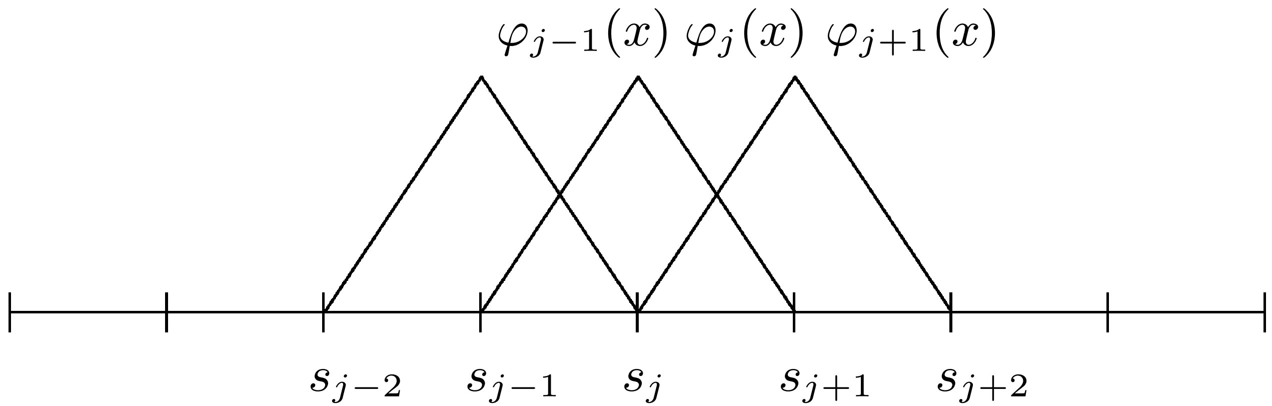
$$\int_0^T (\dot{u}, v) + a(u, v)\, dt = \int_0^T (f, v)\, dt,$$
$$u(0, x) = u_0,$$

for every $v \in \mathcal{W}$.

Finite Element Approximation

- Let $\mathcal{V}^p \subset H^1(\Omega)$ denote the space of piecewise continuous functions of order p.

  Standard nodal basis of $\mathcal{V}^1$.



- On each *space-time* slab $S_N = I_N \times \Omega$, define

$$\mathcal{W}_N^q = \{w : w = \sum_{j=0}^{q} t^j v_j(s),\ v_j \in \mathcal{V}^p,\ (t,s) \in S_N\}.$$

Finite Element, cnt

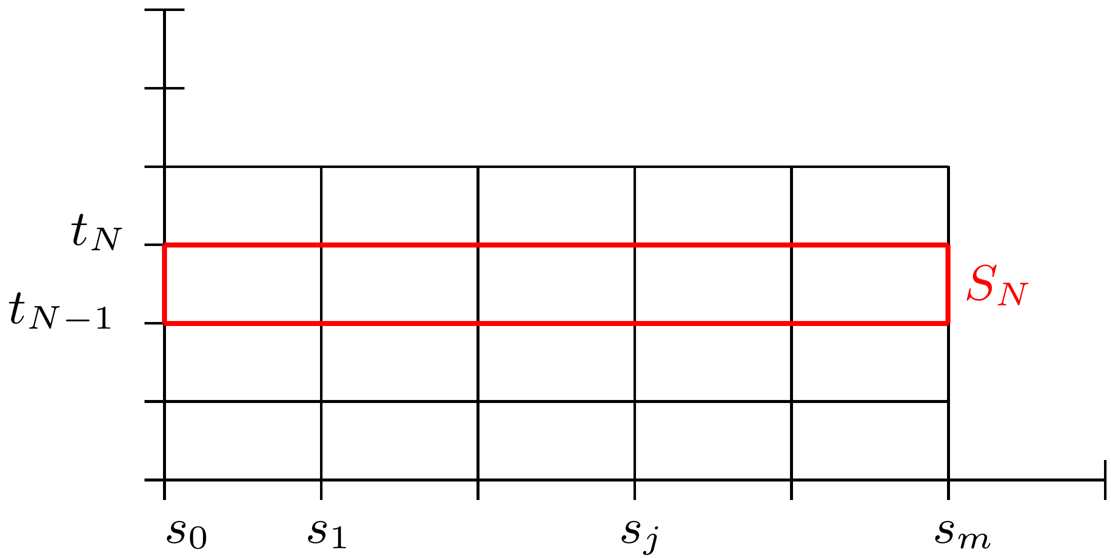


Figure: *Space-time* discretization.

- Let $\mathcal{W}^q \subset \mathcal{W}$ denote the space of functions on $[0, T] \times \Omega$ such that $v\,|_{S_N} \in \mathcal{W}_N^q$ for every N.

Finite Element Problem

FEM: Find $U \in \mathcal{W}^q$ such that

$$\int_{I_n} (\dot{U}, v) + a(U, v)\, dt = \int_{I_n} (f, v)\, dt,$$
$$U^+(t_N) - U^-(t_N) = 0,$$
$$U^+(t_0) = u_0,$$

for all $v \in \mathcal{W}_N^{q-1}$. Here $U^\pm(t_N) = \lim_{\epsilon \to 0} U(t_N \pm \epsilon)$.

Matrix Problem

Example: *Assume* $q = 1$.

Looking only at interval I_N *iteration form is derived*

$$M(U_N - U_{N-1}) + \frac{k_N}{2} S(U_N + U_{N-1}) = F_N,$$

where matrix and vector entries are given by

$$S_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx, \qquad M_{ij} = \int_\Omega \varphi_i \varphi_j \, dx,$$

$$F_{N,j} = \int_{I_N} \int_\Omega f \varphi_j \, dx dt.$$

Wave Equation

Extend to second order equations in time.

Consider for simplicity 1D *Wave Equation*.

Seek u such that

$$\ddot{u} - u'' = 0, \quad 0 \leq x \leq 1, \quad t > 0$$
$$u = 0, \quad u' = 0, \quad t > 0$$
$$u(x,0) = u_0, \quad \dot{u}(x,0) = v_0,$$

where g, u_0 and v_0 are given indata.

Wave Equation, cnt

Substitute $\dot{u} = v$ and write as a system

$$\begin{cases} \dot{u} - v &= 0, \\ \dot{v} - u'' &= 0. \end{cases}$$

Make the cG(1)cG(1) *ansatz*

$$U(x,t) = U_{N-1}(x)\psi_{N-1}(t) + U_N(x)\psi_N(t)$$
$$V(x,t) = V_{N-1}(x)\psi_{N-1}(t) + V_N(x)\psi_N(t)$$

where $U_N(x) = \sum_{J=1}^{m} \xi_{N,J}\varphi_J(x)$ etc.

Wave Equation, cnt

Note that $\dot{u} - v = 0$ implies

$$\int_{I_n} \int_0^1 \dot{u}\eta \, dxdt - \int_{I_n} \int_0^1 v\eta \, dxdt = 0,$$

for all $\eta(x, t)$.

Also,

$$\int_{I_n} \int_0^1 \dot{v}\eta \, dxdt + \int_{I_n} \int_0^1 u'\eta' \, dxdt = 0,$$

for all $\eta(x, t)$ such that $\eta(0, t) = 0$.

Wave Equation, cnt

Find $U(x,t)$ and $V(x,t)$ such that

$$\int_{I_N} \int_0^1 \frac{U_N - U_{N-1}}{k_N} \varphi_j \, dxdt$$

$$- \int_{I_N} \int_0^1 (V_{N-1}\psi_{N-1} + V_N \psi_N) \varphi_j \, dxdt = 0,$$

and

$$\int_{I_N} \int_0^1 \frac{V_N - V_{N-1}}{k_N} \varphi_j \, dxdt$$

$$+ \int_{I_N} \int_0^1 (U'_{N-1}\psi_{N-1} + U'_N \psi_N) \varphi'_j \, dxdt = 0.$$

Wave Equation, cnt

Previous problem reduces to

$$\int_0^1 U_N \varphi_j z \, dx - \frac{k_N}{2} \int_0^1 V_N \varphi_j \, dx$$
$$= \int_0^1 U_{N-1} \varphi_j \, dx + \frac{k_N}{2} \int_0^1 V_{N-1} \varphi_j \, dx,$$

and

$$\int_0^1 V_N \varphi_j \, dx + \frac{k_N}{2} \int_0^1 U_N' \varphi_j' \, dx$$
$$= \int_0^1 V_{N-1} \varphi_j \, dx - \frac{k_N}{2} \int_0^1 U_{N-1}' \varphi_j' dx.$$

Wave Equation, cnt

Vectors U_N and V_N are determined by

$$\left\{ \begin{array}{rcl} MU_N - \frac{k_N}{2} MV_N & = & MU_{N-1} + \frac{k_N}{2} MV_{N-1} \\ \frac{k_N}{2} SU_N + MV_N & = & MV_{N-1} - \frac{k_N}{2} SU_{N-1} \end{array} \right. ,$$

where matrix elements are given by

$$S_{ij} = \int_0^1 \varphi_i' \varphi_j' \, dx, \qquad M_{ij} = \int_0^1 \varphi_i \varphi_j \, dx.$$

Obtains iteration scheme

$$\begin{bmatrix} M & -\frac{k_N}{2} M \\ \frac{k_N}{2} S & M \end{bmatrix} \begin{bmatrix} U_N \\ V_N \end{bmatrix} = \begin{bmatrix} M & \frac{k_N}{2} M \\ -\frac{k_N}{2} S & M \end{bmatrix} \begin{bmatrix} U_{N-1} \\ V_{N-1} \end{bmatrix}.$$

Double Slit Diffraction 1

Example: *Simulation showing a diffracting wave*

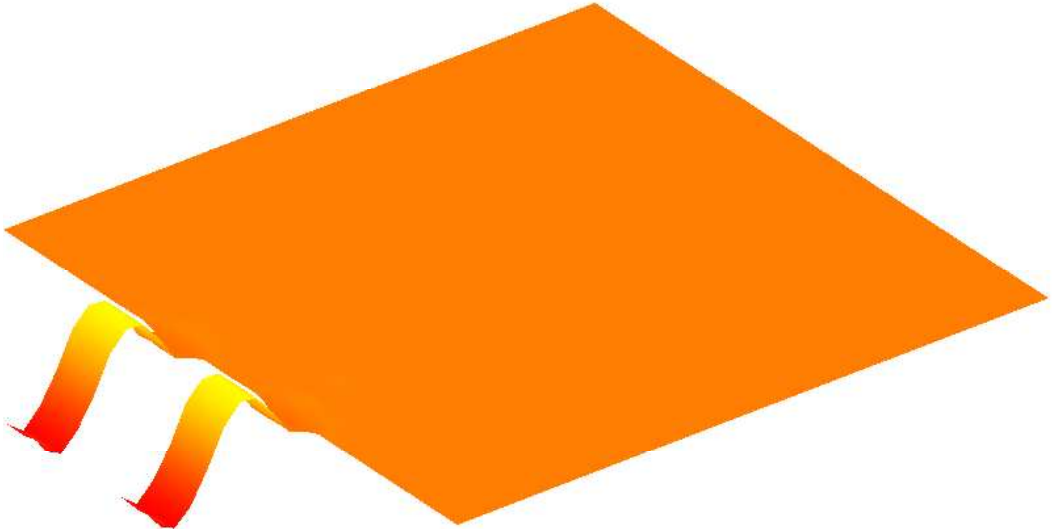


Figure: Light waves encounter a double slit.

Double Slit Diffraction 2

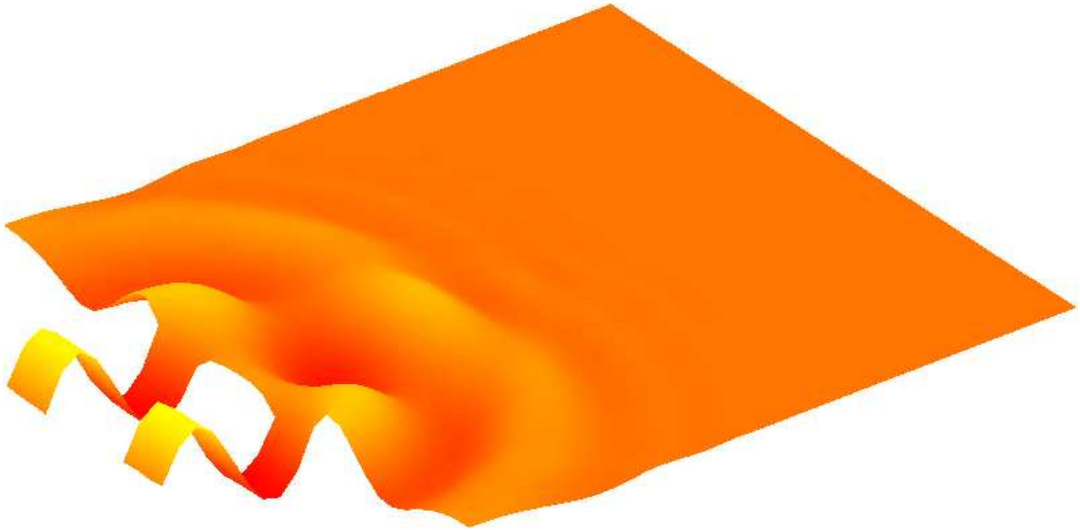


Figure: Slit causes diffraction of waves.

Double Slit Diffraction 3

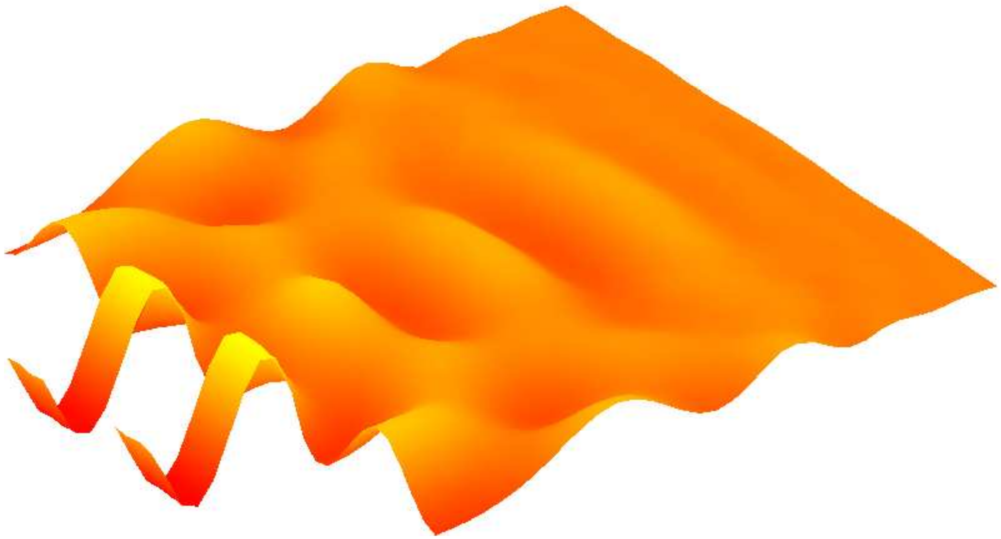


Figure: Superposition of waves give diffraction pattern.

Stabilized Methods

Consider the abstract problem

$$\mathcal{L}u = f, \text{ in } \Omega, \qquad u = 0, \text{ on } \partial\Omega.$$

Variational statement reads find $u \in V$ such that

$$a(u, v) = (f, v)$$

for all $v \in V$. Here $a(u, v) = (\mathcal{L}u, v) = (u, \mathcal{L}^*, v)$.

Standard Galerkin: Find $u \in V_h$ such that

$$a(u, v) = (f, v)$$

for all $v \in V_h$.

Stabilized Methods, cnt

Stabilized Galerkin: Find $u \in V_h$ such that

$$a(u,v) + (\tau(\mathcal{L}u - f), \mathbb{L}v)_{\mathcal{K}} = (f,v)$$

for all $v \in V_h$. Here $\mathbb{L}$ is a stabilizing operator, e.g.

- $\mathcal{L}$ - **(GLS)**.
- $\mathcal{L}_{adv}$ - **(SUPG)**.
- $-\mathcal{L}^*$.

Parameter τ determines size of stabilization. Further,

$$(v,w)_{\mathcal{K}} = \sum_{K \in \mathcal{K}} (v,w)_K.$$

Stabilized Methods, cnt

Example: *Convection-Diffusion Problem*

$$\begin{aligned} \beta \cdot \nabla u - \epsilon \Delta u &= f, \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega. \end{aligned}$$

Galerkin Least Squares method

GLS: Find $u \in V_h$ such that

$$\begin{aligned} (f, v) = (\beta \cdot \nabla u, v) - (\epsilon \nabla u, \nabla v) \\ + (\tau(\beta \cdot \nabla u - \epsilon \Delta u - f), (\beta \cdot \nabla v - \epsilon \Delta v))_K. \end{aligned}$$

for all $v \in V_h$.

Stabilized Methods, cnt

Streamline Upwind Petrov Galerkin method

SUPG: Find $u \in V_h$ such that

$$\begin{aligned}(f, v) &= (\beta \cdot \nabla u, v) - (\epsilon \nabla u, \nabla v) \\ &\quad + (\tau(\beta \cdot \nabla u - \epsilon \Delta u - f), (\beta \nabla \cdot v))_K\end{aligned}$$

for all $v \in V_h$.

Mixed Methods

Consider Stokes problem

$$\begin{aligned} -\Delta u + \nabla p &= f, \text{ in } \Omega, \\ \nabla \cdot u &= 0, \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega. \end{aligned}$$

Here u is velocity, p pressure and f given data.

Note that pressure is not unique and that we may add

$$\int_\Omega p = 0.$$

Mixed Methods, cnt

We may seek $u \in V = H^1_0$ and $p \in Q = L_2$ such that

$$
\begin{aligned}
(\nabla u, \nabla v) - (p, \nabla \cdot u) &= (f, v) \\
(q, \nabla \cdot u) &= 0,
\end{aligned}
$$

for all $v \in V$ and $q \in Q$.

Mixed Methods, cnt

Introduce subspaces $V_h \subset V$ and $Q_h \subset Q$.

Mixed Galerkin: Find $U \in V_h$, $P \in Q_h$ such that

$$
\begin{aligned}
(\nabla U, \nabla v) - (P, \nabla \cdot U) &= (f, v) \\
(q, \nabla \cdot U) &= 0,
\end{aligned}
$$

for all $v \in V_h$ and $q \in Q_h$.

Not all spaces (V_h, Q_h) give stable method.

Combination $V_h = Q_h = P_1$ is instable, for instance.

Mixed Methods, cnt

The Babuska-Brezzi condition

$$\sup_{v \in V_h} \frac{(q, \nabla \cdot u)}{\|v\|_1} \geq \|q\|,$$

gurarantees that the pair (V, Q) is stable.

Can then prove the estimate

$$\|u - U\|_1 + \|p - P\| \leq C(\|u - v\|_1 + \|p - q\|),$$

for all $v \in V_h$ and $q \in Q_h$.

dG Methods - A Model Problem

Problem: Find $u : \Omega \subset \mathbf{R}^d \to \mathbf{R}$ such that

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \Gamma = \partial\Omega. \end{aligned}$$

Exist unique weak solution $u \in H_0^1$ if $f \in H^{-1}$.

Discontinuous spaces

Let $\mathcal{V}$ be the space of

discontinuous piecewise polynomials

of degree p defined on a partition $\{K\} = \mathcal{K}$ of Ω.

$$\mathcal{V} = \bigoplus_{K \in \mathcal{K}} \mathcal{P}_{p_K}(K).$$

May replace $\mathcal{P}_{p_K}(K)$ by a finite dimensional function space $\mathcal{V}_K$ on K.

Averages and jumps

For all $v \in \mathcal{V}$ we define

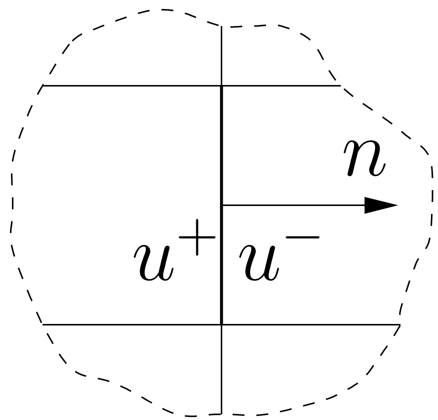
$$\langle v \rangle = \frac{v^+ + v^-}{2},$$

$$[v] = v^+ - v^-,$$

where

$$v^\pm(x) = \lim_{s \to 0^+} v(x - n_E s).$$

and n is a fixed unit normal.



Derivation of a dG method

Multiplying

$$-\Delta u = f,$$

by $v \in \mathcal{V}$ and integrating by parts yields

$$\sum_K (\nabla u, \nabla v)_K - (n_K \cdot \nabla u, v)_{\partial K} = (f, v).$$

Since $[n \cdot \nabla u] = 0$, this may be written

$$\sum_K (\nabla u, \nabla v)_K - \sum_E (n_E \cdot \nabla u, [v])_E = (f, v).$$

The dG method

Find $U \in \mathcal{V}$ such that

$$\boxed{a(U,v) = (f,v) \text{ for all } v \in \mathcal{V}}$$

Here

$$\begin{aligned} a(v,w) = &\sum_K (\nabla v, \nabla w)_K - \sum_E (\langle n \cdot \nabla v \rangle, [w])_E \\ &+ \alpha \sum_E ([v], \langle n \cdot \nabla w \rangle)_E + \beta \sum_E (h_E^{-1}[v],[w])_E, \end{aligned}$$

with α and β real parameters.

Other terms like $([n \cdot \nabla v], [n \cdot \nabla w])_E$ are also possible.

Conservation

Continuous case: For $\omega \subset \Omega$ we have

$$\int_\omega f + \int_{\partial\omega} n \cdot \nabla u = 0,$$

Discrete case: For each element K we have

$$\int_K f + \int_{\partial K} \Sigma_n(U) = 0,$$

with the numerical flux $\Sigma_n(U)$ defined by

$$\Sigma_n(U) = \langle n \cdot \nabla U \rangle - \frac{\beta}{h_E}[U].$$

Remarks on the dG method

Method is consistent for all values of α and β.

Special cases:

- *Nitsche's method*: $\alpha = -1$, β large.
- *Nonsymmetric without penalty* [a]: $\alpha = 1$, $\beta = 0$.
- *Stabilized nonsymmetric*: $\alpha = -1$, $\beta > 0$.

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[a] Babuska, Oden, and Bauman

Effect of β

Example:

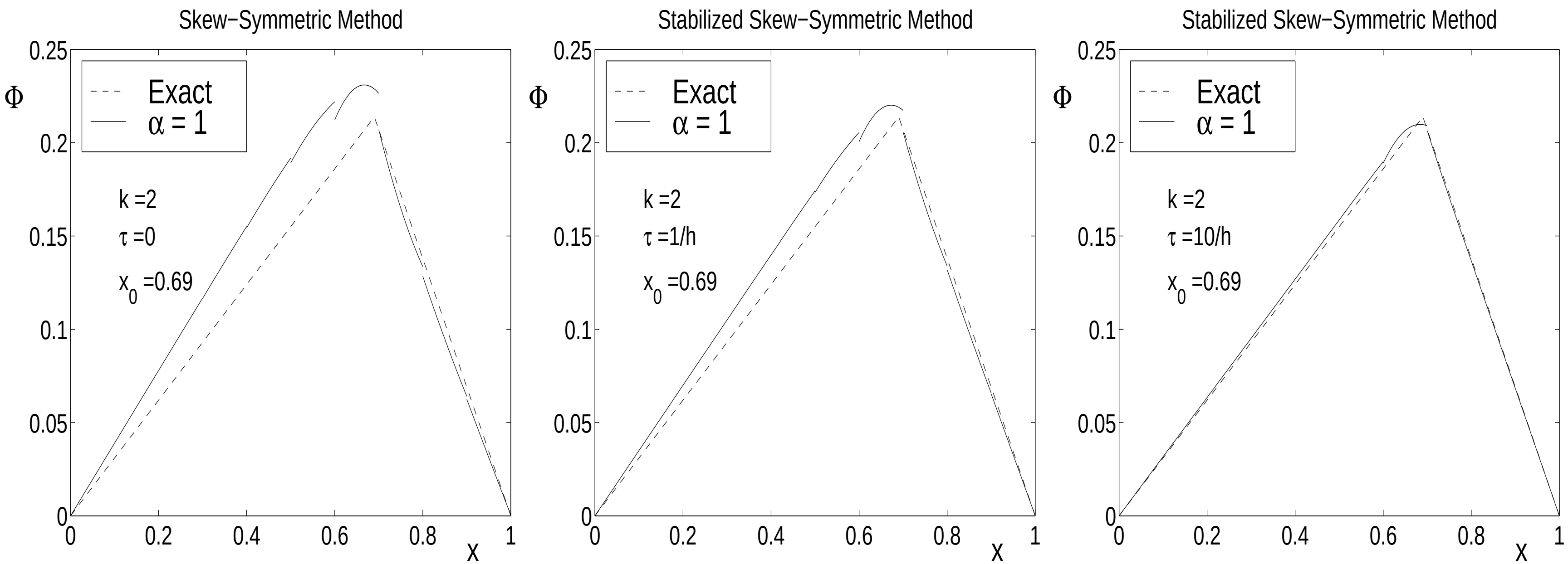

Figure: Quadratic dG, with $\alpha = 1$ and $\beta = 0, 1, 10$. ($\tau = \beta/h$)

Weak Dirichlet conditions

Weak statement of $u = g_D$ on $\partial\Omega$:

$$(\nabla u, \nabla v)_\Omega - (n \cdot \nabla u, v)_{\partial\Omega} + \mu(u, v)_{\partial\Omega} = \mu(g_D, v)_{\partial\Omega} + (f, v)_\Omega.$$

- *Nitsche's method* based on this form is consistent ($\mu = \beta/h$)
- *Stiff springs* obtained by neglecting

$$(n \cdot \nabla u, v)_{\partial\Omega},$$

is not consistent. Corresponds to

$$u = g_D \approx u + \mu^{-1} n \cdot \nabla u = g_D.$$

dG versus cG: Advantages

- Very flexible framework for adaptivity and construction of approximation spaces.
- Not sensitive to the use of triangles, bricks or other types of elements.
- Easy implementation of hp-spaces, hanging nodes and nonmatching polynomial orders.
- Special basis functions not required. Element basis functions can be $x^\alpha y^\beta$ with $0 \leq \alpha + \beta \leq p$.
- Can be used to glue together solutions on nonmatching grids.
- Elementwise conservation property.

dG versus cG: Disadvantages

- Lots of more degrees of freedom.
- Efficient iterative solvers need to be developed.
- Complicated to implement compared to basic cG.

dG versus cG: Number of dof

Number of unknowns for the dG method as a multiple of the number of unknowns for the cG method for various elements and orders of polynomials.
For $p = 0$ normalization is with respect to the unknowns of the cG with $p = 1$.

p	Quad	Tri	Hex	Tet
0	1	2	1	5
1	4	6	8	20
2	2.25	3	3.38	7.14
3	1.78	2.22	2.37	4.35
∞	1	1	1	1

Classical results

Energy norm: Define

$$|||v|||^2 = \sum_K \|\nabla v\|_K^2 + \sum_E \|h_E^{-1/2}[v]\|_E^2 + \sum_E \|h_E^{1/2}\langle n \cdot \nabla v\rangle\|_E^2.$$

Coercivity: If $\beta > 0$ large there is $m > 0$ such that

$$m|||v|||^2 \leq a(v,v) \quad \forall v \in \mathcal{V}.$$

Error estimate: If $\beta > 0$ sufficiently large we have

$$|||u - U||| \leq Ch^p \|u\|_{H^{p+1}}.$$

Systems - Linear Elasticity

Elastic Problem: Find the symmetric stress tensor σ_{ij}, and displacement vector $u_i : \Omega \rightarrow \mathbb{R}^3$ such that

$$
\begin{aligned}
-\frac{\partial \sigma_{ij}}{\partial x_j} &= f_i, \quad \text{in } \Omega, \\
u_i &= 0, \quad \text{on } \Gamma_1, \\
\sigma_{ij} n_j &= g_i, \quad \text{on } \Gamma_2.
\end{aligned}
$$

Here n_i is normal to $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and f_i and g_i are given loads.

$$\sigma_{ij} = C_{ijkl} \frac{\partial u_k}{\partial x_l},$$

where C_{ijkl} is tensor of elastic coefficients.

Linear Elasticity,cnt

Internal work $a(u,v)$ and external load $l(v)$ given by

$$a(u,v) = \int_\Omega \frac{\partial u_i}{\partial x_j} C_{ijkl} \frac{\partial v_k}{\partial x_l}\, dx,$$

$$l(v) = \int_\Omega f_i v_i \, dx + \int_{\Gamma_2} g_i v_i \, ds.$$

Variational Form: Find u such that

$$a(\rho, u, v) = l(v),$$

for all $u, v \in V = \{v \in H^1 : v = 0 \text{ on } \Gamma_1\}$.

Linear Elasticity,cnt

Example: *Stress caused by volume load.*

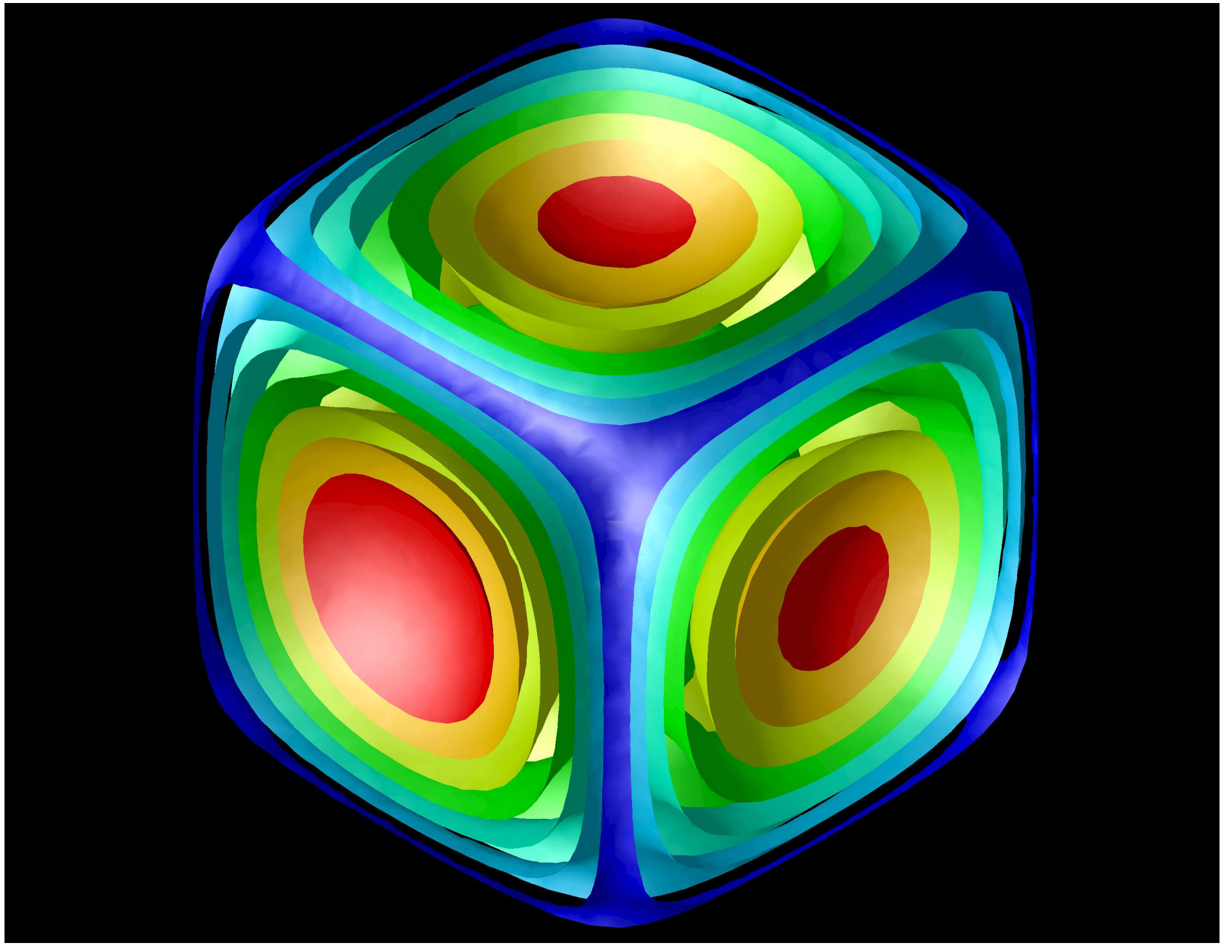

Figure: von Mises stress contours in a cube.

Linear Elasticity, cnt

Example: *Stress in a hoistfitting due to point load.*

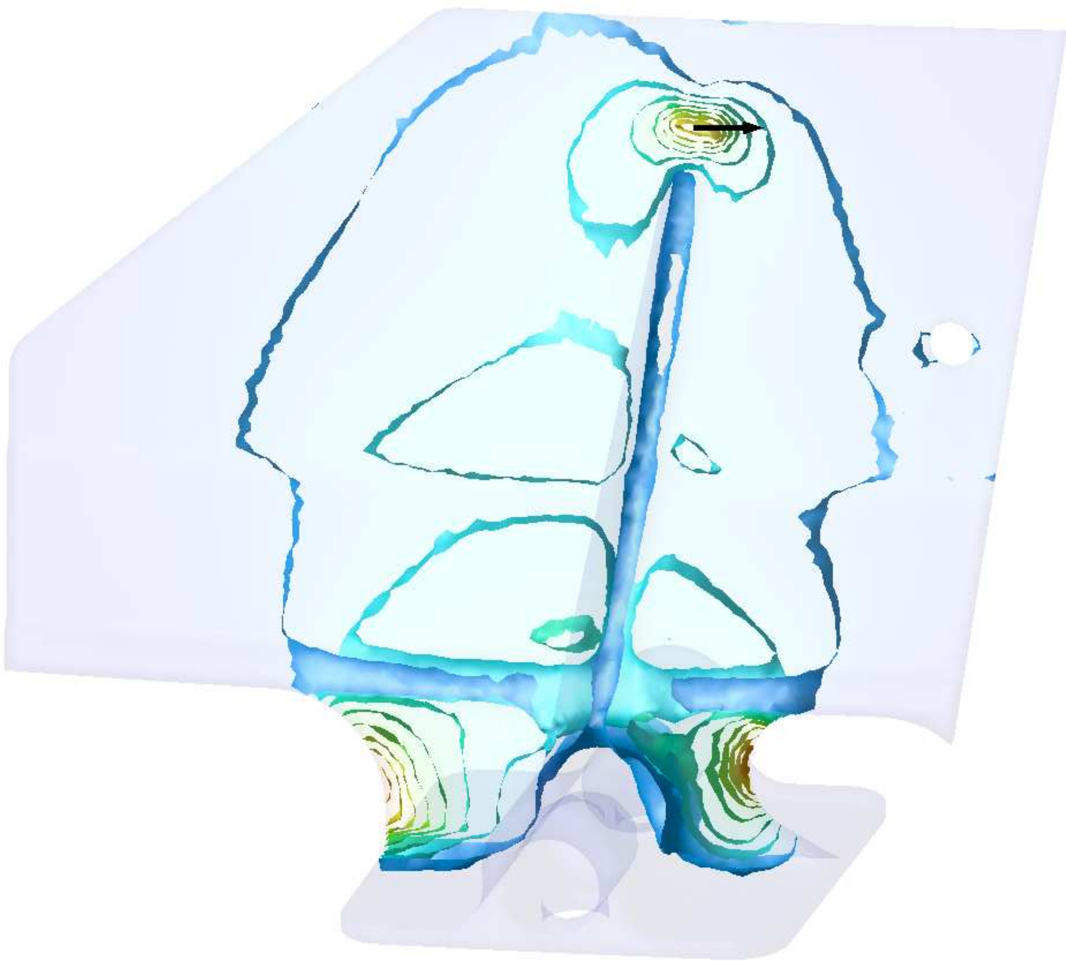

Figure: von Mises stress contours in a hoistfi tting.

Navier Stokes Equations

Motion of incompressible fluids governed by

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \Delta \boldsymbol{u} - \frac{1}{\rho} \nabla p,$$

$$\nabla \cdot \boldsymbol{u} = \boldsymbol{f},$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is velocity and $p(\boldsymbol{x}, t)$ pressure of fluid.

Navier Stokes, cnt

Example: *Dual solution of Navier Stokes equations*

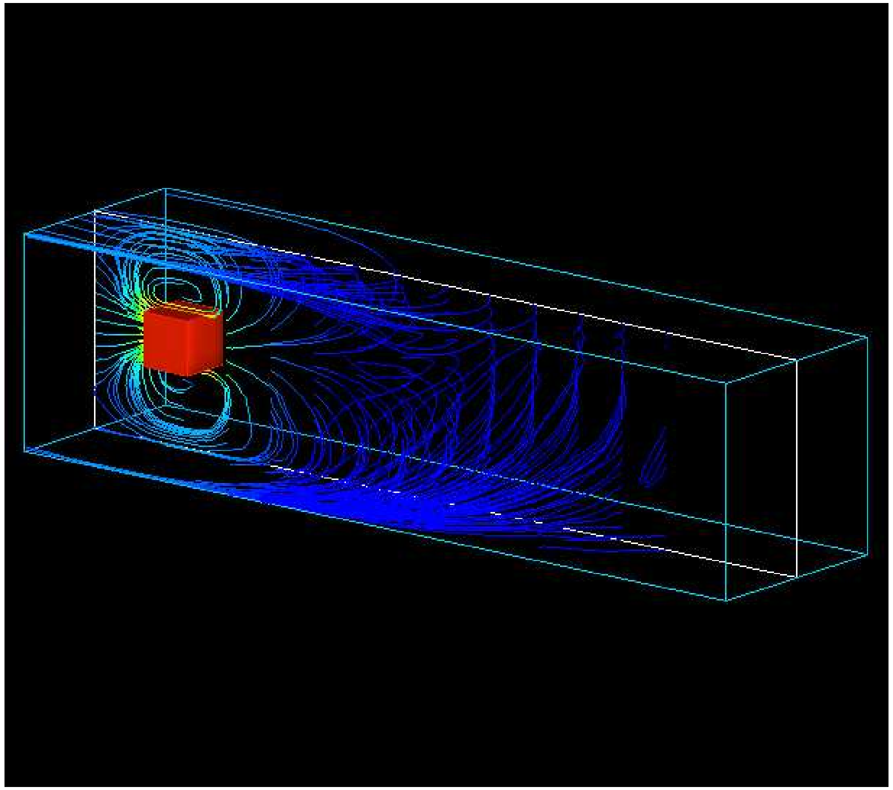

Figure: Streamlines around a solid body.