# Fast divergence-conforming reduced order models for flow

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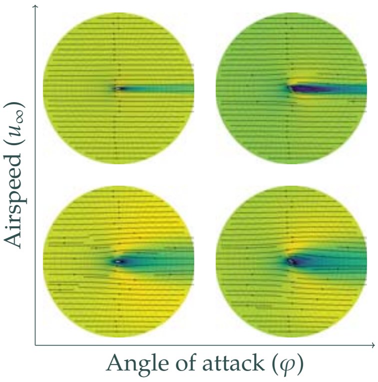



**Problem:** Repetitive solutions of parametrized flow problems (see left) can be quite demanding, each solution involving up to $10^6$–$10^9$ degrees of freedom and hours or days of computational time.

**Answer:** Reduced Order Modelling (ROM) offers solutions with lower accuracy but dramatic speedups. When tied to a divergence-conforming high-fidelity method, the gains can be even greater.

**OFFLINE**

- Parametrized PDE
- High-fidelity discr. $A_h(\boldsymbol{\mu})\boldsymbol{u}_h(\boldsymbol{\mu}) = \boldsymbol{f}_h(\boldsymbol{\mu})$, $A_h = \sum_q \xi_q(\boldsymbol{\mu}) A_h^q$, $\boldsymbol{f}_h = \sum_q \chi_q(\boldsymbol{\mu}) \boldsymbol{f}_h^q$
- Snapshot solutions
- Projection matrix $\boldsymbol{V}$
- Projection $A_N^q = \boldsymbol{V}^\intercal A_h^q \boldsymbol{V}$, $\boldsymbol{f}_N^q = \boldsymbol{V}^\intercal \boldsymbol{f}_h^q$

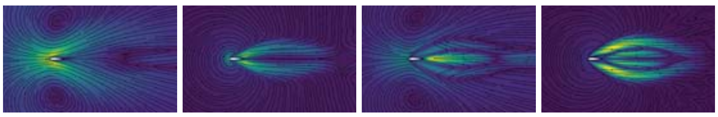
**ONLINE**

- $\boldsymbol{\mu}$
- RB system assembly $A_N = \sum_q \xi_q(\boldsymbol{\mu}) A_N^q$, $\boldsymbol{f}_N = \sum_q \chi_q(\boldsymbol{\mu}) \boldsymbol{f}_N^q$
- RB system solution $A_N(\boldsymbol{\mu})\boldsymbol{u}_N(\boldsymbol{\mu}) = \boldsymbol{f}_N(\boldsymbol{\mu})$
- Post-processing, recovery, visualization, evaluate functionals

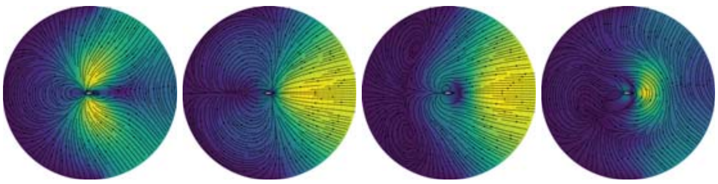
## Problem specifics

High fidelity simulations of *stationary Navier-Stokes* were performed of flow around a NACA0015 airfoil with chord length of 1 m. The inflow velocity $u_\infty$ varied from 1 to 20 m/s, and the angle of attack $\varphi$ varied from −35 to 35°. The viscosity was fixed at $^1/_6$. Snapshots were evaluated at the 15 × 15 Gauss points on the parameter domain, and reduced models created with $N = 10, 20, \ldots, 50$ degrees of freedom.

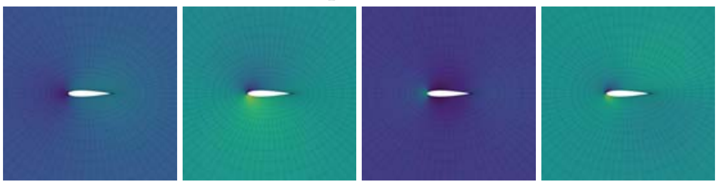
**First four velocity modes**



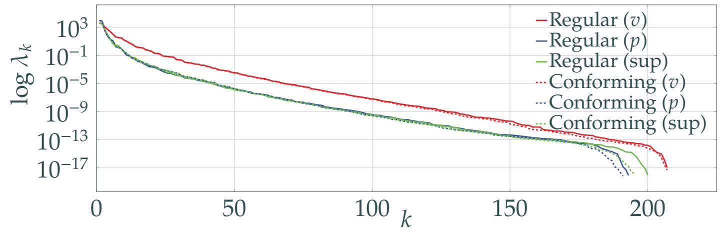
**First four supremizer modes**



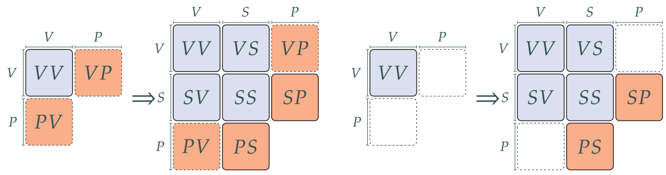
**First four pressure modes**



**Energy spectrum of snapshots**

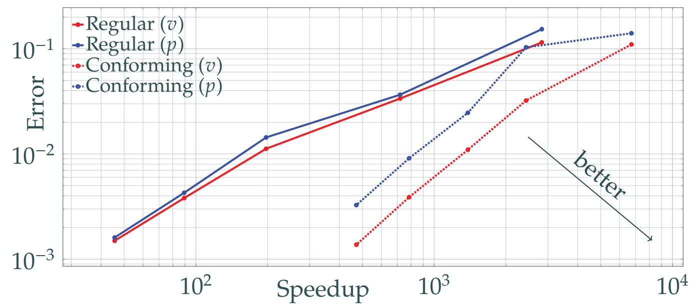



**Regular** **Divergence-conforming**

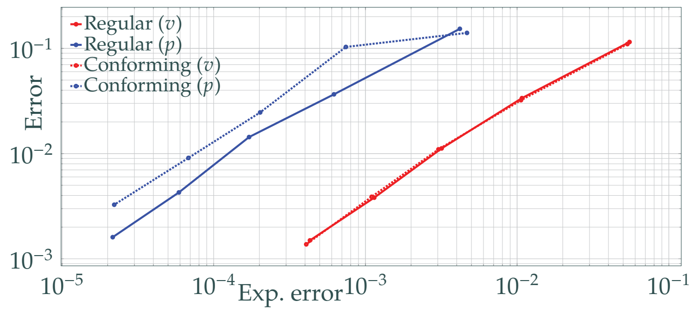



The system matrix (size $2N$) will usually have a rank-deficient velocity-pressure block (VP, indicated with dashed lines). Enriching the velocity space with so-called *supremizers* ensures a full-rank system matrix with size $3N$. A divergence-conforming method will produce a fully divergence-free basis, so the VP-block vanishes, giving a block-triangular system, solvable as two size-$N$ systems instead of one size-$3N$ system.

**Error as a function of speed**




**Actual error as a function of expected error**




**Mean solver time usage**

| | Hi-Fi | $N = 10$ | $N = 20$ | $N = 30$ | $N = 40$ | $N = 50$ |
|---|---|---|---|---|---|---|
| Regular | 104 s | 29 ms | 126 ms | 503 ms | 1.02 s | 2.51 s |
| Conforming | 165 s | 21 ms | 54 ms | 104 ms | 183 ms | 284 ms |

## Discussion

- ROMs are able to deliver results within two to three orders of magnitude at dramatic speedups.
- Divergence-conforming ROMs can deliver higher speeds, up to one order of magnitude faster in the present examples, by exploiting specific properties of the velocity bases.