# Molecular-Flow Effects in Evaporation and Condensation Phenomena

Tor Ytrehus

Professor
Dept. of Energy and Process Engineering
NTNU

Scientific advisor
SINTEF, Materials and Chemistry

# Content

- Remarks on the validity of continuum fluid dynamic descriptions
- Boltzmann Equation and kinetic boundary layers
- Moment solution of Boltzmann Eq. for evaporation/condensation
- Matching to external continuum flow
- Three cases
    1. Evaporation into shear flow
    2. Explosive boiling of liquid droplet
    3. Laser ablation of carbon
- Concluding remarks

# Validity of continuum fluid dynamics on Navier-Stokes level

- $$\frac{\lambda}{c}\left|\frac{\partial u_i}{\partial x_j}\right|_{\max} \ll 1, \qquad \frac{\lambda}{T}\left|\frac{\partial T}{\partial x_j}\right|_{\max} \ll 1$$
  from Chapman-Enskog expansion of the Boltzmann Equation, where

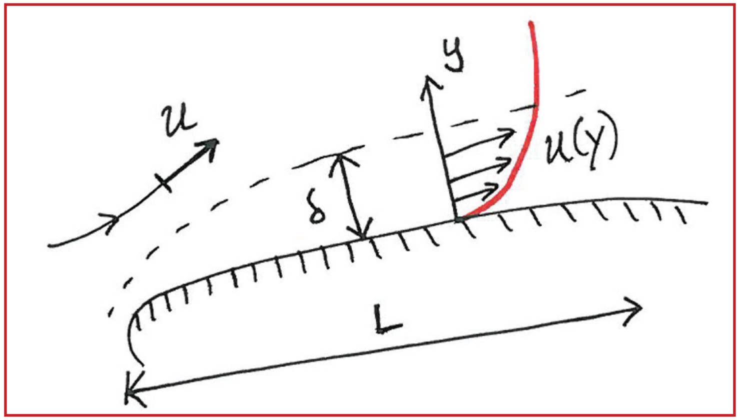
- $\lambda \approx \dfrac{\mu}{\rho c}$ : Molecular mean free path

  $c \approx \sqrt{RT}$ : Molecular random thermal velocity

- Also: $$\mathrm{Kn} = \frac{\lambda}{L} \approx \frac{\mu}{\rho c L} = \frac{\mu}{\rho U L} \cdot \frac{U}{c} \approx \frac{M}{Re}$$

Knudsen number

# Macroscopic gradients in Boundary Layer




- Thickness of Prandtl boundary layer: $\delta \approx \dfrac{1}{\sqrt{\mathrm{Re}}}$
- $$\frac{\lambda}{c}\left|\frac{\partial u}{\partial y}\right| \approx \frac{\lambda}{c}\frac{U}{\delta} \approx \frac{\lambda}{c}\frac{U}{L}\sqrt{\mathrm{Re}} \approx \frac{\lambda}{L}M\sqrt{\mathrm{Re}} \approx \frac{M^2}{\sqrt{\mathrm{Re}}}$$
- since $\dfrac{\lambda}{L} \approx \dfrac{M}{\mathrm{Re}}$ (von Karmann)

# **Limiting cases** for which $\frac{\lambda}{c}\left|\frac{\partial u_i}{\partial x_j}\right| \approx \frac{M^2}{\sqrt{\mathrm{Re}}} \not\ll 1$ :

- Rarefied Gas Dynamics: $\frac{\lambda}{L} \not\ll 1$
  (1950 → Grad, Cercignani, Kogan …)

- Hypersonic flight: $M \gtrsim 7$
  (1960 → Hays & Probstein …)

- Shock wave structure: $\left|\frac{\partial U}{\partial x}\right| \sim \frac{U}{\lambda}$
  (1950 → Gilbarg & Paulucci, Mott-Smith …)

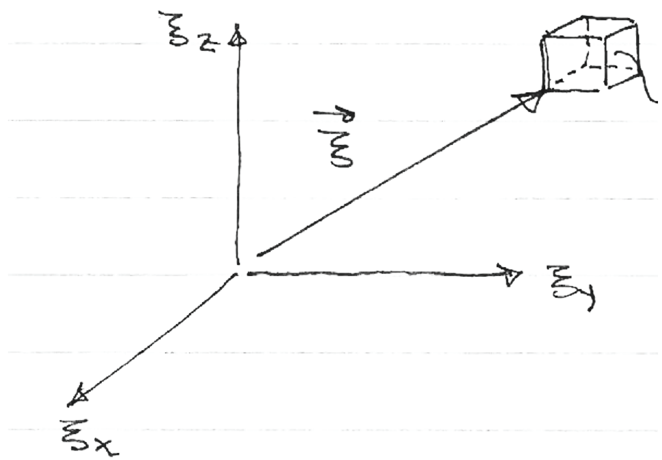
- Evaporation/condensation
  in single component systems: $\left|\frac{\partial V}{\partial y}\right| \sim \frac{c}{\lambda}$
  (1970 → Sone, Kogan, Aoki …)

# The Boltzmann Equation (1867) for molecular distribution function

- $$\frac{\partial f}{\partial t} + \xi_j \frac{\partial f}{\partial x_j} = \int (f' f_1' - f f_1) |\boldsymbol{\xi}_1 - \boldsymbol{\xi}| \mathrm{d}\Sigma \, \underline{\mathrm{d}\xi}_1 \qquad (1)$$

  binary collisions

- $f \underline{\mathrm{d}\xi} \stackrel{\text{def}}{=}$ number of molecules per unit volume with velocities in the range $\boldsymbol{\xi}$, $\boldsymbol{\xi} + \mathrm{d}\boldsymbol{\xi}$



$\underline{\mathrm{d}\xi} = \mathrm{d}\xi_x \mathrm{d}\xi_y \mathrm{d}\xi_z$

- $\int_{\boldsymbol{\xi}} f \underline{\mathrm{d}\xi} = n(t, \mathbf{x})$ ; number density of molecules

# Evaporation/Condensation Knudsen layer

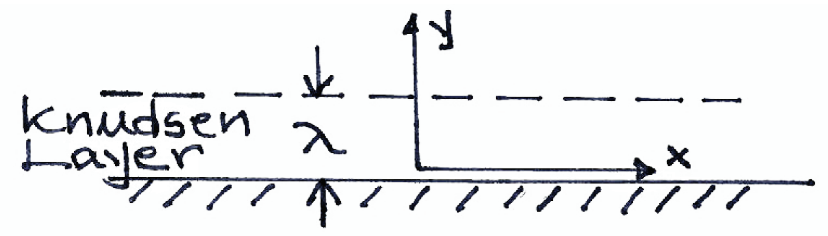
- Mathematics: Nondimensional Boltzmann Equation for 2D steady flow:

$$\epsilon\left(\xi_x\frac{\partial f}{\partial x}+\xi_y\frac{\partial f}{\partial y}\right)=\mathcal{C}(f f_1)_{\text{coll}}$$

$$\mathcal{O}(1/\epsilon)$$

- Singular → Regular perturbation ε→ 0:

$$\xi_y\frac{\partial f}{\partial y}=\mathcal{C}(f f_1)_{\text{coll}} \qquad (2)$$

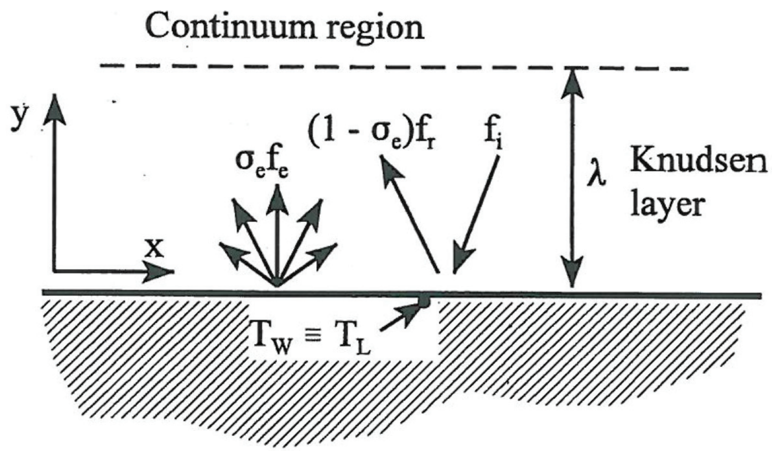



- $\epsilon = \frac{\lambda}{L}$ : Knudsen number

Sone & Onishi (1978), Kogan (1992), Cercignani (2000)

# Evaporation/Condensation Knudsen layer

- Physics:




- Two-sided molecular distribution

$$f_e(\boldsymbol{\xi}, T_L) \text{ for } \xi_y > 0$$
$$f_i(\boldsymbol{\xi}, T_\infty) \text{ for } \xi_y < 0$$

- Typically:

$$f_e = \frac{n_e(T_L)}{(2\pi R T_L)^{\frac{3}{2}}} \exp\left(-\frac{\xi^2}{2RT_L}\right) \qquad (3)$$

- $n_e(T_L) \equiv n_{\mathrm{sat}}(T_L)$

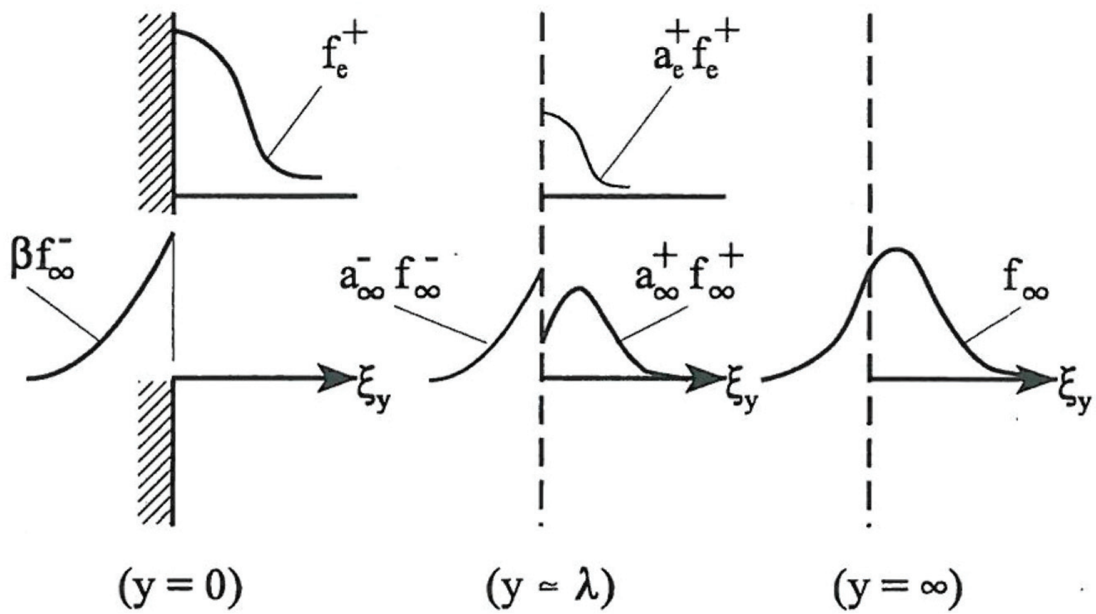
# Solution of Boltzmann equation for evaporation

- Basic "Ansatz" for distribution function:

$$f(y,\boldsymbol{\xi}) = \sum_{k=1}^{3} a_k(y) f_k(\boldsymbol{\xi}, T_k) \qquad (4)$$

- Boundary conditions:

$$y=0: \quad \begin{aligned} a_e^+ &= 1 \\ a_\infty^+ &= 0 \\ a_\infty^- &= \beta \end{aligned} \qquad y \to \infty: \quad \begin{aligned} a_e^+ &= 0 \\ a_\infty^+ &= 1 \\ a_\infty^- &= 1 \end{aligned} \qquad (5)$$




Anisimov (1968),
Ytrehus (1977, 1997)

# Conservation equations

- Moments of the Boltzmann Equation for the collision invariant *m*, $m\xi_y$, ½$m\xi^2$ over the velocity space:

$$\frac{\partial}{\partial y}\int \xi_y \begin{pmatrix} m \\ m\xi_y \\ \frac{1}{2}m\xi^2 \end{pmatrix} f \mathrm{d}\underline{\xi} = \int \begin{pmatrix} m \\ m\xi_y \\ \frac{1}{2}m\xi^2 \end{pmatrix} \mathcal{C}(f f_1)\mathrm{d}\underline{\xi} = 0$$

- Leads to the classical conservation eqs., but with $\tau_{ij}$ (stresses) and $q_i$ (heat flux) determined by the distribution function.

- For instance the 1D continuity equation reads:

$$\rho_e\sqrt{\frac{RT_L}{2\pi}}a_e^+ + \rho_\infty\sqrt{\frac{RT_\infty}{2\pi}}F^+a_\infty^+(y) - \rho_\infty\sqrt{\frac{RT_\infty}{2\pi}}F^-a_\infty^-(y) = \rho_\infty v_\infty \qquad (6)$$

$$F^\pm = \sqrt{\pi}S_\infty(\pm 1 + \mathrm{erf}\, S_\infty) + e^{-S_\infty^2}$$

# Gas Kinetic Connection Problem

- Evaluated between the interphase surface ($y$=0) and external equilibrium ($y \to \infty$) according to Eq. (5), the conservation eqs. Read:

$$\rho_e \sqrt{\frac{RT_L}{2\pi}} - \rho_\infty \sqrt{\frac{RT_\infty}{2\pi}} \beta F^- = \rho_\infty v_\infty \qquad (7a)$$

$$\frac{1}{2}\rho_e RT_L + \frac{1}{2}\rho_\infty RT_\infty \beta G^- = \rho_\infty v_\infty^2 + \rho_\infty RT_\infty \qquad (7b)$$

$$2\rho_e RT_L \sqrt{\frac{RT_L}{2\pi}} - 2\rho_\infty RT_\infty \sqrt{\frac{RT_\infty}{2\pi}} \beta H^- = \rho_\infty v_\infty \left( \frac{1}{2} v_\infty^2 + \frac{5}{2} RT_\infty \right) \qquad (7c)$$

with $F^-$, $G^-$, $H^-$ being transcendental functions of the speed ratio

$$S_\infty = \frac{v_\infty}{\sqrt{2RT_\infty}} = \sqrt{\frac{\gamma}{2}} M_\infty$$

as given in Ytrehus (1977, 1997).

# Solution of connection problem

- Solution of conservation Eqs (7) is expressed in terms of the single free control parameter

$$z = \frac{p_e}{p_\infty} \equiv \frac{p_{\text{sat}}(T_L)}{p_\infty} \quad (8)$$

as follows:

$$\sqrt{\frac{T_\infty}{T_L}} = -\frac{\sqrt{\pi}}{8} S_\infty + \sqrt{1 + \frac{\pi}{64} S_\infty^2} \quad (9a)$$

$$z_e = \frac{2 e^{S_\infty^2}}{F^- + \sqrt{\frac{T_\infty}{T_L}} G^-} \quad (9b)$$

$$\beta = \frac{2(2 S_\infty^2 + 1)\sqrt{\frac{T_\infty}{T_L}} - 2\sqrt{\pi} S_\infty}{F^- + \sqrt{\frac{T_\infty}{T_L}} G^-} \quad (9c)$$

# Tabulated results

- Table 1 Gas dynamic parameters in evaporation

| $S_\infty$ | $p_e / p_\infty$ | $\rho_\infty / \rho_e$ | $T_\infty / T_L$ | $\beta$ | $\dot{m} / \dot{m}_e$ |
|---|---|---|---|---|---|
| 0.0 | 1.000 | 1.000 | 1.000 | 1.000 | 0.000 |
| 0.1 | 1.231 | 0.849 | 0.957 | 1.020 | 0.294 |
| 0.2 | 1.500 | 0.728 | 0.915 | 1.060 | 0.494 |
| 0.3 | 1.812 | 0.630 | 0.876 | 1.135 | 0.627 |
| 0.4 | 2.170 | 0.550 | 0.838 | 1.271 | 0.714 |
| 0.5 | 2.577 | 0.484 | 0.802 | 1.511 | 0.768 |
| 0.6 | 3.037 | 0.429 | 0.767 | 1.928 | 0.800 |
| 0.7 | 3.553 | 0.383 | 0.734 | 2.644 | 0.815 |
| 0.8 | 4.127 | 0.345 | 0.703 | 3.862 | 0.820 |
| 0.9 | 4.764 | 0.312 | 0.673 | 5.932 | 0.816 |
| 0.907* | 4.813 | 0.310 | 0.671 | 6.132 | 0.816 |

* Critical value.

- Evaporation: $\left(\frac{p_e}{p_\infty}\right)_{\text{crit}} = 4.813$ Nozzle: $\left(\frac{p_0}{p_\infty}\right)_{\text{crit}} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} = 2.053$
- Linearized result:

$$\frac{v_\infty}{\sqrt{2RT_L}} = \frac{1}{2\sqrt{\pi}}\frac{32\pi}{32+9\pi}\frac{p_e - p_\infty}{p_e} \simeq 0.4705\frac{p_e - p_\infty}{p_e} \quad (10)$$

# Spatial Structure

Introducing a non-conserved moment ($m\xi_y^2$) into the Boltzmann Eq. and combining with local conservation eqs. of the type Eq. (6), a bilinear form results:

- $$\frac{\partial a_\infty^-}{\partial x} = -\frac{P(S_\infty)}{\lambda_e}(a_\infty^- - 1)\left[a_\infty^- - r(S_\infty)\right] \quad (11)$$

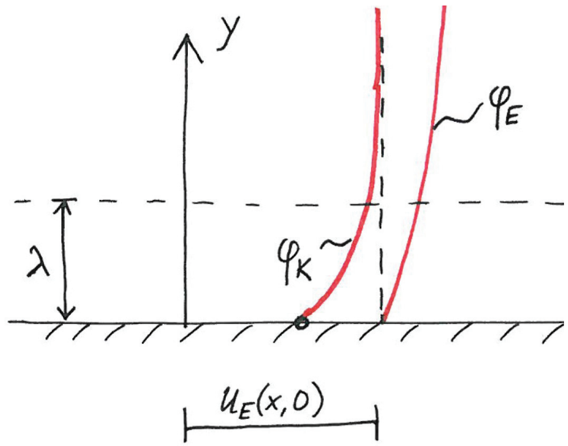
  Relation to external equilibrium $a_\infty=1$ only if

  $r \leq 1$, implying $$S_\infty \leq 0.907...$$ $$M_\infty \leq 0.994...$$

- Evaporation Knudsen layer restricted to subsonic flow normal to interphase surface.

# Matching to External Continuum Flow

Matching principle:




$\varphi_K$: Euler
$\varphi_E$: Knudsen layer

$$\lim_{y_K \to \infty} \varphi_K = \lim_{y_E \to 0} \varphi_E \quad (12)$$

| Outer limit of Inner solution | Inner limit of outer solution |
|---|---|

Aoki *et al.* (1991), Abramov & Kogan (1984):

$$U_E(x, y = 0) = 0 \quad \text{Evaporation}$$

$$U_E(x, y = 0) \neq 0 \quad \text{Condensation}$$

# Why not Navier-Stokes?

- Navier-Stokes component normal to interphase surface:

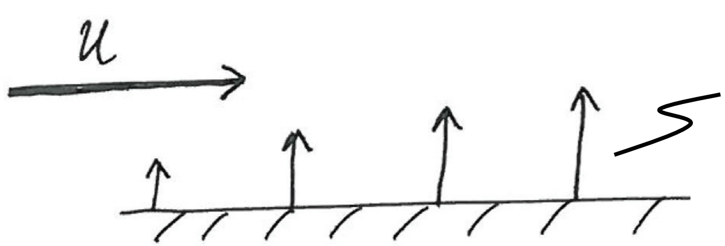
$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$v\Delta v/l \qquad\qquad \nu\Delta v/l^2$$

- $$\Longrightarrow l \sim \frac{\nu}{v} \sim \frac{c\lambda}{v} \sim \frac{\lambda}{M_y}$$

  that is: Similar scale as Knudsen layer and already accounted for.
  Also: $v(y=0)$ = ?

- Shear layer "blown off" in evaporation




$$v_\infty \simeq \sqrt{RT}\frac{\Delta p}{p} \gg v_{\mathrm{BL}}$$

Lateral diffusion: $u/\sqrt{\mathrm{Re}}$

# Case 1: Evaporation into shear flow

- System: Water/water vapor at conditions such that

- $\left|\frac{p_\infty - p_e}{p_e}\right| = \mathcal{O}(M_y) \ll 1$
- $\frac{U_\infty}{\sqrt{2RT_\infty}} \ll 1$

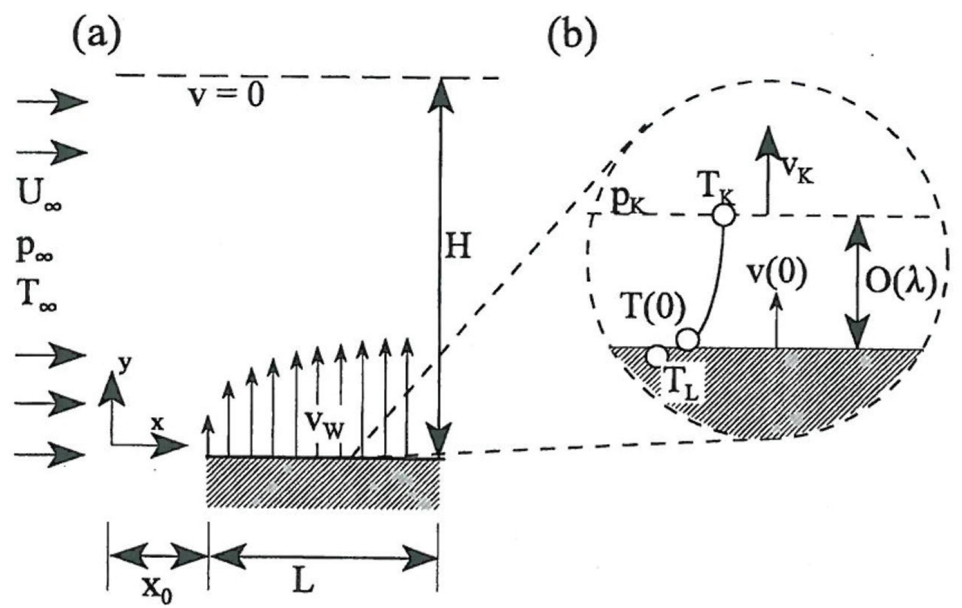



**Figure 21.** Definition sketch for evaporation into a shear flow. (a) External flow domain, and (b) details from the Knudsen layer domain.

# Governing equations in external flow

- Description on Navier-Stokes level, mainly to detect eventual influence of turbulence

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\nu_{eff}\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_{eff}\frac{\partial U}{\partial y}\right) \qquad (13a)$$

$$U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(\nu_{eff}\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_{eff}\frac{\partial V}{\partial y}\right) \qquad (13b)$$

$$U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial y}=\frac{\partial}{\partial x}\left(\gamma_{eff}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\gamma_{eff}\frac{\partial T}{\partial y}\right) \qquad (13c)$$

Here $\nu_{\text{eff}} = \nu + \nu_t$, $\gamma_{\text{eff}} = \gamma + \gamma_t = \nu/P_r + \nu_t/P_{r_t}$ where $P_{r_t}$ is turbulent Prandtl number

$$\nu_t = C\frac{k^2}{\epsilon} \quad , C=0.09$$

in standard k-ε model, Patankar (1980)

# Boundary conditions

- On reactive boundary; $y$=0, $x \in [x_0, x_0+L]$:

$$V = v_w = \sqrt{2RT_L}\frac{4}{\sqrt{\pi}\beta_c}\frac{p_e - p_w}{p_e}, \quad \beta_c = \frac{32+9\pi}{4\pi} \tag{14a}$$

$$T = T_w = T_L - \frac{T_L}{\beta_c}\frac{p_e - p_w}{p_e}, \tag{14b}$$

$$U = 0, \ \text{Evaporation}$$
$$U \neq 0, \ \text{Condensation}$$

- $p_w$: Inner limit of external solution, to be determined.

- Iteration on the parameter $(p_e\text{-}p_w)/p_e$ after initial guess $p_w$=$p_\infty$.

- Extent of domain:
  $x_0$=0.5m, $L$=0.55m, $T_L$=373.16 K, $p_e(T_L)$=1 atm, $R$=461.96 J/K/kg,
  $p_\infty \in [0.98, 1.03]$atm, $U_\infty \in [20, 50]$m/s.

# Computed results

- Obtained by TEACH-T code using SIMPLE algorithm, Patankar (1980)

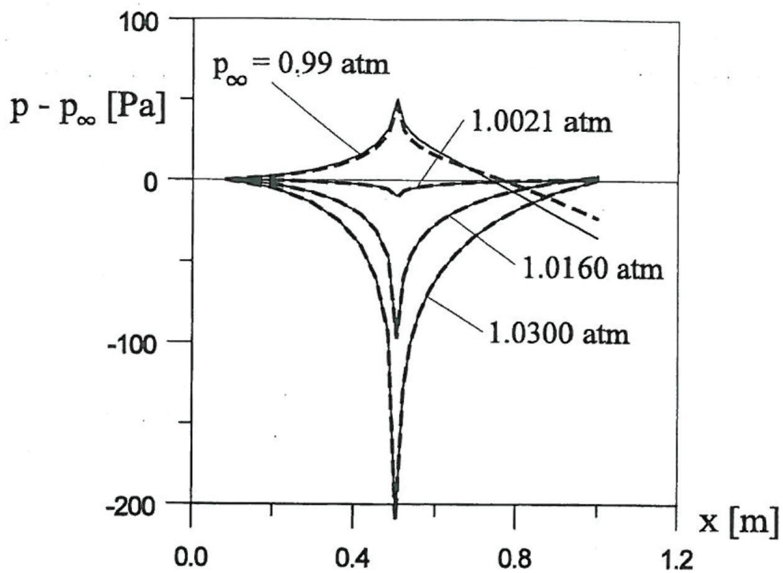



**Figure 22.** Differential pressure $p - p_\infty$ upstream of and along interphase surface in shear flow, $U_\infty = 20$ m/s. Dashed line: laminar flow, solid line: turbulent flow.

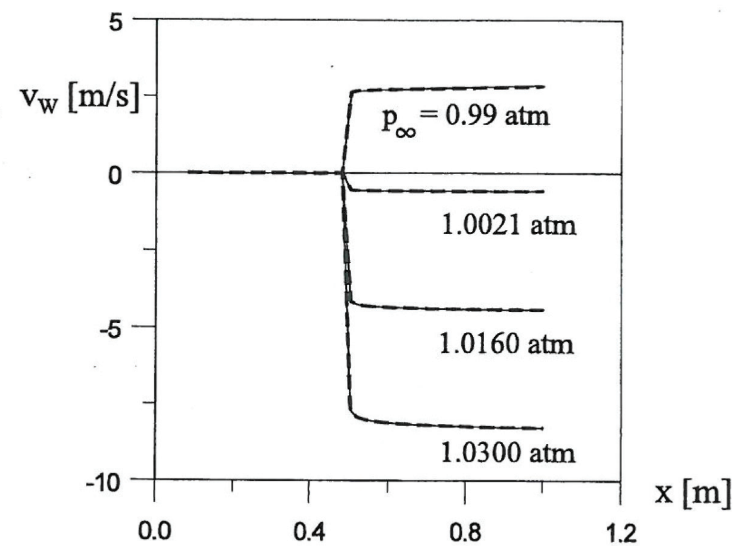



**Figure 23.** Normal velocity upstream of and along interphase surface in shear flow, $U_\infty = 20$ m/s. Dashed line: laminar flow, solid line: turbulent flow.

# Computed results (contd.)

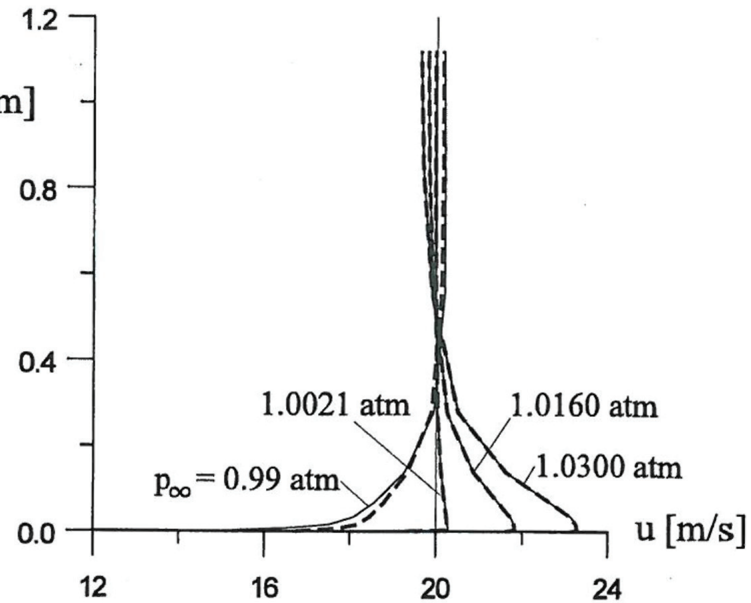



**Figure 24.** Tangential velocity profiles at x = 0.58 m on interphase suface in shear flow, $U_\infty$ = 20 m/s. Dashed line: laminar flow, solid line: turbulent flow.

- Thickness of "shear profile", $\delta$=0.35m, of same order as geometric scale, $L$=0.55m; EULER result.
- Also EULER slip flow at wall $U_E(x,0) > U_\infty$ for condensation.

# Case 2: Explosive boiling of liquid droplet

- Rapid phase transition from highly superheated liquid to vapor.
- Typical situations:
  - Droplet immersed in hot medium
  - High intensity laser radiation
  - Passage through shock wave
  - Sudden decompression of liquid
- Bacground references and current understanding of problem in Shusser & Weihs (1999), Shusser *et al*. (2000).

# Explosive boling (contd. I)

- Governing eqs. by Shusser & Weihs (1999)

  - Extension of classical theories by Rayleigh (1917), Prosperetti & Plesset (1978).

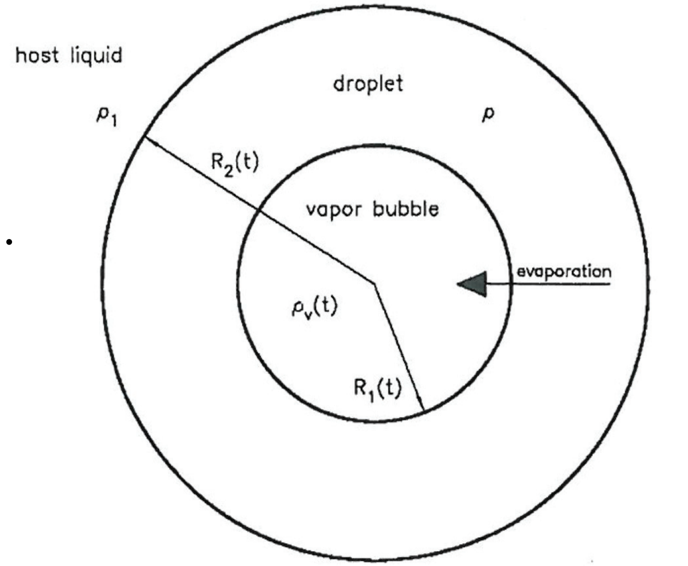



Fig. 1. Schematic illustration of explosive boiling of a liquid droplet.

$$\left(R_1\ddot{R}_1 + 2\dot{R}_1^2 - 2\dot{R}_1\frac{J}{\rho} - R_1\frac{\dot{J}}{\rho}\right)\left[1 - \frac{R_1}{R_2}\left(1 - \frac{\rho_\ell}{\rho}\right)\right]$$
$$+\frac{1}{2}\left(\dot{R}_1^2 - 2\dot{R}_1\frac{J}{\rho} + \frac{J^2}{\rho^2}\right)\left[\frac{R_1^4}{R_2^4}\left(1 - \frac{\rho_\ell}{\rho}\right) - 1\right] - \frac{J^2}{\rho_v\rho} + \frac{J^2}{\rho^2} = \frac{1}{\rho}\left[p_i - p_0 - \frac{2\sigma_\ell}{R_1}\right], \quad (15)$$

$$\dot{R}_2 = \left(\dot{R}_1 - \frac{J}{\rho}\right)\frac{R_1^2}{R_2^2}. \quad (16)$$

# Explosive boling (contd. II)

- Evolution of vapor pressure $p_i$

$$\frac{1}{p_i}\frac{\partial p_i}{\partial t} = -\frac{3\gamma}{R_1}\left(\dot{R}_1 - \frac{J}{\rho_\nu}\right) \tag{17}$$

- $J$ is evaporated mass flux:

$$J = \sigma\rho_e\sqrt{\frac{RT_L}{2\pi}} - \sigma\rho_\infty\sqrt{\frac{RT_\infty}{2\pi}}\beta F^-(S_\infty) \tag{18}$$

$$F^-(S_\infty) = \sqrt{\pi}S_\infty(1+\mathrm{erf}\,S_\infty) + e^{-S_\infty^2}$$

modified for non-unity evaporation coefficient σ.

- Invokes the complete non-linear gas kinetic connection problem generalized to $\sigma \neq 1$, Ytrehus (1997).

# Computed results

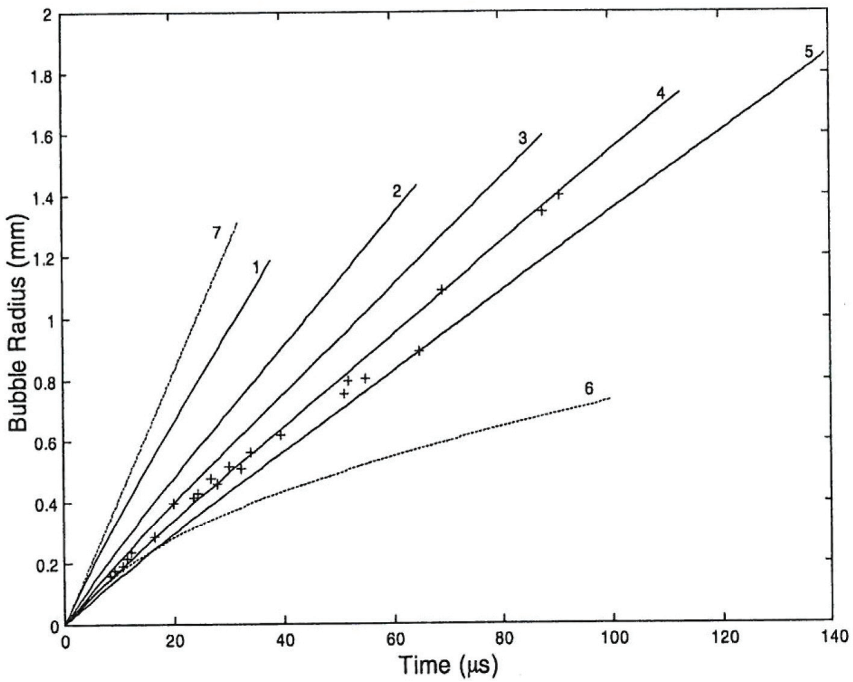
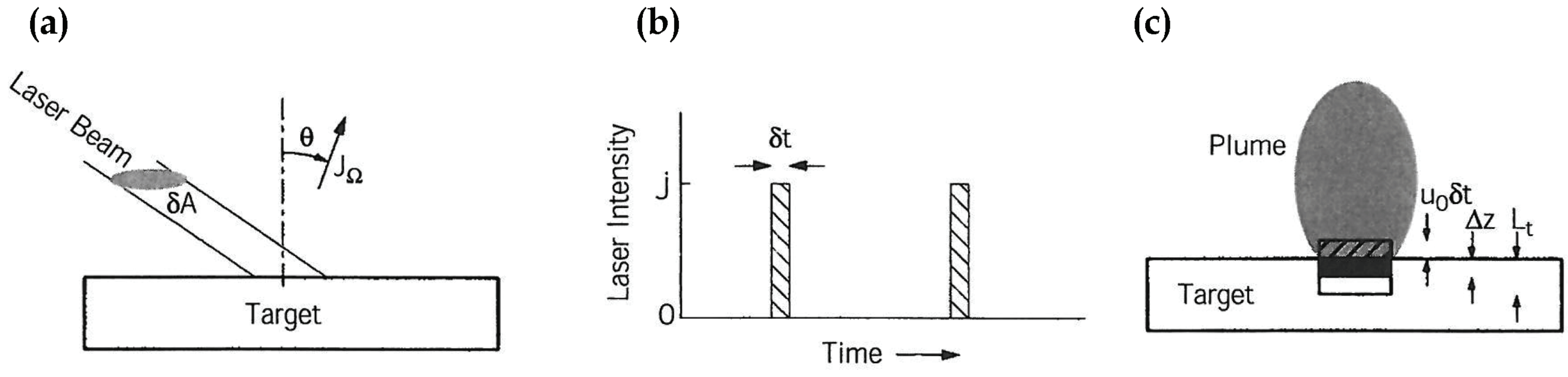



Fig. 2. Bubble radius as a function of time during explosive boiling for several values of the evaporation coefficient: (1) $a_w = 1$; (2) $a_w = 0.2$; (3) $a_w = 0.1$; (4) $a_w = 0.06$; (5) $a_w = 0.04$; (6) classical theory (Shepherd and Sturtevant 1982); (7) classical inertial growth rate (Shepherd and Sturtevant 1982); +, experiment (Shepherd and Sturtevant 1982).

- Butane droplet, 1mm diameter, immersed in ethylene glycol.
- Best fit with experiments of Shepherd & Sturtevant (1982) for $\sigma = 0.06$.
  - Comparable to molecular-dynamics simulations by Matsumoto (1998).

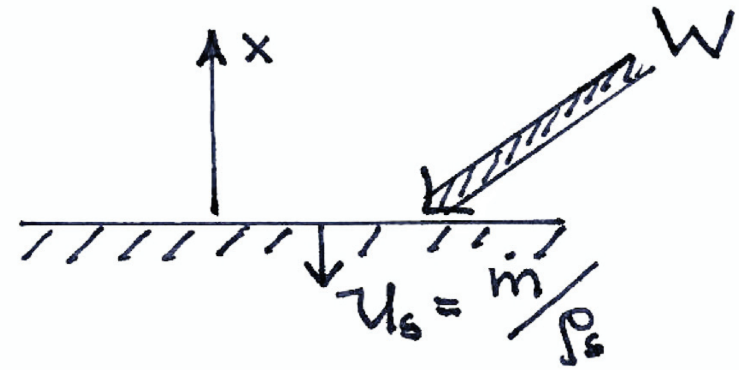
# Case 3: Laser ablation of carbon




Objective: To understand and control the physics of laser ablation and mechanism of carbon nanotube growth, M. Shusser (2007),(2010).

# Heat transfer in the solid

$$\frac{\partial T}{\partial t} - U_s \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2}$$




Initial and boundary conditions:

$$t = 0: \qquad T = T_0$$

$$x = 0: \qquad k\frac{\partial T}{\partial x} = W - \dot{m}L - \epsilon\sigma T^4$$

$$x \to -\infty: \qquad T \to T_0$$

$U_s$: Interface recession velocity
$W$: Laser energy flux
$L$: Latent heat of sublimation

# External gas dynamic problem

- $$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0 \quad &\text{(20a)} \\ \frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) &= 0 \quad &\text{(20b)} \\ \frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x}(\rho u E + u p) &= 0 \quad &\text{(20c)} \end{aligned}$$

- Boundary conditions at $x$=0 supplied from "outer limit" of Knudsen layer solution for evaporation, i.e. Eq. 9(a-c).

- For
  $$\frac{p_{\text{sat}}(T_w)}{p_\infty} \geq 4.81, \quad M_\infty = 1$$
  i.e., sonic outflow from Knudsen layer.

# Numerical Method

- Heat transfer equation approximated by

$$\frac{T_i(t+\Delta t)-T_i(t)}{\Delta t} = U_s(t)\frac{T_{i+1}(t)-T_(i-1)(t)}{2h} + \alpha\frac{T_{i+1}(t)-2T_i(t)+T_{i-1}}{h^2}$$

- Gas dynamics eqs. solved by first-order Gudonov scheme:

$$\frac{\rho(t+\Delta t)-\rho(t)}{\Delta t} + \frac{(\rho u)_{i+\frac{1}{2}}(t)-(\rho u)_{i-\frac{1}{2}}(t)}{h} = 0.$$

- Flux at cell boundaries from solution of corresponding Riemann problem, Gottlieb & Groth (1988).

# Results

- Calculations performed by Shusser (2007) for conditions corresponding to benchmark experiments by Greendyke *et al.* (2002):
  - Pulse length: 10ns
  - Beam diameter: 5mm
  - Laser intensity: $1.53 \cdot 10^{12}$ W/m$^2$

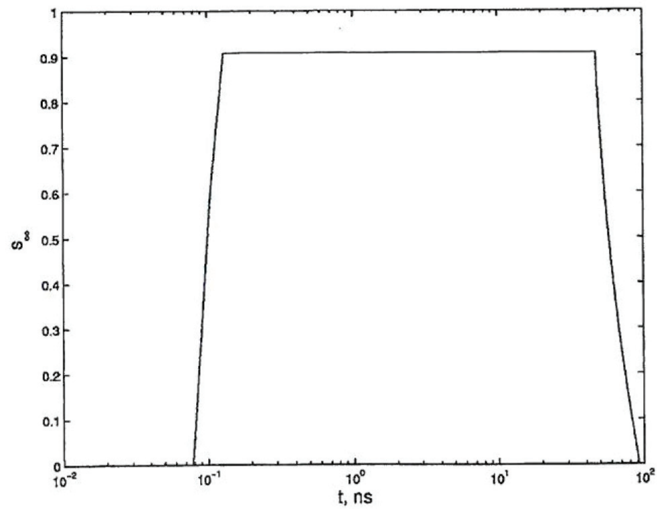



FIG. 1. Time dependence of dimensionless evaporation rate.

- Ablation rate most of the time restricted by kinetic theory limitation $M_\infty \leq 1, \quad (S_\infty \leq 0.907)$

- Considerable postpulse evaporation, $\tau \simeq 80\text{ns}$ .

# Results (contd. I)

- Time dependence of surface temperature

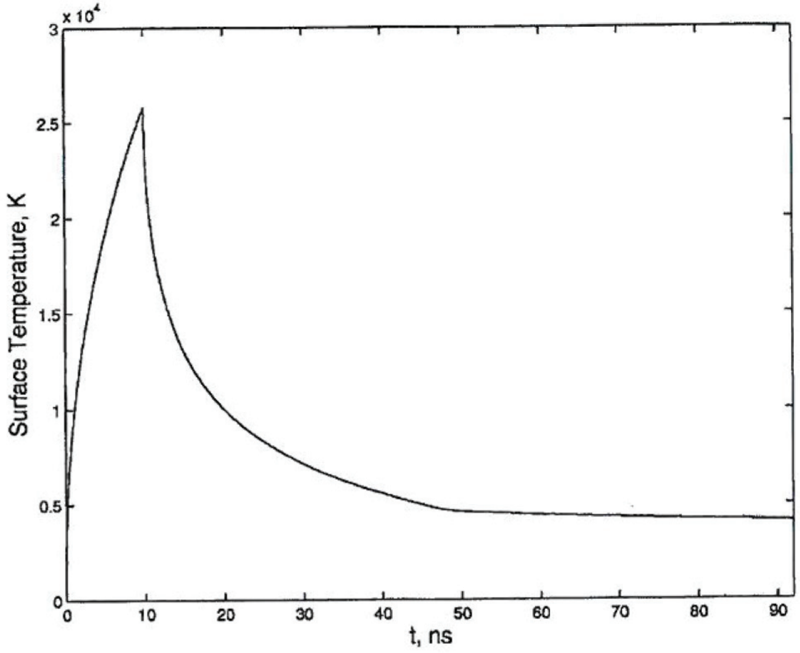



FIG. 2. Time dependence of surface temperature.

- Superheated surface up to 2500°K due to kinetic limit $M_\infty \leq 1$ of mass transfer rate.
  - Also the reason for postpulse evaporation.

# Results (contd. II)

- Time dependence of ablated mass

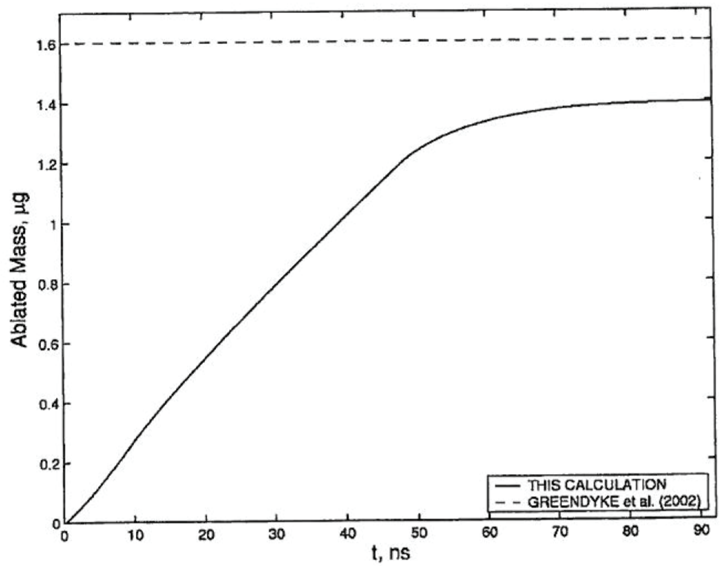



FIG. 5. Time dependence of ablated mass.

- Total ablated mass after postpulse period is within 15% of measured value.

- Accuracy limited by 1D modelling of external gas dynamics, and of heat transfer in solid.

# Concluding remarks

- Criteria for validity of continuum macroscopic flow description are generally violated for evaporation/condensation in single-component systems
- Current understanding of such phenomena is based on kinetic theory and the Boltzmann equation.
- A kinetic boundary layer (Knudsen layer) adjacent to the interphase surface should be matched to the external flow.
- Composite solutions are demonstrated for several cases, including nanotechnology applications.

# References


- Abramov, AA and Kogan, MN, *Sov. Phys. Dokl.* **29**, 763 (1984).
- Anisomov, SI, *J. Eksperim. Teor. Fisika* **54**, 339 (1968).
- Aoki, K, Nishino, K, Sone, Y. and Sugimoto, H, *Phys. Fluids* **A3,** 2260 (1991).
- Cercignani, C, *Rarefied Gas Dynamics: From Basic Concepts to Actual Calculations* (Cambridge Univ. Press, 2000).
- Gottlieb, IJ, and Groth, CPT, *J. Comput. Phys.* **78**, 437 (1988).
- Kogan, MN, *Prog. Aerospace Sci.* **29,** 271 (1992).
- Matsumoto, M, *Fluid Phase Equil.* **144**, 307 (1998).
- Patankar, S *Numrical Heat Transfer and fluid flow* (Hemisphere Books, 1980)
- Prosperetti, A, and Plesset, MS, *J. Fluid Mech.* **85**, 349 (1978).
- Rayleigh, L, *Philos. Mag.* **34**, 94 (1917).
- Shepherd, JE, and Sturtevant, B, *J. Fluid Mech.* **121**, 379 (1982).
- Shusser, M, *J. Appl. Phys.* **101**, 033529 (2007).
- Shusser, M, *J. Appl. Phys.* **105**, 093534 (2009).
- Shusser, M, and Weihs, D, *Int. J. Multiphase Flow* **25**, 1561 (1999).
- Shusser, M, Ytrehus, T, and Weihs, D, *Fluid Dyn. Res.* **27**, 353 (2000).
- Sone Y, and Onishi, Y, *J. Phys. Soc. Jpn.* **44**, 1981 (1978).
- Ytrehus, T, in L Potter (ed.) *Rarefied Gas Dynamics*, vol 51 of *Progr. Aeronaut.* (AIAA, 1977), p. 1197.
- Ytrehus, T, *Multiphase Sci. Technol.* **9**, 205 (1997).