Using one Thousand GPUs to Understand the Euler Equations

Kjetil Lye (ETH Zurich)

joint work with:
Ulrik S. Fjordholm (University of Oslo)
Siddhartha Mishra (ETH Zurich)

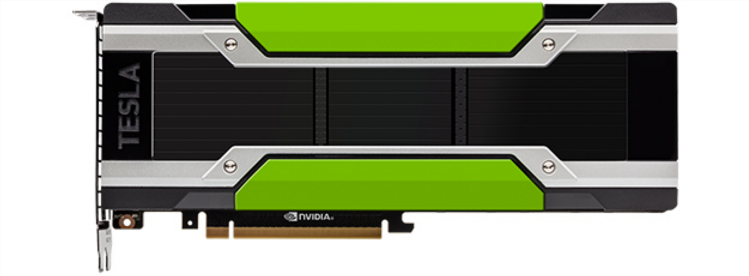
Geilo Winter School, 2019

Funded by the ERC SPARCCLE project

Compute time on CSCS through s665, s839

In this talk...

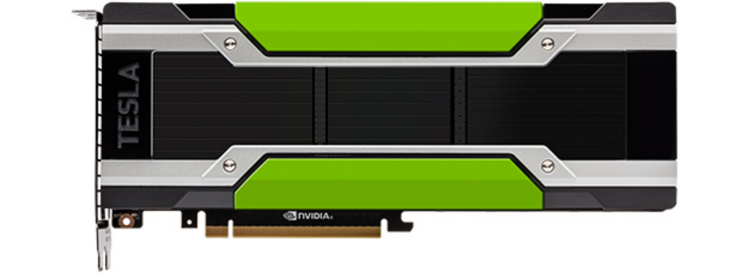
We will use a thousand of these



price tag: 15 000 EUR

In this talk. . .

We will use a thousand of these

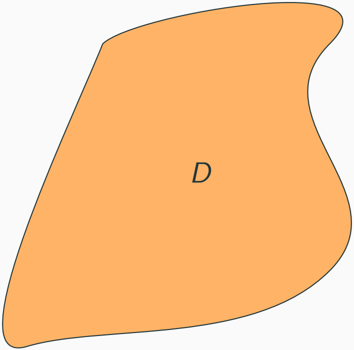


. . . to understand these

$$
\begin{aligned}
\rho_t + \nabla_x \cdot (\rho \mathbf{v}) &= 0 \\
(\rho \mathbf{v})_t + \nabla_x \cdot (\rho \mathbf{v} \otimes \mathbf{v} + pI) &= 0 \\
E_t + \nabla_x \cdot ((E + p)\mathbf{v}) &= 0
\end{aligned}
$$

The Compressible Euler Equations

A model for gas



$$\begin{aligned} \rho_t + \nabla_x \cdot (\rho \mathbf{v}) &= 0 \\ (\rho \mathbf{v})_t + \nabla_x \cdot (\rho \mathbf{v} \otimes \mathbf{v} + pI) &= 0 \\ E_t + \nabla_x \cdot ((E + p)\mathbf{v}) &= 0 \end{aligned}$$

with suitable equation of state

Hyperbolic conservation laws

$$u_t + \nabla_x \cdot f(u) = 0 \qquad \text{on } D \times \mathbb{R}^+$$

here $D \subset \mathbb{R}^d$ and $u : D \times \mathbb{R}^+ \to \mathbb{R}^N$.

- d: dimension
- N: number of unknowns

Hyperbolic conservation laws

$$u_t + \nabla_x \cdot f(u) = 0 \qquad \text{on } D \times \mathbb{R}^+$$

here $D \subset \mathbb{R}^d$ and $u : D \times \mathbb{R}^+ \to \mathbb{R}^N$.

- d: dimension
- N: number of unknowns

The compressible **Euler equations** ($d = 3, N = 5$):

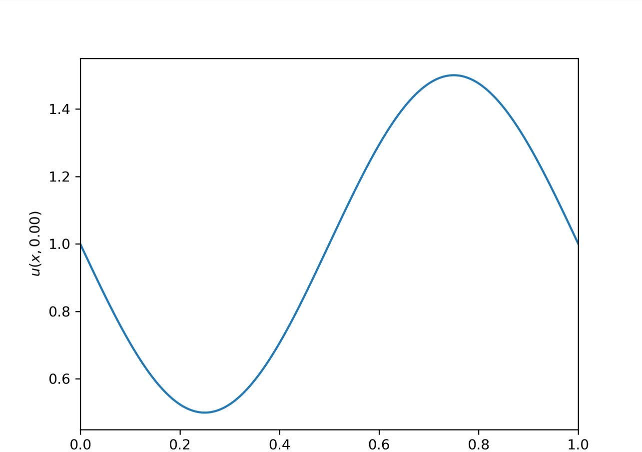
$$\begin{pmatrix} \rho_t \\ (\rho \mathbf{v})_t \\ E_t \end{pmatrix} + \nabla_x \cdot \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + pI \\ (E + p)\mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Are non-linear hyperbolic conservation laws well-posed?

Burgers' equation with smooth initial data

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \qquad \text{on } D \times \mathbb{R}^+$$

- Discontinuities form



Are non-linear hyperbolic conservation laws well-posed?

Go to **weak formulation**

$$\int_{D\times\mathbb{R}^+} u\phi_t + \nabla_x\phi \cdot f(u)\,dx + \int_D u(x,0)\phi(x,0)\,dx = 0$$

for all $\phi \in C_c^\infty(D \times \mathbb{R}^+)$

- Weak solutions are not unique

Are non-linear hyperbolic conservation laws well-posed?

$$\int_{D\times\mathbb{R}^+} u\phi_t + \nabla_x\phi \cdot f(u)\,dx + \int_D u(x,0)\phi(x,0)\,dx = 0$$

for all $\phi \in C_c^\infty(D \times \mathbb{R}^+)$

Introduce **entropy conditions**

$$\eta(u)_t + \nabla_x \cdot q(u) \leq 0 \ \text{ in the sense of distributions}$$

- Well-posedness for **scalar** equations

Are non-linear hyperbolic conservation laws well-posed?

$$\int_{D\times\mathbb{R}^+} u\phi_t + \nabla_x\phi \cdot f(u)\, dx + \int_D u(x,0)\phi(x,0)\, dx = 0$$

for all $\phi \in C_c^\infty(D\times\mathbb{R}^+)$

Introduce **entropy conditions**

$$\eta(u)_t + \nabla_x \cdot q(u) \leq 0 \quad \text{in the sense of distributions}$$

- Well-posedness for **scalar** equations
- Well-posedness for systems **in 1D** with small initial data. [Glimm, 1965; Bressan et al, 2000]

Are non-linear hyperbolic conservation laws well-posed?

$$\int_{D\times\mathbb{R}^+} u\phi_t + \nabla_x\phi \cdot f(u)\, dx + \int_D u(x,0)\phi(x,0)\, dx = 0$$

for all $\phi \in C_c^\infty(D \times \mathbb{R}^+)$

Introduce **entropy conditions**

$$\eta(u)_t + \nabla_x \cdot q(u) \leq 0 \quad \text{in the sense of distributions}$$

- Well-posedness for **scalar** equations
- Well-posedness for systems **in 1D** with small initial data. [Glimm, 1965; Bressan et al, 2000]

However, DeLellis et al constructed *infinitely* many entropy solutions to 2D Euler, **and** the Shallow Water equations

Are non-linear hyperbolic conservation laws well-posed?

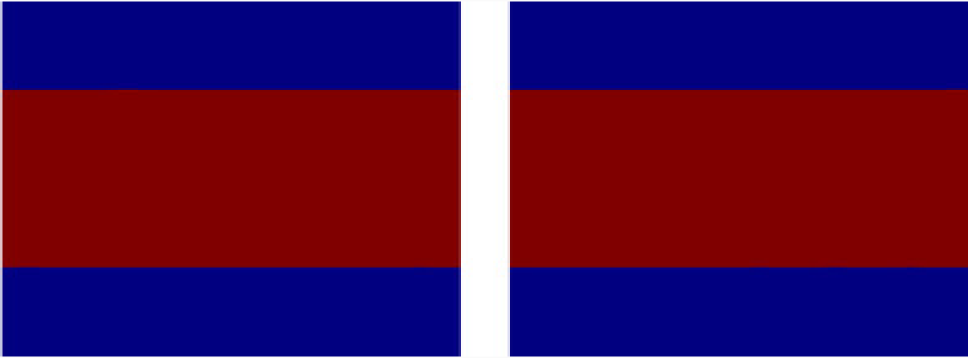
Only if $N = 1$ (scalar) or $d = 1$ (1 spatial dimension).

Are non-linear hyperbolic conservation laws well-posed?

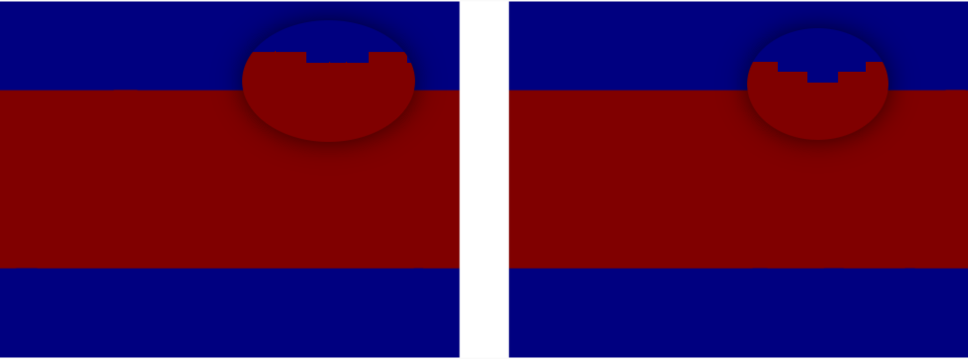
Only if $N = 1$ (scalar) or $d = 1$ (1 spatial dimension).

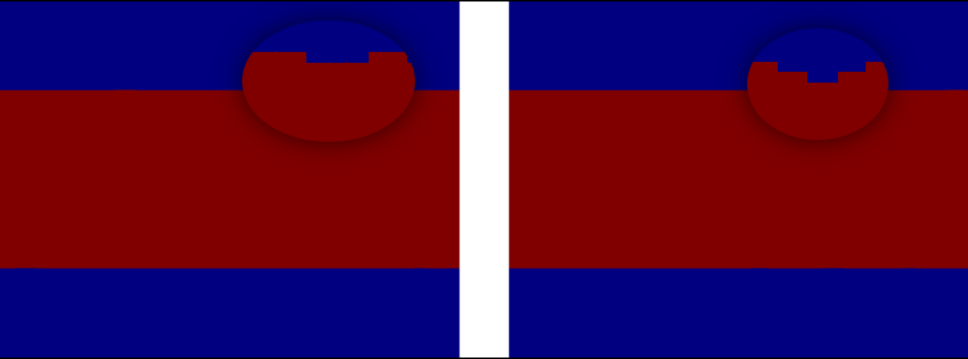
Does it matter in practice?

Kelvin-Helmholtz: Numerical experiment

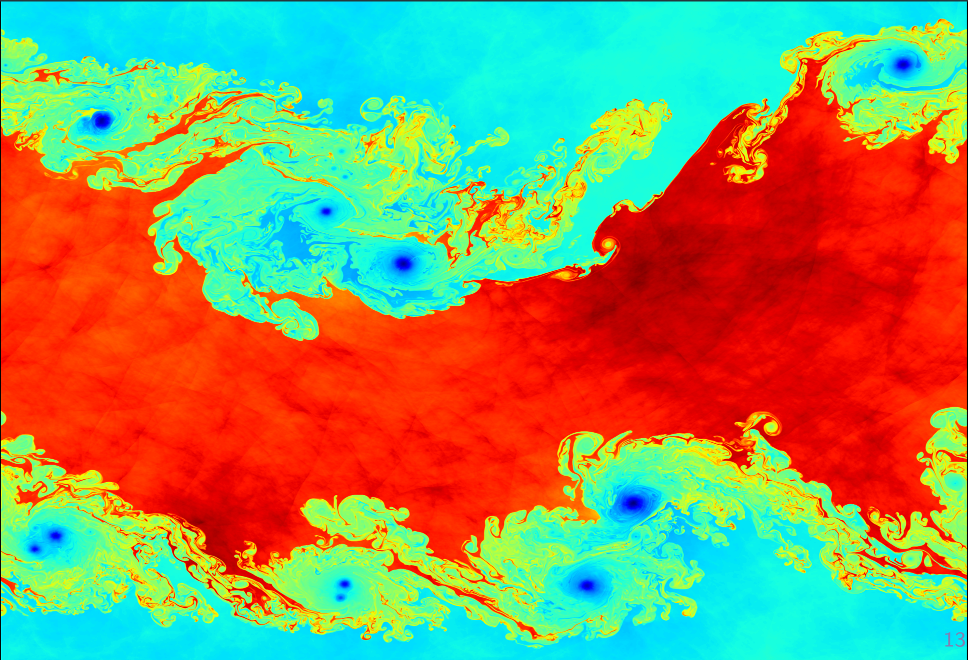


Kelvin-Helmholtz: Numerical experiment





Infinitely small structures appear (16384 x 16384)



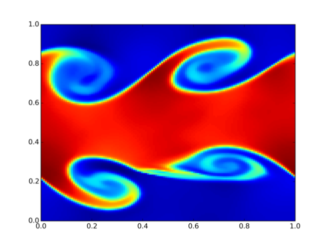
Appears in clouds

Source: **Wikipedia** http://en.wikipedia.org/wiki/Image:Wavecloudsduval.jpg

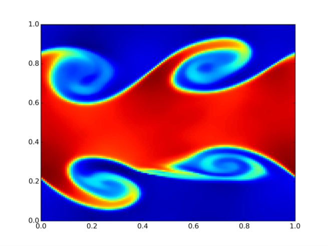
...and Norwegian slow TV

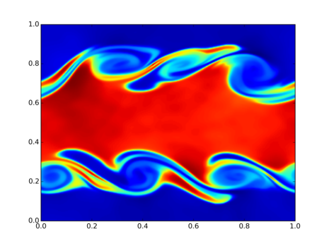
Source: `NRK.no`, Saltstraumen minutt for minutt, Bodø, Norway

Kelvin-Helmholtz: Varying mesh resolution

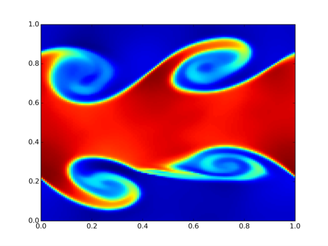


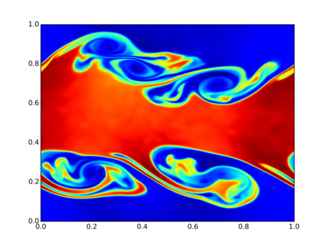
Kelvin-Helmholtz: Varying mesh resolution

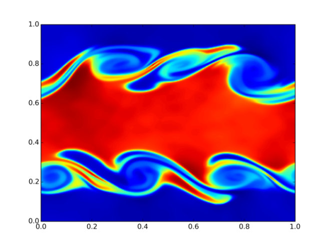




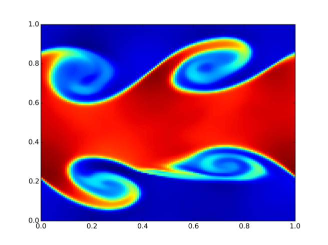
Kelvin-Helmholtz: Varying mesh resolution

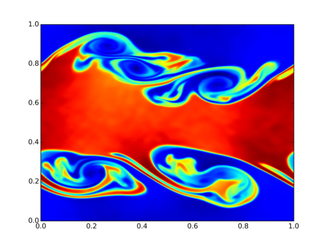


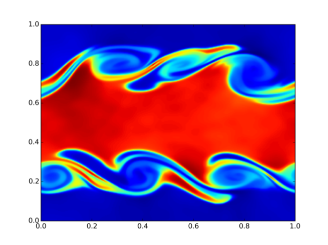


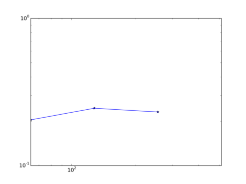


Kelvin-Helmholtz: Varying mesh resolution





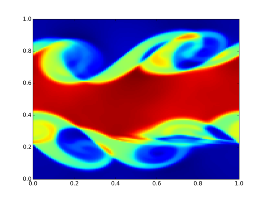


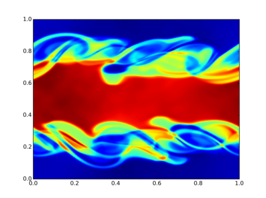


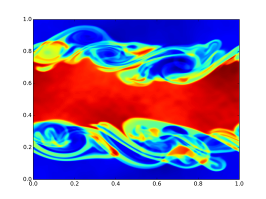
Cauchy convergence ($L^1(D)$)

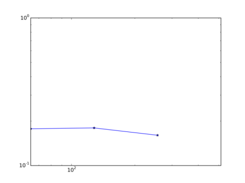
Kelvin-Helmholtz: Ensemble averages[1]

$$\bar{\rho}(x,t) = \frac{1}{2}\left(\rho_1(x,t) + \rho_2(x,t)\right).$$







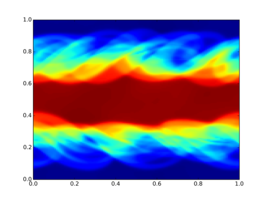


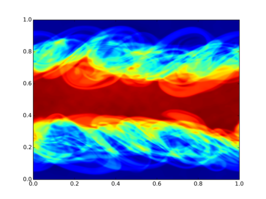
Cauchy convergence ($L^1(D)$)

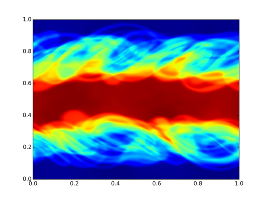
[1] [Fjordholm, Käppeli, Mishra, Tadmor, 2014]

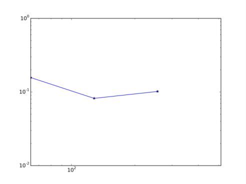
Kelvin-Helmholtz: Ensemble averages[1]

$$\bar{\rho}(x,t) = \frac{1}{8}\sum_{k=1}^{8} \rho_k(x,t).$$







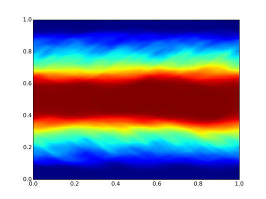


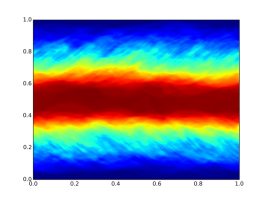
Cauchy convergence ($L^{\mathbf{1}}(D)$)

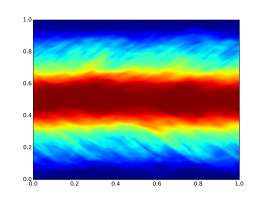
[1] [Fjordholm, Käppeli, Mishra, Tadmor, 2014]

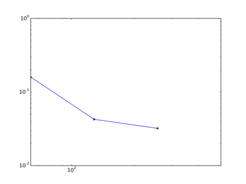
Kelvin-Helmholtz: Ensemble averages[1]

$$\bar{\rho}(x,t) = \frac{1}{64}\sum_{k=1}^{64} \rho_k(x,t).$$







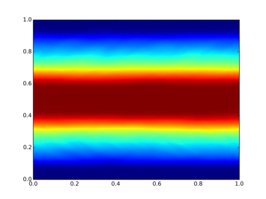


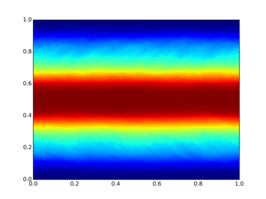
Cauchy convergence ($L^1(D)$)

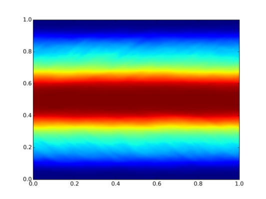
[1] [Fjordholm, Käppeli, Mishra, Tadmor, 2014]

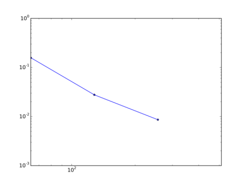
Kelvin-Helmholtz: Ensemble averages[1]

$$\bar{\rho}(x,t) = \frac{1}{1024}\sum_{k=1}^{1024} \rho_k(x,t).$$









Cauchy convergence ($L^{\mathbf{1}}(D)$)

[1] [Fjordholm, Käppeli, Mishra, Tadmor, 2014]

Statistics of the solution are well-behaved[2]

[2] For *one point statistics*, theory and experiments were done in [Fjordholm, Käppeli, Mishra, Tadmor, 2014]

Statistical solutions

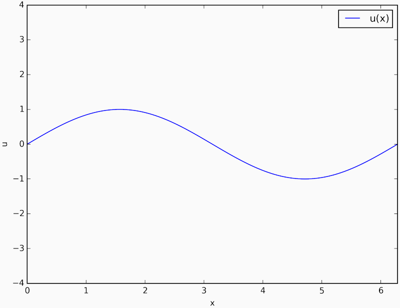
The epistemological value of probability theory is based on the fact that chance phenomena, considered collectively and on a grand scale, create non-random regularity.

– Andrey Kolmogorov,

Limit Distributions for Sums of Independent Random Variables (1954)

Statistical solutions

- Classical: Function $u : D \times [0, T) \to \mathbb{R}^N$

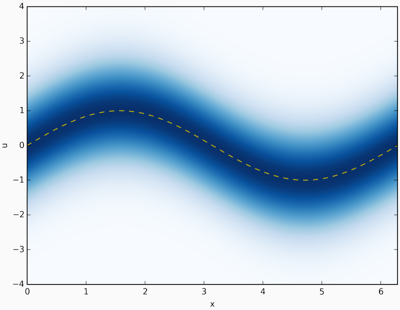


Statistical solutions

- Classical: Function $u : D \times [0, T) \to \mathbb{R}^N$
- Statistical solution: $\mu_t \in \text{Prob}(L^1(D, \mathbb{R}^N))$

Duality

$$\underbrace{\mu \in \text{Prob}(L^p(D, U))}_{\text{Measure on function space}} \qquad \text{and} \qquad \underbrace{\nu = \left(\nu^1_{x_1}, \nu^2_{x_1,x_2}, \ldots\right)}_{\text{correlation measure}}$$



A formal computation

$$u_t + f(u)_x = 0$$

Second moment evolves as

$$\partial_t(u(x_1,t)u(x_2,t))$$

.

A formal computation

$$u_t + f(u)_x = 0$$

Second moment evolves as

$$\partial_t(u(x_1,t)u(x_2,t)) + \overbrace{\partial_{x_2}\Big(u(x_1,t)f(u(x_2,t))\Big)}^{=u(x_1,t)u(x_2,t)_t} + \underbrace{\partial_{x_1}\big(u(x_2,t)f(u(x_1,t))\big)}_{=u(x_1,t)_t u(x_2,t)} = 0.$$

Generalized to

$$\partial_t(u(x_1,t)\cdots u(x_k,t)) + \sum_{i=1}^{k} \partial_{x_i}\Big(u(x_1,t)\cdots f(u(x_i,t))\cdots u(x_k,t)\Big) = 0.$$

Statistical solutions for hyperbolic conservation laws

A $\nu^t = \left(\nu^t_{x_1}, \nu^t_{x_1,x_2}, \ldots\right)$ is a statistical solution if for every $k \in \mathbb{N}$

$$\partial_t \langle \nu^t_{x_1,\ldots,x_k}, \xi_1 \cdots \xi_k \rangle + \sum_{i=1}^{k} \partial_{x_i} \langle \nu^t_{x_1,\ldots,x_k}, \xi_1 \cdots f(\xi_i) \cdots \xi_k \rangle = 0,$$

in the sense of distributions.

Numerical algorithm

A man provided with paper, pencil, and rubber, and subject to strict discipline, is in effect a universal machine.

– Alan Turing

Numerical algorithm for statistical solutions[3]

Let $\mu_0 \in \mathrm{Prob}(L^1(D, \mathbb{R}^N))$ be the initial data. Let $\mathcal{S}^\Delta$ be a numerical evolution operator

[3] Based on the FKMT algorithm [Fjordholm, Käppeli, Mishra, Tadmor, 2014]

Numerical algorithm for statistical solutions[3]

Let $\mu_0 \in \mathrm{Prob}(L^1(D, \mathbb{R}^N))$ be the initial data. Let $\mathcal{S}^\Delta$ be a numerical evolution operator

1. Draw M i.d.d. samples u_0^k from μ_0

u_0^1 ... u_0^k ... u_0^M

Numerical algorithm for statistical solutions[3]

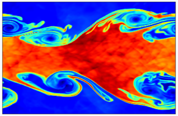
Let $\mu_0 \in \mathrm{Prob}(L^1(D, \mathbb{R}^N))$ be the initial data. Let $\mathcal{S}^\Delta$ be a numerical evolution operator

1. Draw M i.d.d. samples u_0^k from μ_0
2. For $k = 1, \dots, M$ evolve

$$u^{k,\Delta} = \mathcal{S}^\Delta(u_0^k)$$

u_0^k

$$\xrightarrow{\text{Use FVM/DG/}\dots}$$



$u^{k,\Delta}$

Numerical algorithm for statistical solutions[3]

Let $\mu_0 \in \mathrm{Prob}(L^1(D, \mathbb{R}^N))$ be the initial data. Let $\mathcal{S}^\Delta$ be a numerical evolution operator

1. Draw M i.d.d. samples u_0^k from μ_0
2. For $k = 1, \ldots, M$ evolve
$$u^{k,\Delta} = \mathcal{S}^\Delta(u_0^k)$$
3. Estimate statistical solution using Monte-Carlo:
$$\mu^{\Delta,M} = \frac{1}{M}\sum_{k=1}^{M} \delta_{u^{k,\Delta}}.$$

[3] Based on the FKMT algorithm [Fjordholm, Käppeli, Mishra, Tadmor, 2014]

Goal: show that $\mu^{\Delta,M}$ converges

1. Draw M i.d.d. samples u_0^k from μ_0
2. For $k = 1, \dots, M$ evolve
$$u^{k,\Delta} = \mathcal{S}^{\Delta}(u_0^k)$$
3. Estimate statistical solution using Monte-Carlo:
$$\mu^{\Delta,M} = \frac{1}{M}\sum_{k=1}^{M} \delta_{u^{k,\Delta}}.$$

[3] Based on the FKMT algorithm [Fjordholm, Käppeli, Mishra, Tadmor, 2014]

Theory for Systems

I summed up all systems in a phrase, and all existence in an epigram.

– Oscar Wilde

Convergence of correlation measures

Theorem (Fjordholm, Lye, Mishra, in preparation)
Assume *that ν_n is a sequence of correlation measures satisfying*

$$\sup_{n\in\mathbb{N}}\langle \nu_n^1, |\xi|^p\rangle \leq c^p$$

$$\lim_{r\to 0}\sup_{n\in\mathbb{N}}\left(\int_D \fint_{B_r(x)} \langle \nu_{n,x,y}^2, |\xi_1-\xi_2|^p\rangle\, dydx\right)^{1/p} = 0$$

for some $c > 0$.

Convergence of correlation measures

Theorem (Fjordholm, Lye, Mishra, in preparation)
Assume *that ν_n is a sequence of correlation measures satisfying*

$$\sup_{n\in\mathbb{N}} \langle \nu^1_n, |\xi|^p \rangle \leq c^p$$

$$\lim_{r\to 0} \sup_{n\in\mathbb{N}} \left(\int_D \fint_{B_r(x)} \langle \nu^2_{n,x,y}, |\xi_1 - \xi_2|^p \rangle \, dydx \right)^{1/p} = 0$$

for some $c > 0$.

Then *there exists a weak-$*$ convergent subsequence of ν_n*

Convergence of correlation measures

Theorem (Fjordholm, Lye, Mishra, in preparation)
Assume *that ν_n is a sequence of correlation measures satisfying*

$$\sup_{n\in\mathbb{N}} \langle \nu_n^1, |\xi|^p \rangle \leq c^p$$

$$\lim_{r\to 0} \sup_{n\in\mathbb{N}} \left(\int_D \fint_{B_r(x)} \langle \nu_{n,x,y}^2, |\xi_1 - \xi_2|^p \rangle \, dy dx \right)^{1/p} = 0$$

for some $c > 0$.

Then *there exists a weak-$*$ convergent subsequence of ν_n*

If *we have some boundedness assumptions on ν^n,*
then *observables converge strongly:*

$$\lim_{j\to\infty} \int_{D^k} \left| \langle \nu_{n_j,x}^k, g(x) \rangle - \langle \nu_x^k, g(x) \rangle \right| dx = 0$$

Theorem (Fjordholm, Lye, Mishra, in preparation)
Assume $\{\mu_t^{\Delta_\lambda}\}_\lambda$ *converges weakly-$*$ to* μ_t *in* $Prob(L^1(\mathbb{R}^d, \mathbb{R}^N))$ *where* $\Delta_\lambda \to 0$, *and assume*

$$\underbrace{\int_{L^1(D)} TV(\{\overline{v}_{\mathbf{i}}\})\, d\mu_t^{\Delta_\lambda}(u)}_{\text{Average total variation}} \leq C\Delta_\lambda^{-s},$$

for some $s < 1$.

Then μ_t *is a statistical solution.*

We have to use numerical experiments

The last two theorems translate to:

If

$$\lim_{r\to 0} \sup_{n\in\mathbb{N}} \left(\int_D \fint_{B_r(x)} \langle \nu^2_{n,x,y}, |\xi_1 - \xi_2|^p \rangle \, dydx \right)^{1/p} = 0$$

then

$$\mu^{\Delta,M} \text{ converges to a statistical solution}$$

We have to use numerical experiments

The last two theorems translate to:

If

$$\lim_{r\to 0}\sup_{n\in\mathbb{N}}\left(\int_D \fint_{B_r(x)} \langle \nu^2_{n,x,y}, |\xi_1-\xi_2|^p\rangle \, dydx\right)^{1/p} = 0$$

then

$$\mu^{\Delta,M} \text{ converges to a statistical solution}$$

We will use

$$\left(\int_D \fint_{B_r(x)} \langle \nu^2_{n,x,y}, |\xi_1-\xi_2|^p\rangle \, dydx\right)^{1/p} \approx \frac{1}{M}\sum_k S_h^p(u_k^\Delta)$$

We have to use numerical experiments

The last two theorems translate to:

If

$$\lim_{r\to 0}\sup_{n\in\mathbb{N}}\left(\int_D \fint_{B_r(x)} \langle \nu^2_{n,x,y}, |\xi_1-\xi_2|^p\rangle\, dydx\right)^{1/p} = 0$$

then

$$\mu^{\Delta,M} \text{ converges to a statistical solution}$$

We will use

$$\left(\int_D \fint_{B_r(x)} \langle \nu^2_{n,x,y}, |\xi_1-\xi_2|^p\rangle\, dydx\right)^{1/p} \approx \frac{1}{M}\sum_k S^p_h(u^\Delta_k)$$

Recall: S^p_h is a FVM evolution!

Can we do better than Monte-Carlo?

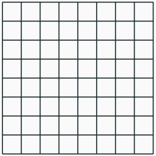
Multilevel Monte Carlo

''Will you walk a little faster?'' said a whiting to a snail. ''There's a porpoise close behind us, and he's treading on my tail.''

– Lewis Carroll,

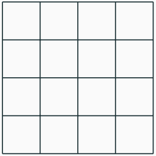
Alice's Adventures in Wonderland.

Multi-level Monte Carlo [Heinrich, 2001, Giles, 2008]

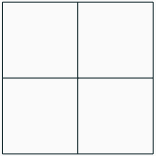


$\ell = 2$

Levels $\ell = 0, \ldots, L$, approximations u_k^ℓ.

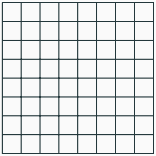


$\ell = 1$



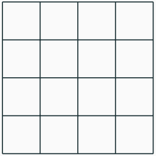
$\ell = 0$

Multi-level Monte Carlo [Heinrich, 2001, Giles, 2008]



$\ell = 2$

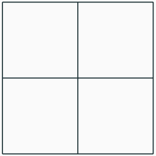
Levels $\ell = 0, \ldots, L$, approximations u_k^ℓ.



$$\mu^L = \mu^0 + \sum_{\ell=1}^{L} (\mu^\ell - \mu^{\ell-1})$$

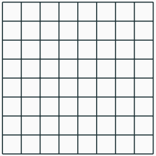
$\ell = 1$

.



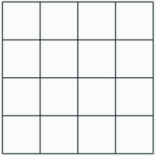
$\ell = 0$

Multi-level Monte Carlo [Heinrich, 2001, Giles, 2008]



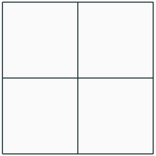
$\ell = 2$

Levels $\ell = 0, \dots, L$, approximations u_k^ℓ.



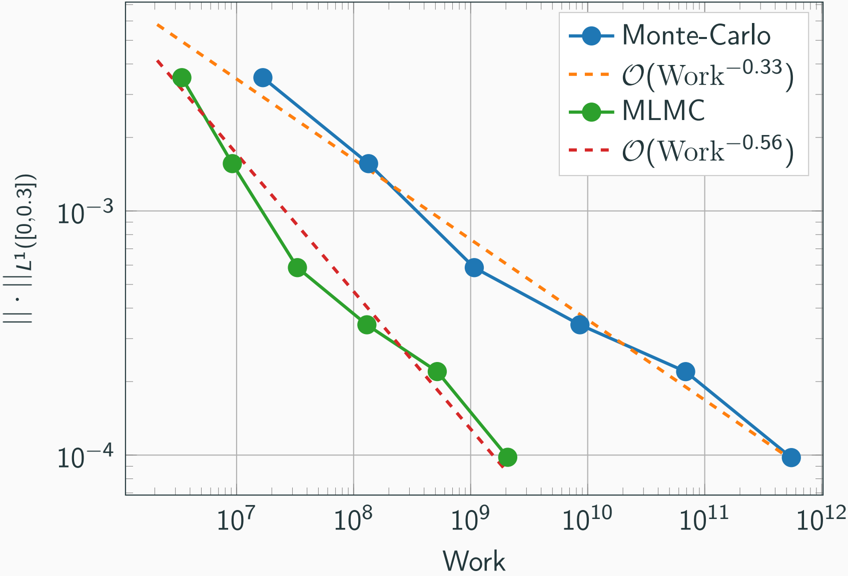
$\ell = 1$

$$\begin{aligned}\mu^L &= \mu^0 + \sum_{\ell=1}^{L} (\mu^\ell - \mu^{\ell-1}) \\ &\approx \frac{1}{M_0} \sum_{k=1}^{M_0} \delta_{u_k^0} + \sum_{\ell=1}^{L} \frac{1}{M_l} \sum_{k=1}^{M_\ell} (\delta_{u_k^\ell} - \delta_{u_k^{\ell-1}}) \\ &=: \mu_{\Delta x, L}.\end{aligned}$$

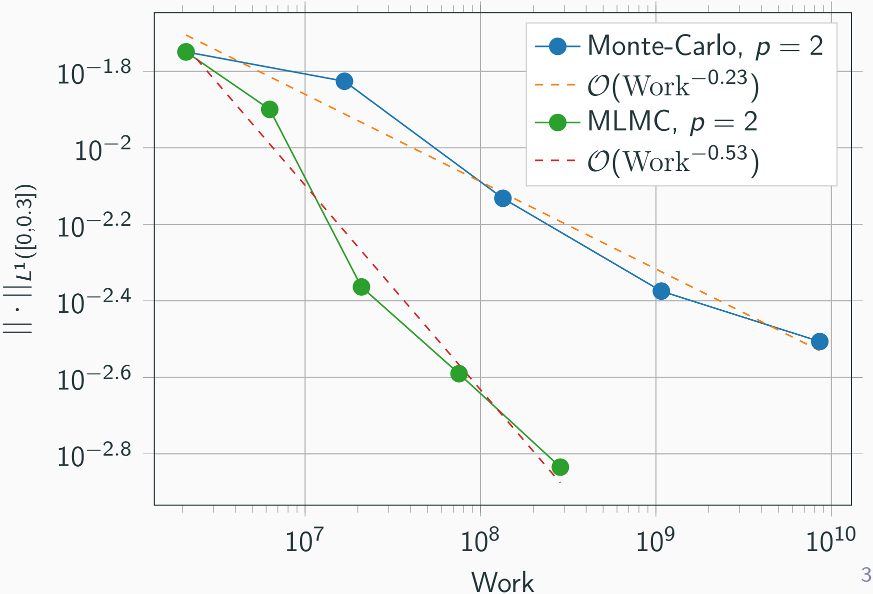


$\ell = 0$

Burgers': Uncertain shock location: MLMC

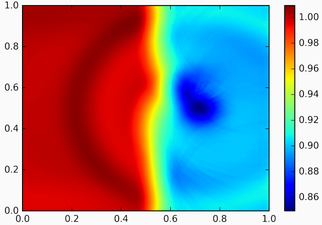


Burgers': Brownian initial data: MLMC

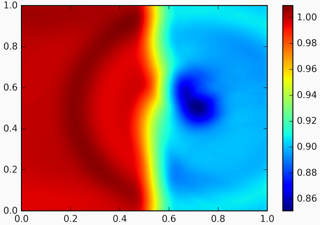


MLMC test: Shock-Vortex

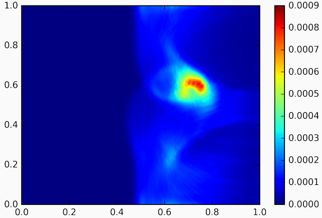
MLMC test: Shock-Vortex



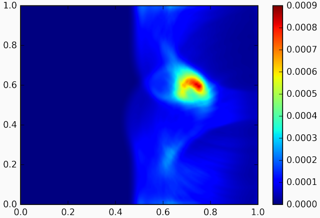
Mean, MLMC



Mean, MC



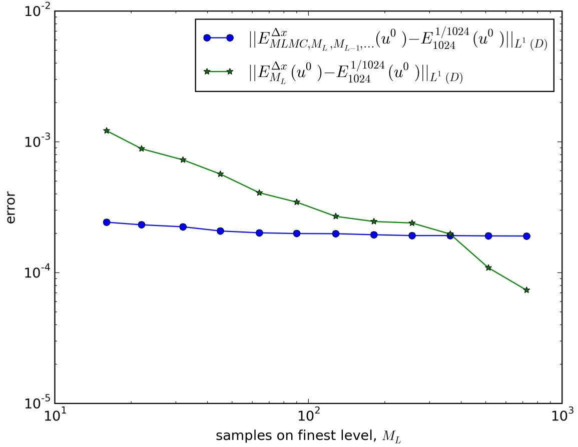
Variance, MLMC



Variance, MC

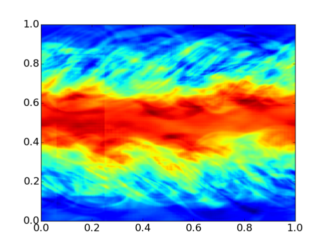
MLMC-EMVS: Error analysis Shock-Vortex

Reference solution: Monte-Carlo 1024x1024 using 1024 samples

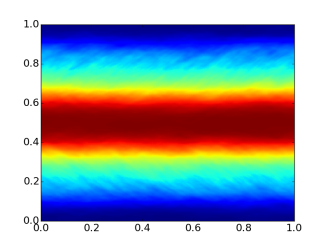


Kelvin-Helmholtz + MLMC

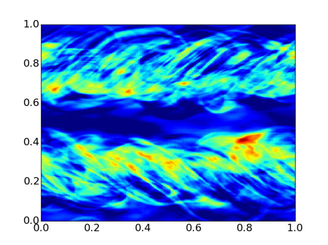
Kelvin-Helmholtz + MLMC



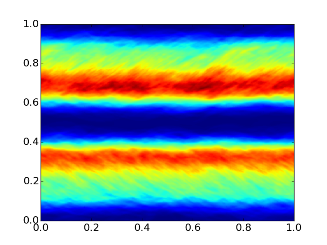
Mean, MLMC



Mean, MC



Variance, MLMC



Variance, MC

Convergence

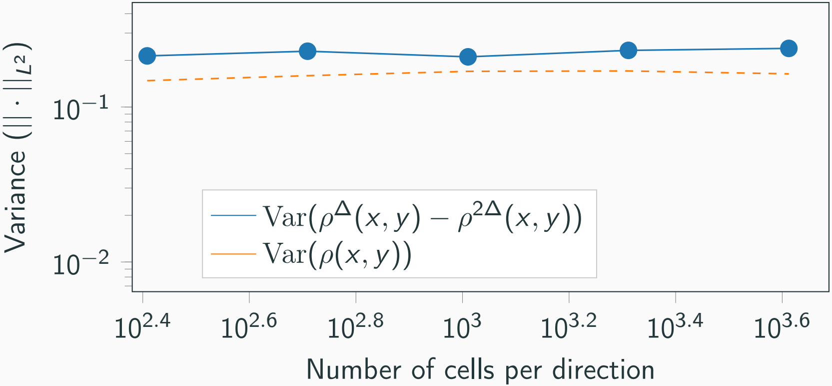
For MLMC to be faster, we need

$$\mathrm{Var}(\rho^{\Delta} - \rho^{2\Delta}) < \mathrm{Var}(\rho)$$

Convergence

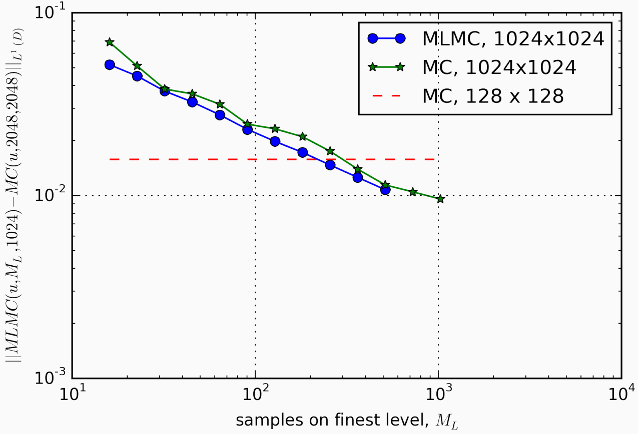
For MLMC to be faster, we need

$$\mathrm{Var}(\rho^{\Delta} - \rho^{2\Delta}) < \mathrm{Var}(\rho)$$



MLMC-EMVS: Error analysis

Reference solution: Monte-Carlo 2048x2048 using 2000 samples



Can we use MLMC for systems?

Yes, but we only get speed-up for well-behaved data.

All further experiments are done with single level Monte Carlo

Alsvinn

Árvakr ok Alsviðr,
þeir skulu upp héðan
svangir sól drag

Early-woke, All-swift,
hence must these horses
wearily draw up the sun

– Grímnismál

The Alsvinn Simulator

- alsvinn.github.io [GPLv3]

The Alsvinn Simulator

- `alsvinn.github.io` [GPLv3]
- Multi-GPU Finite Volume simulator

The Alsvinn Simulator

- `alsvinn.github.io` [GPLv3]
- Multi-GPU Finite Volume simulator
- Written in C++/CUDA

Reproducibility

The best is the enemy of the good.

– Voltaire

Reproducibility of Alsvinn experiments

1. Open source [GPLv3]

Reproducibility of Alsvinn experiments

1. Open source [GPLv3]
2. Unit tests

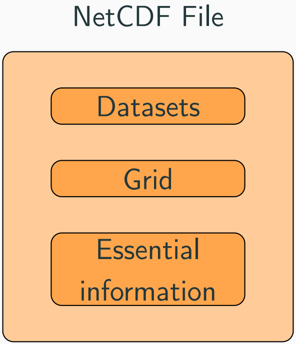
```
[ RUN      ] ReconstructionTests/ReconstructionConvergenceTest.ReconstructionTest/11
[       OK ] ReconstructionTests/ReconstructionConvergenceTest.ReconstructionTest/11 (8 ms)
[----------] 12 tests from ReconstructionTests/ReconstructionConvergenceTest (120 ms total)

[----------] 2 tests from MeanVarConvergenceTests/MeanVarTest
[ RUN      ] MeanVarConvergenceTests/MeanVarTest.MeanVarTest/0
[       OK ] MeanVarConvergenceTests/MeanVarTest.MeanVarTest/0 (1768 ms)
[ RUN      ] MeanVarConvergenceTests/MeanVarTest.MeanVarTest/1
[       OK ] MeanVarConvergenceTests/MeanVarTest.MeanVarTest/1 (729 ms)
[----------] 2 tests from MeanVarConvergenceTests/MeanVarTest (2497 ms total)

[----------] Global test environment tear-down
[==========] 139 tests from 40 test cases ran. (3819 ms total)
[  PASSED  ] 139 tests.
```

Reproducibility of Alsvinn experiments

1. Open source [GPLv3]
2. Unit tests
3. Essential information stored in output:



Reproducibility of Alsvinn experiments

1. Open source [GPLv3]
2. Unit tests
3. Essential information stored in output:
    - commit id

Reproducibility of Alsvinn experiments

1. Open source [GPLv3]
2. Unit tests
3. Essential information stored in output:
    - commit id
    - CPU/GPU used

Reproducibility of Alsvinn experiments

1. Open source [GPLv3]
2. Unit tests
3. Essential information stored in output:
    - commit id
    - CPU/GPU used
    - library versions

Reproducibility of Alsvinn experiments

1. Open source [GPLv3]
2. Unit tests
3. Essential information stored in output:
    - commit id
    - CPU/GPU used
    - library versions
    - solver settings (flux, grid, etc)

Reproducibility of Alsvinn experiments

1. Open source [GPLv3]
2. Unit tests
3. Essential information stored in output:
   - commit id
   - CPU/GPU used
   - library versions
   - solver settings (flux, grid, etc)
4. Docker

Numerical experiments for systems

I think that it is a relatively good approximation to truth – which is much too complicated to allow anything but approximations – that mathematical ideas originate in empirics.

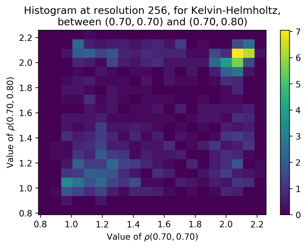
– John von Neumann

What converges?

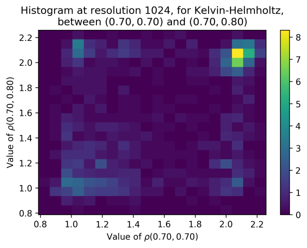
Kelvin-Helmholtz: 2 point histograms

$N = 128$

$N = 512$

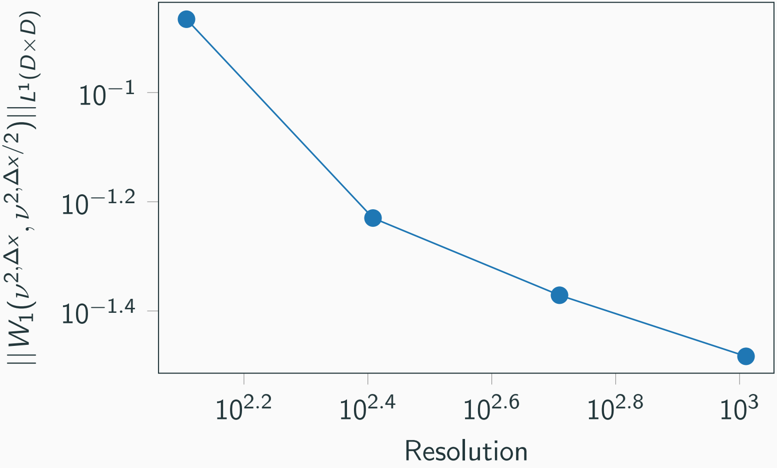


$N = 256$

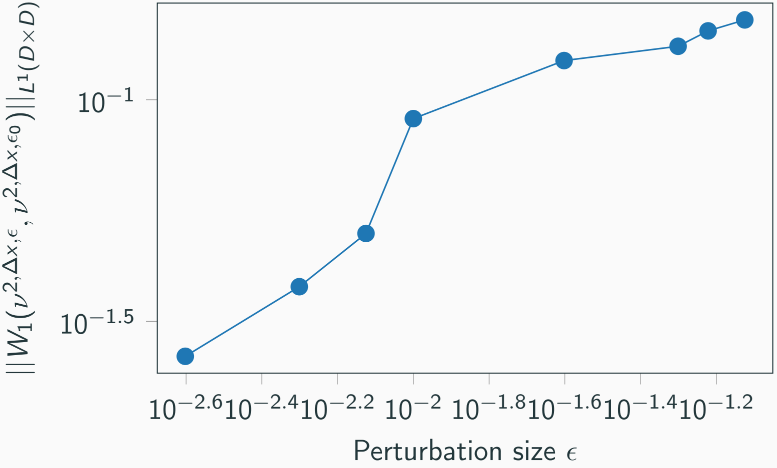


$N = 1024$

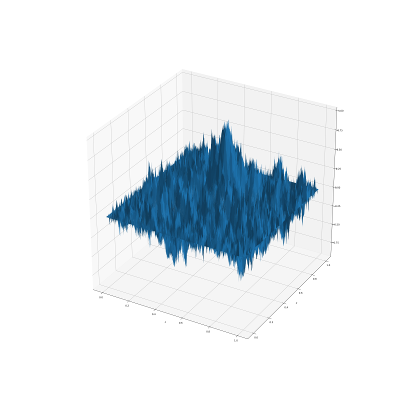
Kelvin-Helmholtz: Wasserstein convergence



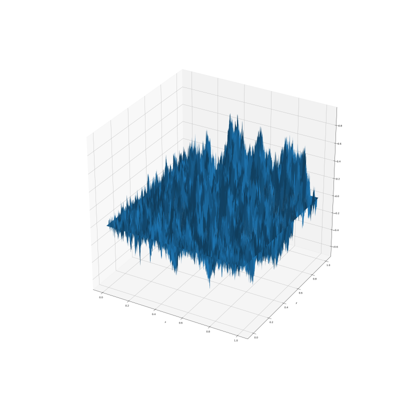
Kelvin-Helmholtz: varying perturbation size (stability)



Brownian initial data

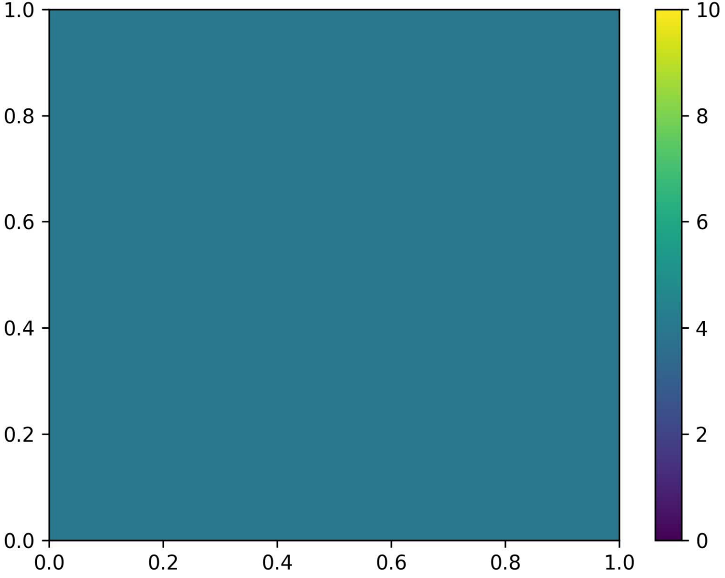


u_x at $t = 0$

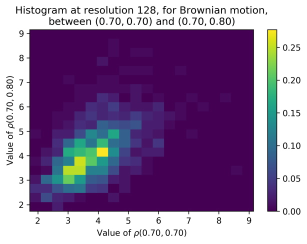


u_y at $t = 0$

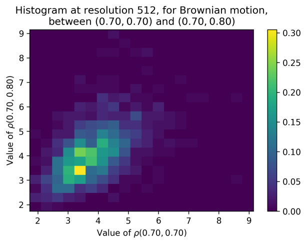
$\rho_0 = 4$ and $p_0 = 2.5$


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
10
8
6
4
2
0

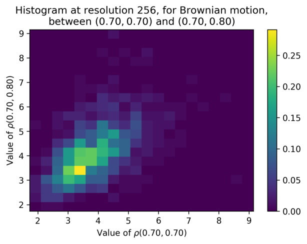
Brownian initial data: 2 point histograms



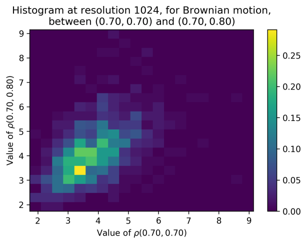
$N = 128$



$N = 512$

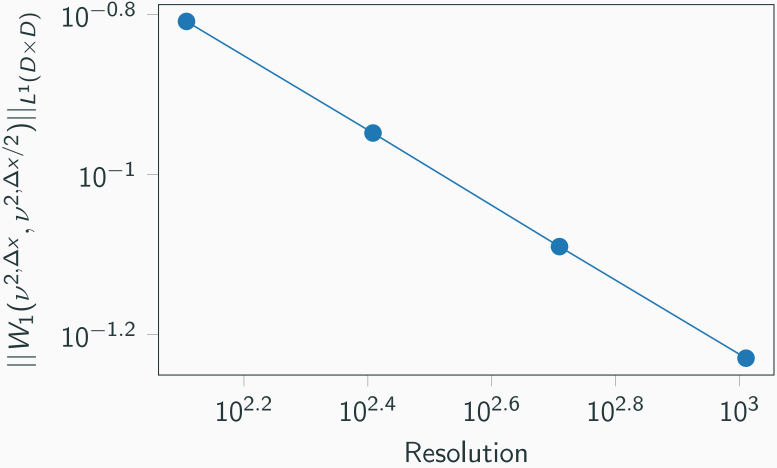


$N = 256$

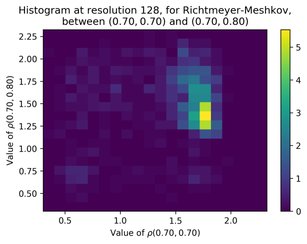


$N = 1024$

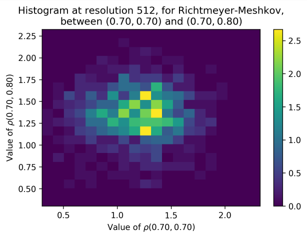
Brownian initial data: Wasserstein convergence



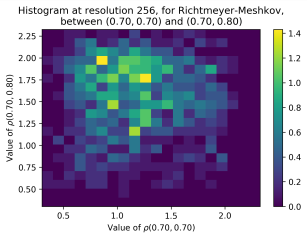
Richtmeyer-Meshkov: 2 point histograms



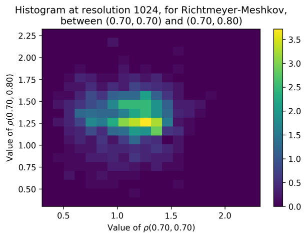
$N = 128$



$N = 512$



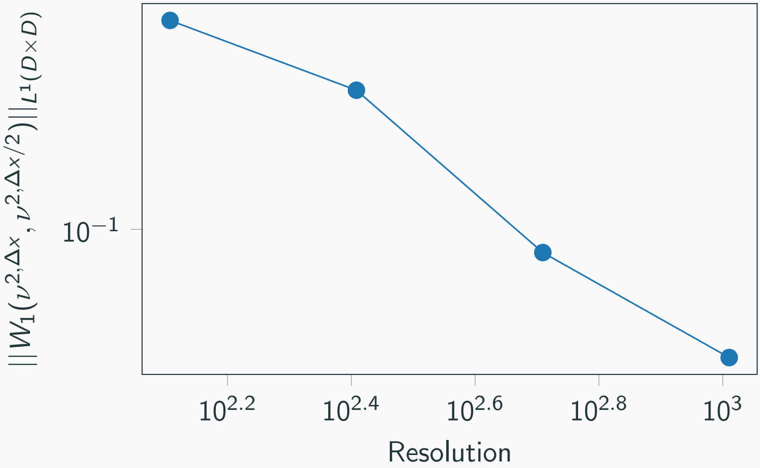
$N = 256$



$N = 1024$

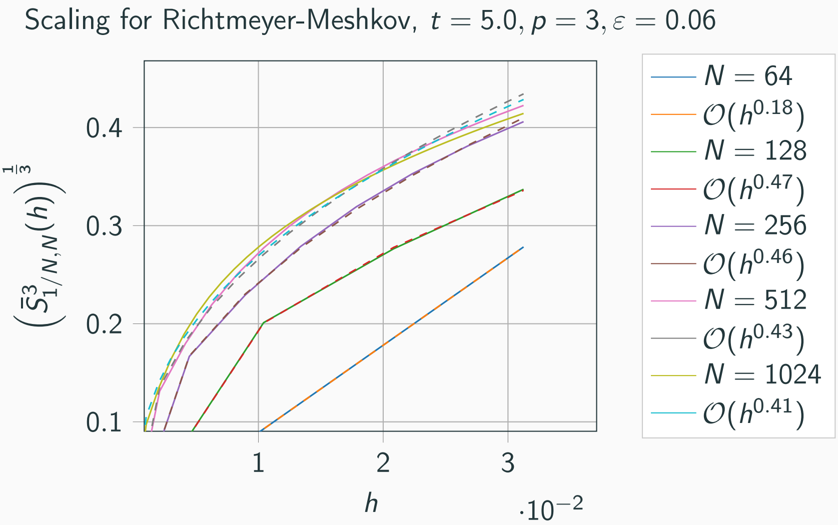
Richtmeyer-Meshkov: Wasserstein convergence

Wasserstein convergence for Richtmeyer-Meshkov for second correlation r

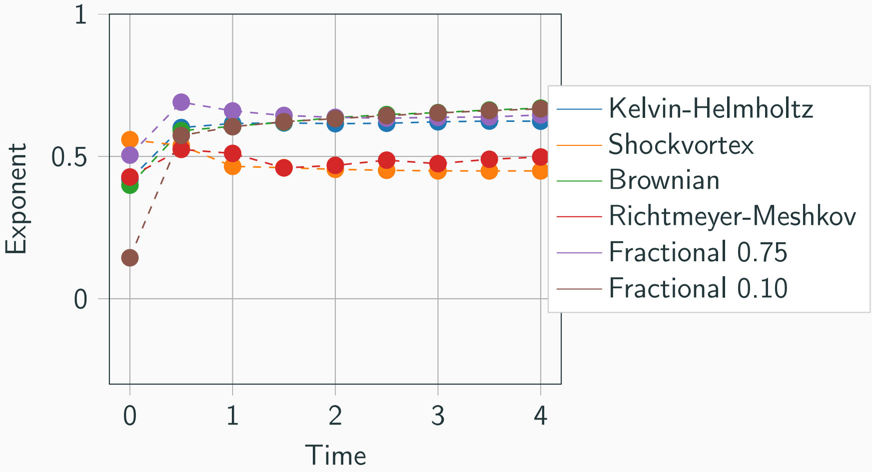


Can we measure the assumptions for convergence?

$$\lim_{r \to 0} \sup_{n \in \mathbb{N}} \left(\underbrace{\int_D \fint_{B_r(x)} \langle \nu^2_{n,x,y}, |\xi_1 - \xi_2|^p \rangle \, dydx}_{:= \bar{S}^p_n(r)} \right)^{1/p} = 0?$$


Scaling for Richtmeyer-Meshkov, $t = 5.0, p = 3, \varepsilon = 0.06$
$\left(\bar{S}^3_{1/N,N}(h)\right)^{\frac{1}{3}}$
0.4
0.3
0.2
0.1
1
2
3
h
$\cdot 10^{-2}$
$N = 64$
$\mathcal{O}(h^{0.18})$
$N = 128$
$\mathcal{O}(h^{0.47})$
$N = 256$
$\mathcal{O}(h^{0.46})$
$N = 512$
$\mathcal{O}(h^{0.43})$
$N = 1024$
$\mathcal{O}(h^{0.41})$

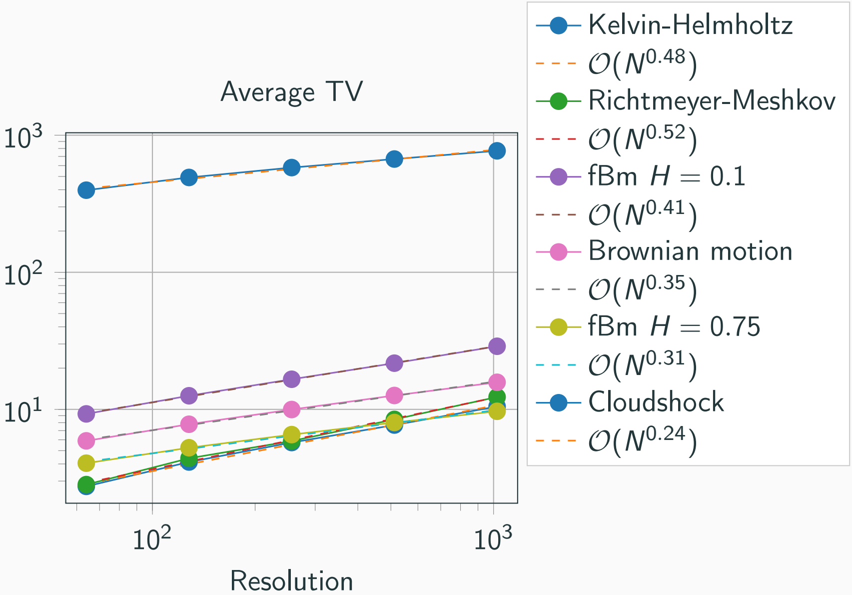
Time evolution, $p = 2$



Can we measure the assumptions for Lax-Wendroff?

$$\int_{L^1(D)} \mathrm{TV}(\{\overline{v}_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^d})\, d\mu_t^{\Delta}(u) \leq CN^s, \qquad 0 \leq s < 1?$$

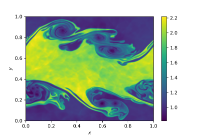
Total variation



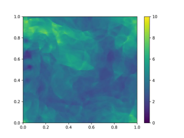
Summary:

Statistical solutions are observed to be well-behaved

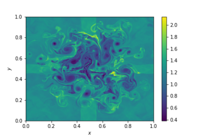
Statistical solutions are observed to be well-behaved



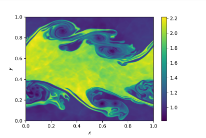
Kelvin-Helmholtz



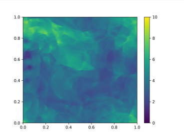
Brownian



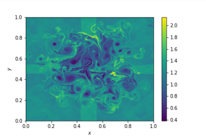
Richtmeyer-Meshkov



Kelvin-Helmholtz



Brownian



Richtmeyer-Meshkov

Thank you.

Kjetil Lye

<kjetil.lye@sam.math.ethz.ch>

`https://alsvinn.github.io`