Approximative Implicitization

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The GAIA project

- Title: "*Application of approximate algebraic geometry in industrial computer aided geometry*"
- Based on the doctorate thesis of [Dokken 97], results published in the book:

  Mathematical Methods for Curves and Surfaces Oslo 2000, 2001 Vanderbilt University Press

  in the article

  T. Dokken, Approximate Implicitization.

The GAIA project

- **Participants:**
- SINTEF Applied Mathematics, Norway
  - think3 SPA , Italy
  - University of Nice Sophia Antipolis, France
  - University of Oslo, Norway
- **Total cost :** € 160. 001.- ,
- **Community funding** € 100 000.-
- **Project start and duration:** Start October 1st. 2000, Duration 12 months.
- Intention to continue the work in a full FET RTD project. Proposal to be made.

The GAIA project

- We address singularities in NURBS curves and surfaces e.g. self intersection.
- Approach:
  - Approximate implicitization aimed at reproducing singularities.
  - Classification of low degree (3,4,…) algebraic curves and surfaces for possible use when choosing among candidate approximations.
  - Reference sampling based method
  - Industrial testing
- Rational parametric curve and surfaces are the most singular subsets of algebraic curves and surfaces!

Implicit Representation

[Sederberg 84] “Implicit Representation of Parametric Curves and Surfaces”:

- A 2D rational parametric curve $\mathbf{p}(t)$ of degree n in the variable t can be expressed implicitly as a degree n polynomial $q(x,y)$, with $q(\mathbf{p}(\mathrm{t}))=0$.
- A 3D rational parametric surface $\mathbf{p}(s,t)$ of degrees (n_1,n_2) in the variables (s,t) can be expressed implicitly as a degree $2n_1n_2$ polynomial $q(x,y,z)$, with $q(\mathbf{p}(s,t))=0$.
  - *A bicubic surface has algebraic degree 2x3x3=18*

Intersection and implicitization

- Combining implicit and parametric descriptions in intersections algorithms reduce the number of variables, however, the polynomial degree increase
- Let $\mathbf{p}(s,t)$ and $\mathbf{r}(u,v)$ be two parametric surfaces, with implicit descriptions $q_{\mathbf{p}}(x,y,z)=0$, *and* $q_{\mathbf{r}}(x,y,z)=0$. Two alternative formulations of the intersection of $\mathbf{p}(s,t)$ and $\mathbf{r}(u,v)$ are
  - $\mathbf{f}(s,t,u,v) = \mathbf{p}(s,t)-\mathbf{r}(u,v) = 0$
  - $q_{\mathbf{p}}(\mathbf{r}(u,v))=0$ and $q_{\mathbf{r}}(\mathbf{p}(s,t))=0$
- Let $\mathbf{p}(s,t)$ and $\mathbf{r}(u,v)$ be bicubic then $q_{\mathbf{p}}(\mathbf{r}(u,v))$ and $q_{\mathbf{r}}(\mathbf{p}(s,t))$ have degree (3x18,3x18)=(54,54).

Exact implicitization

- Find nontrivial polynomial $q \in P_m(\mathbb{R}^l)$
  such that
  $$q(\mathbf{p}(\mathbf{s})) = 0, \quad \mathbf{s} \in \mathbb{R}^{l-1}$$
- For curves $l=2$ for surface $l=3$
- Assumption:
  - Exact arithmetic
  - High polynomial degrees no problem

Exact implicitization methods

- Resultant based methods
  - Express the implicit equation using determinants, in general the determinants for surfaces are large
- Gröbner bases methods
  - Symbolic computations on polynomials
- Moving curves and surfaces [Sederberg 95,97]. Proof of moving surfaces [Cox 99]
  - Reduced the size of the determinates by of 4
  - Base points reduce the size of the determinant

Resultants using scaled Bernstein basis and floating point

- Recent work by Joab Winkler on resultants using the scaled Bernstein basis show promising results.
  - A resultant matrix for scaled Bernstein polynomials, in Linear Algebra and its Applications, vol. 319, pages 179-191, 2000.
- Using the Bernstein basis makes resultant type methods better suited for floating point arithmetic.

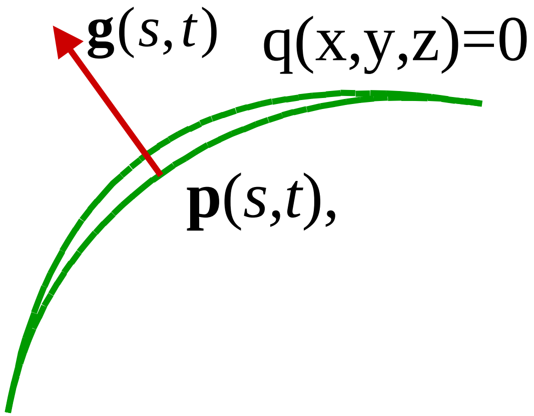
Approximative Implicitization

- Let $\mathbf{p}(s,t)$, $(s,t) \in \Omega \subset \mathbb{R}^2$ be a surface in $\mathbb{R}^3$
- Find nontrivial $q \in P_m(\mathbb{R}^3)$ such that

$$q(\mathbf{p}(s,t) + \eta(s,t)\mathbf{g}(s,t)) = 0, \quad (s,t) \in \Omega$$

- with direction for error measurement $\|\mathbf{g}(s,t)\|_2 = 1.$ and error

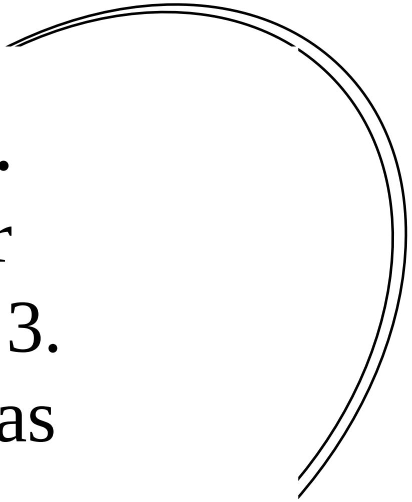
$$|\eta(s,t)| \leq \varepsilon, \quad (s,t) \in \Omega.$$



- Assumption
  - Floating point can be used
  - Degrees can be kept low

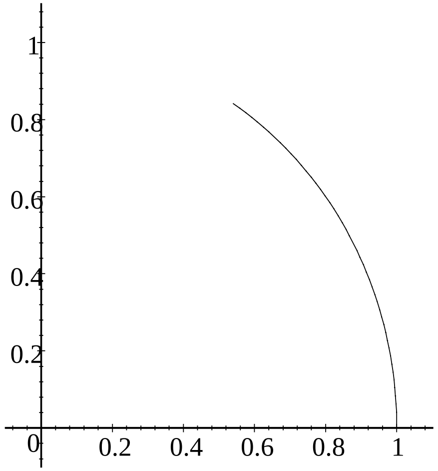
Approximate implicitization and intersection

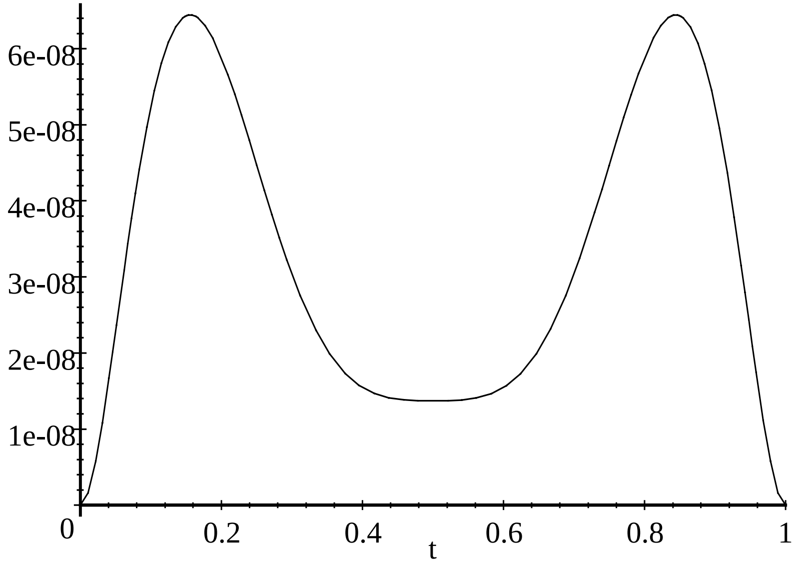
- Example separation of two near parallel not intersecting parametric surfaces $\mathbf{p}(s,t)$ and $\mathbf{r}(u,v)$ is difficult.
  - Approximation $\mathbf{r}(u,v)$ with a low order algebraic surface $q(x,y,z)=0$ of degree 3.
  - Make the combination $q(\mathbf{p}(s,t))$, this has degrees (9,9).
  - Now the geometry is flattened out, check for possible zeroes of $q(\mathbf{p}(s,t))$
- This approach has run successfully in industrial code for five years.



Error of 4th degree unit circle approximation opening angle 1

- Is within a tolerance of 10^{-7} a circle e.g. algebraic degree 2.
- However, the approximation has algebraic degree 4





Ray-tracing and approximative implicitization

- [Sederberg 99] showed that ray-tracing of Bezier surfaces using approximative implicitization is 2-3 times faster than using Bezier clipping methods.
- By approximating the Bezier surface with a monoid algebraic surface of degree 4, the ray will only intersect the approximation once in addition to the multiple intersection in the chosen monoid point.
- Thus, finding the intersection is just solving a fourth degree equation.

The approximative implicitization factorization

- [Dokken 97,01] Approximate Implicitization
- Assume that the surface $\mathbf{p}(s,t)$ has degree (n_1,n_2)
- Assume that q has total degree m and that $\mathbf{b}$ is a vector containing the unknown coefficients of q
- The combination $q(\mathbf{p}(s,t))$ is a polynomial of degrees (mn_1, mn_2)
- Collect basis functions of degree (mn_1,mn_2) in $\alpha(s,t)$
- Then $q(\mathbf{p}(s,t))$ can be factorized

$$q(\mathbf{p}(s,t)) = (\mathbf{D}\mathbf{b})^T \boldsymbol{\alpha}(s,t).$$

Motivation approximative implicitization

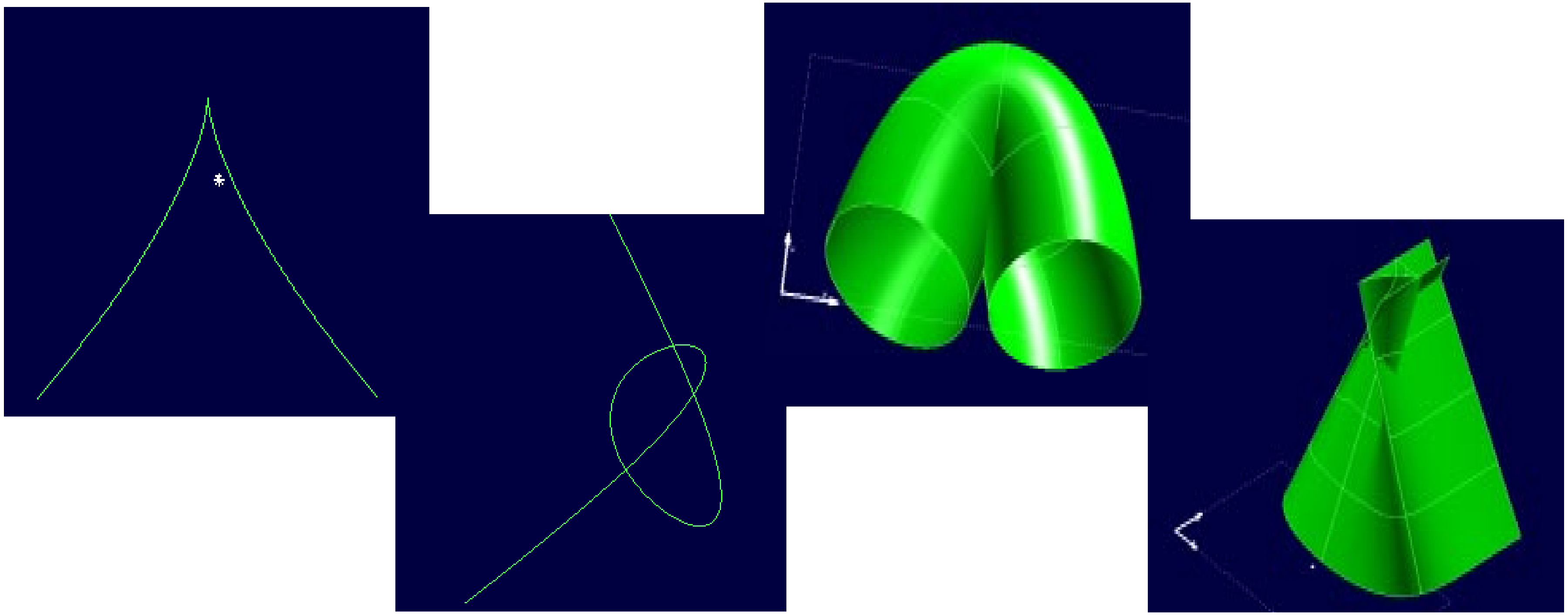
- CAD type application use floating point
- Methods based on exact arithmetic will, when implemented in floating point, in certain situations behave differently with minor changes in input data.
  - E.g. noise of relative size 10^{-16} change the classification from singular to nonsingular
- Methods that are based on approximation will allow controlled behavior:
  - Approximate and try to avoid singularities
  - Approximate and try to produce singularities

Initial idea for use of approximate implicitization

- Approximate with algebraic surfaces and make sure that singular points are far away from the region of interest.
- The number of singular points for a given algebraic degree is known from classical algebraic geometry.
- Alternative:
  - Give interpolation condition ensuring controlled behavior of singularities [Sederberg 99]
  - Choose solutions that have controlled behavior of gradients in region of interest

Idea in the EU GAIA project

- Use approximative implicitization to detect singularities is NURBS curves and surfaces.
  - Make approximations that aim at enhancing singular behavior.



The factorization

$$q(\mathbf{p}(s,t)) = (\mathbf{Db})^T \boldsymbol{\alpha}(s,t).$$

- The matrix **D** is built from products of the coefficients of **p**(s,t).
- An element in **D** is the product of a maximum of m such coefficients, with m the total degree of q.
- If **p**(s,t) was described in a Bernstein basis of degree (n_1, n_2) then $\alpha(s,t)$ contains the Bernstein basis of degree (mn_1, mn_2).
- The first step of moving curves and surfaces use the same factorization.

Properties of the factorization

$$q(\mathbf{p}(s,t)) = (\mathbf{Db})^T \boldsymbol{\alpha}(s,t).$$

- If **Db**=0 and **b≠0** then $q(\mathbf{p}(s,t))=0$ and **b** describes an implicitization q of $\mathbf{p}(s,t)$).
- If $\alpha(s,t)$ describes a Bernstein basis then

$$\|\boldsymbol{\alpha}(s,t)\|_2 \le 1.$$

- Then the following inequality is valid

$$|q(\mathbf{p}(s,\mathrm{t}))| = \left|(\mathbf{Db})^{\mathrm{T}} \boldsymbol{\alpha}(s,t)\right| \le \|\mathbf{Db}\|_2 .$$

- Also valid if $\alpha(s,t)$ is a partition of unity, thus including NURBS with positive weights.

Properties of the inequality

$$\left|q(\mathbf{p}(s,\mathrm{t}))\right| = \left|(\mathbf{Db})^{T}\boldsymbol{\alpha}(s,t)\right| \leq \|\mathbf{Db}\|_2.$$

- Let σ_1 be the smallest singular value of **D** then

$$\min_{\|\mathbf{b}\|_2=1} \max_{(s,t)\in\Omega} \left|q(\mathbf{p}(s,t))\right| \leq \sigma_1.$$

- Singular value decomposition (SVD) can be used to find approximative solutions of the implicitization problem.
- The singular values can be regarded as a criteria for sorting the approximations.

$y=x^3$ between (-1-1) and (1,1)

$$\mathbf{p}(s) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}(1-s)^3 + \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} 3(1-s)^2 s + \begin{pmatrix} \frac{1}{3} \\ -1 \end{pmatrix} 3(1-s)s^2 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s^3$$

- Implicit factorization

  $$q(\mathbf{p}(s)) = (\mathbf{Db})^T \boldsymbol{\alpha}(s).$$

- Singular values of **D**: 3.9, 3.5, 2.6, 1.9, 1.2 0.8, 0.69, 0.3, 0.28, 0
- Sequence of terms in algebraic expression and vector for 0 value

$$\mathbf{D} = \begin{pmatrix}
-1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-\frac{1}{3} & \frac{1}{9} & \frac{5}{9} & 1 & \frac{5}{9} & \frac{1}{9} & -\frac{1}{3} & -\frac{7}{9} & -\frac{1}{3} & 1 \\
0 & \frac{1}{9} & -\frac{2}{9} & -1 & \frac{2}{9} & -\frac{1}{9} & 0 & -\frac{5}{9} & 0 & 1 \\
\frac{2}{21} & -\frac{1}{21} & 0 & 1 & 0 & -\frac{1}{21} & \frac{2}{21} & -\frac{1}{3} & \frac{2}{21} & 1 \\
\frac{1}{21} & -\frac{1}{21} & \frac{1}{9} & -1 & -\frac{1}{9} & \frac{1}{21} & -\frac{1}{21} & -\frac{1}{9} & \frac{1}{21} & 1 \\
-\frac{1}{21} & \frac{1}{21} & -\frac{1}{9} & 1 & -\frac{1}{9} & \frac{1}{21} & -\frac{1}{21} & \frac{1}{9} & -\frac{1}{21} & 1 \\
-\frac{2}{21} & \frac{1}{21} & 0 & -1 & 0 & -\frac{1}{21} & \frac{2}{21} & \frac{1}{3} & -\frac{2}{21} & 1 \\
0 & -\frac{1}{9} & \frac{2}{9} & 1 & \frac{2}{9} & -\frac{1}{9} & 0 & \frac{5}{9} & 0 & 1 \\
\frac{1}{3} & -\frac{1}{9} & -\frac{5}{9} & -1 & \frac{5}{9} & \frac{1}{9} & -\frac{1}{3} & \frac{7}{9} & \frac{1}{3} & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

$$\begin{array}{cccccccccc}
x^3 & x^2 y & x y^2 & y^3 & x^2 & xy & y^2 & x & y & 1 \\
\sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{1}{2}} & 0
\end{array}$$

$$\sqrt{\frac{1}{2}}\, x^3 - \sqrt{\frac{1}{2}}\, y = 0$$

Example $\mathbf{p}(s) = (s, s^3)$

- Implicit factorization

$$q(\mathbf{p}(s)) = (\mathbf{Db})^T \boldsymbol{\alpha}(s).$$

- Singular values

$\sqrt{2}$, 1, 0.

- Sequence of terms in algebraic expression and vector for 0 value

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{cccccccccc} x^3 & x^2y & xy^2 & y^3 & x^2 & xy & y^2 & x & y & 1 \\ \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{1}{2}} & 0 \end{array}$$

$$\sqrt{\frac{1}{2}}\,x^3 - \sqrt{\frac{1}{2}}\,y = 0$$

Representing *q* by a Bernstein basis over a tetrahedral

- Let *q* be described in a Bernstein basis over a tetrahedral that contains $\mathbf{p}(s,t)$ (Also described in a Bernstein basis). Then
  - All entries in **D** are non-negative
  - The sum of entries in all rows of **D** is one
  - The Frobenius norm of **D** is limited by the number of rows
  - If proper algorithms are used for building **D** the relative rounding errors of **D** are limited by $m\varepsilon$; m the total degree of q, ε the relative rounding error of $\mathbf{p}(\mathbf{s})$

$$\varepsilon_{\mathbf{D}} \approx m\varepsilon^{\mathbf{p}}_{\max}.$$

  - Bernstein basis representation of algebraic curves was introduced by [Sederberg 84] in the paper “Planar piecewise algebraic curves.”

Constraining the approximation

- Constraints can be added imposing
  - Interpolation of points, curves,....
  - Tangent direction in points and along curves,...
  - Normal (gradient) in points and along curves,...
- Direct elimination to impose constraints.
- Direct elimination alternative to SVD for finding approximative null space.

Convergence Rate of Approximative Implicitization

- Curves in $\mathbb{R}^2$ with convergence $O(h^{\frac{(m+1)(m+2)}{2}-1})$.

Algebraic degree	1	2	3	4	5	6	7	8	9	10
Convergence rate	2	5	9	14	20	27	35	44	54	65

- Curves in $\mathbb{R}^3$ with convergence $O(h^{\frac{(m+1)(m+2)(m+3)}{6}-1})$.

Algebraic degree	1	2	3	4	5	6	7	8	9	10
Convergence rate	3	9	19	34	55	83	119	164	219	285

- Surfaces in $\mathbb{R}^3$ with $O(h^{\left\lfloor \frac{1}{6}\sqrt{(9+12m^2+72m^2+132m)}-\frac{1}{2} \right\rfloor})$.

Algebraic degree	1	2	3	4	5	6	7	8	9	10
Convergence rate	2	3	5	7	10	12	14	17	20	23

Accuracy and selection of solution

- Accuracy dependent on value and gradient of *q*

$$\rho(s,t)=\frac{q(\mathbf{p}(s,t))}{\nabla\mathbf{q}(\mathbf{p}(s,t)-\theta\mathbf{g}(s,t))\cdot\mathbf{g}(s,t)}.$$

- Find approximative null space of **D**.
- Select solution in the approximative null space of **D** by finding the maximal value of

$$\int_{\Omega}\int_{-\varepsilon}^{\varepsilon}(\nabla q(\mathbf{p}(s,t)-\theta\mathbf{g}(s,t)))\cdot\mathbf{g}(s)\,d\theta\,ds\,dt,$$

or an approximation there of.

Piecewise polynomials can be approximated

- Approximation of multiple manifolds

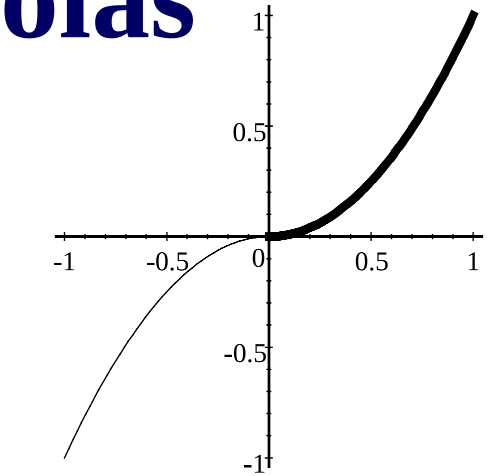
$$\sum_{i=1}^{r} \left(q(\mathbf{p}_i(s_i, t_i))\right)^2 \leq \left\| \begin{pmatrix} \mathbf{D}_1 \\ \vdots \\ \mathbf{D}_r \end{pmatrix} \mathbf{b} \right\|_2^2 .$$

- Separation of two manifolds by approximative implicitization of one of the manifolds.

Example two parabolas

$$\mathbf{p}_1(s) = (-1,-1)(1-s)^2 + \left(-\tfrac{1}{2},0\right)2(1-s)s + (0,0)s^2$$

$$\mathbf{p}_2(s) = (0,0)(1-s)^2 + \left(\tfrac{1}{2},0\right)2(1-s)s + (1,1)s^2$$



- We want to approximate both curve segments at the same time with one algebraic curve of degree 3.
- Thus we will make
  - $q(\mathbf{p}_1(s))=(\mathbf{D}_1\mathbf{b})\alpha_1(s)$
  - $q(\mathbf{p}_2(s))=(\mathbf{D}_2\mathbf{b})\alpha_2(s)$
- And combine the matrices $\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{pmatrix}$

The combined matrix

Contribution from $\mathbf{D}_1$

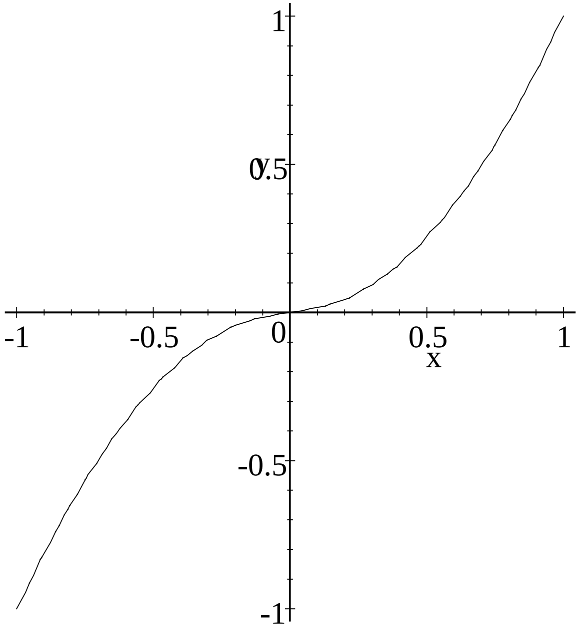
$$
\begin{pmatrix} \mathbf{D}_1 \\ \hline \mathbf{D}_2 \end{pmatrix} =
\left(\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{1}{15} & 0 & 0 & \frac{1}{3} & \frac{1}{15} & 1 \\
\frac{1}{20} & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{20} & 0 & \frac{1}{2} & \frac{1}{5} & 1 \\
\frac{1}{5} & \frac{1}{15} & 0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{1}{15} & \frac{2}{3} & \frac{2}{5} & 1 \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} & \frac{5}{6} & \frac{2}{3} & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
-1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-\frac{1}{2} & -\frac{1}{3} & -\frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} & -\frac{5}{6} & -\frac{2}{3} & 1 \\
-\frac{1}{5} & -\frac{1}{15} & 0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{1}{15} & -\frac{2}{3} & -\frac{2}{5} & 1 \\
-\frac{1}{20} & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{20} & 0 & -\frac{1}{2} & -\frac{1}{5} & 1 \\
0 & 0 & 0 & 0 & \frac{1}{15} & 0 & 0 & -\frac{1}{3} & -\frac{1}{15} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Contribution from $\mathbf{D}_2$

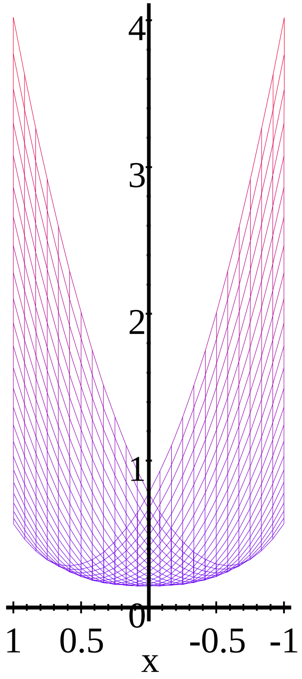
Implicit from combined D

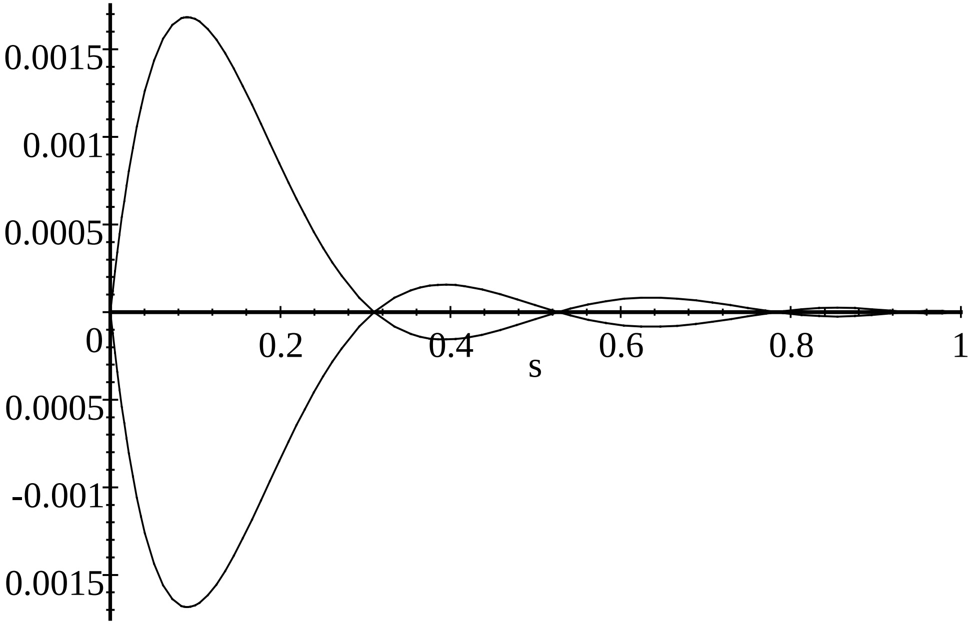
- Eigenvalues 4.25, 3.91, 1.98, 1.31, 0.38, 0.37, 0.11, 0.05, 0.03 and 0.007937. Combining eigenvector with basis functions and plot implicit

$$\begin{pmatrix} x^3 \\ x^2 y \\ x y^2 \\ y^3 \\ x^2 \\ xy \\ y^2 \\ x \\ y \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -.4602815334 \\ .6983314313 \\ -.5087222523 \\ .1433800204 \\ 0 \\ 0 \\ 0 \\ -.0170228764 \\ .1443197267 \\ 0 \end{pmatrix} = 0$$



Studying the quality of approximation





Length of gradient of q

Error of $\mathbf{p}_1(s)$ and $\mathbf{p}_2(s)$

Example parabolas in power basis

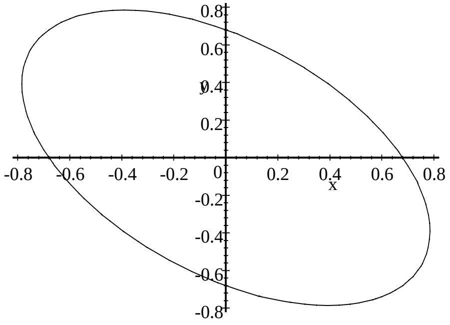
Contribution from $\mathbf{D}_1$

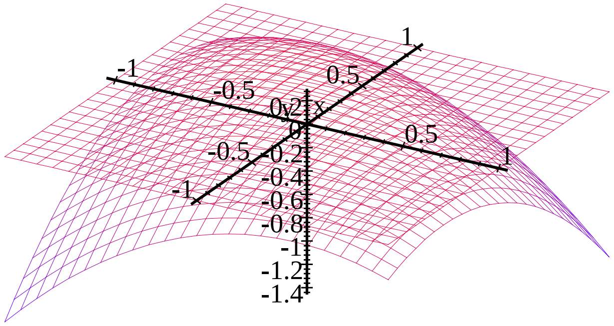
$$\left(\begin{array}{c} \mathbf{D}_1 \\ \hline \mathbf{D}_2 \end{array}\right) = \left(\begin{array}{cccccccccc}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

Contribution from $\mathbf{D}_2$

Implicit combined D power basis

- All eigenvalues larger than one $\{\sqrt{2}, \sqrt{6}+\sqrt{2}, \sqrt{6}-\sqrt{2}\}$
- No good approximation when power basis used for the basis functions of the parametric curves





q is very flat around the origin

Algebraic geometry within CAGD

- Knowledge from algebraic geometry has already contributed significantly to CAGD.
- To accelerate the use of algebraic geometry in CAGD a concentrated research effort is necessary.
- Intels IA-64 with multiple parallel instructions (up to 4 floating point operations per clock cycle) and large cache (4Mbyte L3 cache) will be introduce later in 2001. These are well suited for executing approximative algebraic algorithms used within CAGD.

References

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