Approximate Implicitization

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Two uses of approximate implicitization

- Part 1: Original approach - First part of talk
  - Make approximations that are non-singular.
  - Uses:
    - Separating curves/surfaces
    - Raytracing
- Part 2: GAIA approach - Second part of talk
  - Try to reproduce singularities
  - Uses:
    - Self-intersections of curves and surfaces
    - detect loops on 3D curves

Implicit Representation

[Sederberg 84] "Implicit Representation of Parametric Curves and Surfaces":

- A 2D rational parametric curve $\mathbf{p}(t)$ of degree n in the variable t can be expressed implicitly as a degree n polynomial $q(x,y)$, with $q(\mathbf{p}(t))=0$.
- A 3D rational parametric surface $\mathbf{p}(s,t)$ of degrees (n_1,n_2) in the variables (s,t) can be expressed implicitly as a degree $2n_1n_2$ polynomial $q(x,y,z)$, with $q(\mathbf{p}(s,t))=0$.

Intersection and implicitization

- Combining implicit and parametric descriptions in intersections algorithms reduce the number of variables, however, the polynomial degree increase
- Let $\mathbf{p}(s)$ and $\mathbf{r}(t)$ be two parametric curves, with implicit descriptions $q_{\mathbf{p}}(x,y)=0$, *and* $q_{\mathbf{r}}(x,y)=0$. Two alternative formulations of the intersection of $\mathbf{p}(s)$ and $\mathbf{r}(t)$ are
  - $\mathbf{f}(s,t) = \mathbf{p}(s)-\mathbf{r}(t) = 0$
  - $q_{\mathbf{p}}(\mathbf{r}(t))=0$ and $q_{\mathbf{r}}(\mathbf{p}(s))=0$

Ray-tracing and approximate implicitization

- [Sederberg 99] showed that ray-tracing of Bezier surfaces using approximate implicitization 2-3 times faster than using Bezier clipping methods.
- By approximating the Bezier surface with a monoid algebraic surface of degree 4, the ray will only intersect the approximation once in addition to the multiple intersection in the chosen monoid point.
- Thus, finding the intersection is just solving a fourth degree equation.

Exact implicitization

- Find nontrivial polynomial $q \in P_m(\mathbb{R}^l)$ such that
$$q(\mathbf{p}(\mathbf{s})) = 0, \quad \mathbf{s} \in \mathbb{R}^{l-1}$$
- For curves *l=2* for surface *l=3*
- Assumption:
  - Exact arithmetic
  - High polynomial degrees no problem

Resultants using Bernstein basis and floating point

- Recent work by Joab Winkler on resultants using the Bernstein basis show promising results.
  - |A resultant matrix for scaled Bernstein polynomials' and appeared in Linear Algebra and its Applications, vol. 319, pages 179-191, 2000.
- Using the Bernstein basis make resultant type methods better suited for floating point arithmetic.

Exact implicitization methods

- Resultant based methods
  - Express the implicit equation using determinants, in general the determinants for surfaces are large
- Gröbner bases methods
  - Symbolic computations on polynomials
- Moving curves and surfaces [Sederberg 95,97]. Proof of moving surfaces [Cox 99]
  - Reduced the size of the determinates by a factor of 4
  - Base points reduce the size of the determinant

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Approximate Implicitization

- Let $\mathbf{p}(s,t)$, $(s,t) \in \Omega \subset \mathbb{R}^2$ be a surface in $\mathbb{R}^3$
- Find nontrivial $q \in P_m(\mathbb{R}^3)$ such that

$$q(\mathbf{p}(s,t) + \eta(s,t)\mathbf{g}(s,t)) = 0, \quad (s,t) \in \Omega$$

- with direction for error measurement $\|\mathbf{g}(s,t)\|_2 = 1.$ and error

$$|\eta(s,t)| \leq \varepsilon, \quad (s,t) \in \Omega.$$

- Assumption
  – Floating point can be used
  – Degrees can be kept low

The approximate implicitization factorization

- [Dokken 1997,2001] Approximate Implicitization
- Assume that the surface **p**(s,t) has degree (n_1,n_2)
- Assume that q has total degree m and that **b** is a vector containing the unknown coefficients of q
- The combination $q(\mathbf{p}(s,t))$ is a polynomial of degrees (mn_1, mn_2)
- Collect basis functions of degree (mn_1,mn_2) in $\alpha(s,t)$
- Then $q(\mathbf{p}(s,t))$ can be factorized

$$q(\mathbf{p}(s,t)) = (\mathbf{Db})^T \boldsymbol{\alpha}(s,t).$$

Motivation approximate implicitization

- CAD type application use floating point arithmetic
- Methods based on exact arithmetic will, when implemented in floating point, in certain situations behave differently with minor changes in input data.
  - E.g. noise of relative size 10^{-16} change the classification from singular to nonsingular
- Methods that are based on approximation will allow controlled behavior:
  - Approximate and try to avoid singularities
  - Approximate and try to produce singularities

Initial idea for use of approximate implicitization

- Approximate with algebraic surfaces and make sure that singular points are far away from the region of interest.
- The number of singular points for a given algebraic degree is known from classical algebraic geometry.
- Alternative:
  - Give interpolation condition ensuring controlled behavior of singularities [Sederberg 99]
  - Choose solutions that have controlled behavior of gradients in region of interest

Self-intersection and implicit representation

- In GAIA we use approximate implicit representations to detect self-intersections
- Let $\mathbf{p}(s)$ be a parametric polynomial curve and let $q(x,y)=0$ be the exact implicit representation of $\mathbf{p}(s)$. Let $\mathbf{n}(s)$ be the normal of $\mathbf{p}(s)$.
  - If $\nabla q(\mathbf{p}(s)) \cdot \mathbf{n}(s)$ change sign then the curve has a self-intersection
  - If $\nabla q(\mathbf{p}(s)) \cdot \mathbf{n}(s)=0$ then the curve has a cusp

Properties of the factorization

$$q(\mathbf{p}(s,t)) = (\mathbf{Db})^T \boldsymbol{\alpha}(s,t).$$

- If $\mathbf{Db}=0$ and $\mathbf{b}\neq\mathbf{0}$ then $q(\mathbf{p}(s,t))=0$ and $\mathbf{b}$ describes an implicitization q of $\mathbf{p}(s,t)$).
- If $\alpha(s,t)$ describes a Bernstein basis then
$$\|\boldsymbol{\alpha}(s,t)\|_2 \leq 1.$$
- Then the following inequality is valid
$$|q(\mathbf{p}(s,\mathrm{t}))| = \left|(\mathbf{Db})^{\mathrm{T}} \boldsymbol{\alpha}(s,t)\right| \leq \|\mathbf{Db}\|_2 .$$

The factorization

$$q(\mathbf{p}(s,t)) = (\mathbf{Db})^T \boldsymbol{\alpha}(s,t).$$

- The matrix **D** is built from products of the coefficients of $\mathbf{p}(s,t)$.
- An element in **D** is the product of a maximum of m such coefficients, where m the total degree of q.
- If $\mathbf{p}(s,t)$ was described in a Bernstein basis of degree (n_1, n_2) then $\alpha(s,t)$ contains the Bernstein basis of degree (mn_1, mn_2).
- The first step of moving curves and surfaces use the same factorization.

Properties of the inequality

$$\left|q(\mathbf{p}(s,\mathrm{t}))\right| = \left|(\mathbf{Db})^{\mathrm{T}}\boldsymbol{\alpha}(s,t)\right| \le \|\mathbf{Db}\|_2 .$$

- Let σ_1 be the smallest singular value of **D** then

$$\min_{\|\mathbf{b}\|_2=1} \max_{(s,t)\in\Omega} \left|q(\mathbf{p}(s,t))\right| \le \sigma_1 .$$

- Singular value decomposition (SVD) can be used to find approximate solutions of the implicitization problem.

$y=x^3$ between (-1-1) and (1,1)

$$\mathbf{p}(s) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}(1-s)^3 + \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} 3(1-s)^2 s + \begin{pmatrix} \frac{1}{3} \\ -1 \end{pmatrix} 3(1-s)s^2 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s^3$$

- Implicit factorization

$$q(\mathbf{p}(s)) = (\mathbf{Db})^T \boldsymbol{\alpha}(s).$$

- Singular values of **D**: 3.9, 3.5, 2.6, 1.9, 1.2 0.8, 0.69, 0.3, 0.28, 0
- Sequence of terms in algebraic expression and vector for 0 value

$$\mathbf{D} = \begin{pmatrix}
-1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-\frac{1}{3} & \frac{1}{9} & \frac{5}{9} & 1 & \frac{5}{9} & \frac{1}{9} & -\frac{1}{3} & -\frac{7}{9} & -\frac{1}{3} & 1 \\
0 & \frac{1}{9} & -\frac{2}{9} & -1 & \frac{2}{9} & -\frac{1}{9} & 0 & -\frac{5}{9} & 0 & 1 \\
\frac{2}{21} & -\frac{1}{21} & 0 & 1 & 0 & -\frac{1}{21} & \frac{2}{21} & -\frac{1}{3} & \frac{2}{21} & 1 \\
\frac{1}{21} & -\frac{1}{21} & \frac{1}{9} & -1 & -\frac{1}{9} & \frac{1}{21} & -\frac{1}{21} & -\frac{1}{9} & \frac{1}{21} & 1 \\
-\frac{1}{21} & \frac{1}{21} & -\frac{1}{9} & 1 & -\frac{1}{9} & \frac{1}{21} & -\frac{1}{21} & \frac{1}{9} & -\frac{1}{21} & 1 \\
-\frac{2}{21} & \frac{1}{21} & 0 & -1 & 0 & -\frac{1}{21} & \frac{2}{21} & \frac{1}{3} & -\frac{2}{21} & 1 \\
0 & -\frac{1}{9} & \frac{2}{9} & 1 & \frac{2}{9} & -\frac{1}{9} & 0 & \frac{5}{9} & 0 & 1 \\
\frac{1}{3} & -\frac{1}{9} & -\frac{5}{9} & -1 & \frac{5}{9} & \frac{1}{9} & -\frac{1}{3} & \frac{7}{9} & \frac{1}{3} & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

x^3	x^2y	xy^2	y^3	x^2	xy	y^2	x	y	1
$\sqrt{\frac{1}{2}}$	0	0	0	0	0	0	0	$-\sqrt{\frac{1}{2}}$	0

$$\sqrt{\frac{1}{2}}\,x^3 - \sqrt{\frac{1}{2}}\,y = 0$$

Example p(s) = (s,s³)

- Implicit factorization

$$q(\mathbf{p}(s)) = (\mathbf{Db})^T \boldsymbol{\alpha}(s).$$

- Singular values

$\sqrt{2}$, 1, 0.

- Sequence of terms in algebraic expression and vector for 0 value

$$\mathbf{D} = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

x^3	x^2y	xy^2	y^3	x^2	xy	y^2	x	y	1
$\sqrt{\frac{1}{2}}$	0	0	0	0	0	0	0	$-\sqrt{\frac{1}{2}}$	0

$$\sqrt{\tfrac{1}{2}}\,x^3 - \sqrt{\tfrac{1}{2}}\,y = 0$$

Representing *q* by a Bernstein basis over a tetrahedral

- Let q be described in a Bernstein basis over a tetrahedral that contains $\mathbf{p}(s,t)$ (Also described in a Bernstein basis). Then
  - All entries in **D** are non-negative
  - The sum of entries in all rows of **D** is one
  - The Frobenius norm of **D** is limited by the number of rows
  - If proper algorithms are used for building **D** the relative rounding errors of **D** are limited by $m\varepsilon$; m the total degree of q, ε the relative rounding error of $\mathbf{p}(\mathbf{s})$

$$\varepsilon_{\mathbf{D}} \approx m\varepsilon_{\max}^{\mathbf{p}}.$$

- Bernstein basis representation of algebraic curves was introduced by [Sederberg 84] in the paper "Planar piecewise algebraic curves."

Constraining the approximation

- Constraints can be added imposing
  - Interpolation of points, curves,....
  - Tangent direction in points and along curves,...
  - Normal (gradient) in points and along curves,...
- Direct elimination to impose constraints.
- Direct elimination alternative to SVD for finding approximate null space.

Convergence Rate of approximate Implicitization

- Curves in $\mathbb{R}^2$ with convergence $O(h^{\frac{(m+1)(m+2)}{2}-1})$.

Algebraic degree	1	2	3	4	5	6	7	8	9	10
Convergence rate	2	5	9	14	20	27	35	44	54	65

- Curves in $\mathbb{R}^3$ with convergence $O(h^{\frac{(m+1)(m+2)(m+3)}{6}-1})$.

Algebraic degree	1	2	3	4	5	6	7	8	9	10
Convergence rate	3	9	19	34	55	83	119	164	219	285

- Surfaces in $\mathbb{R}^3$ with $O(h^{\lfloor \frac{1}{6}\sqrt{(9+12m^2+72m^2+132m)}-\frac{1}{2} \rfloor})$.

Algebraic degree	1	2	3	4	5	6	7	8	9	10
Convergence rate	2	3	5	7	10	12	14	17	20	23

Accuracy and selection of solution

- Accuracy dependent on value and gradient of q

$$\rho(s,t) = \frac{q(\mathbf{p}(s,t))}{\nabla \mathbf{q}(\mathbf{p}(s,t) - \theta \mathbf{g}(s,t)) \cdot \mathbf{g}(s,t)}.$$

- Find approximate null space of **D**.
- Select solution in the approximate null space of **D** by finding the maximal value of

$$\int_{\Omega} \int_{-\varepsilon}^{\varepsilon} (\nabla q(\mathbf{p}(s,t) - \theta \mathbf{g}(s,t))) \cdot \mathbf{g}(s) \, d\theta \, ds \, dt,$$

or an approximation there of.

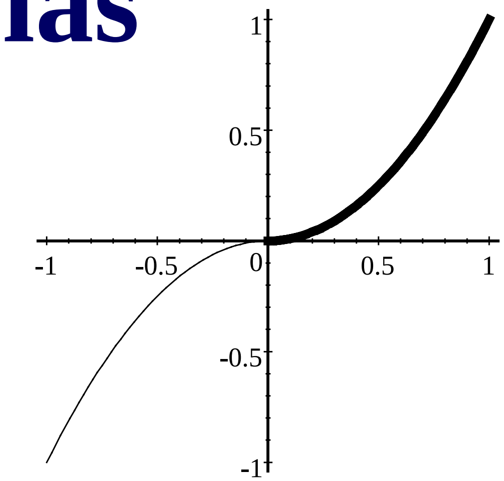
Piecewise polynomials can be approximated

- Approximation of multiple manifolds

$$\sum_{i=1}^{r} \left(q(\mathbf{p}_i(s_i, t_i))\right)^2 \leq \left\| \begin{pmatrix} \mathbf{D}_1 \\ \vdots \\ \mathbf{D}_r \end{pmatrix} \mathbf{b} \right\|_2^2 .$$

- Separation of two manifolds by approximate implicitization of one of the manifolds.

Example two parabolas



$$\mathbf{p}_1(s) = (-1,-1)(1-s)^2 + \left(-\tfrac{1}{2}, 0\right)2(1-s)s + (0,0)s^2$$

$$\mathbf{p}_2(s) = (0,0)(1-s)^2 + \left(\tfrac{1}{2}, 0\right)2(1-s)s + (1,1)s^2$$

- We want to approximate both curve segments at the same time with one algebraic curve of degree 3.
- Thus we will make
  - $q(\mathbf{p}_1(s)) = (\mathbf{D}_1\mathbf{b})\alpha_1(s)$
  - $q(\mathbf{p}_2(s)) = (\mathbf{D}_2\mathbf{b})\alpha_2(s)$
- And combine the matrices: $\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{pmatrix}$

The combined matrix

Contribution from $\mathbf{D}_1$

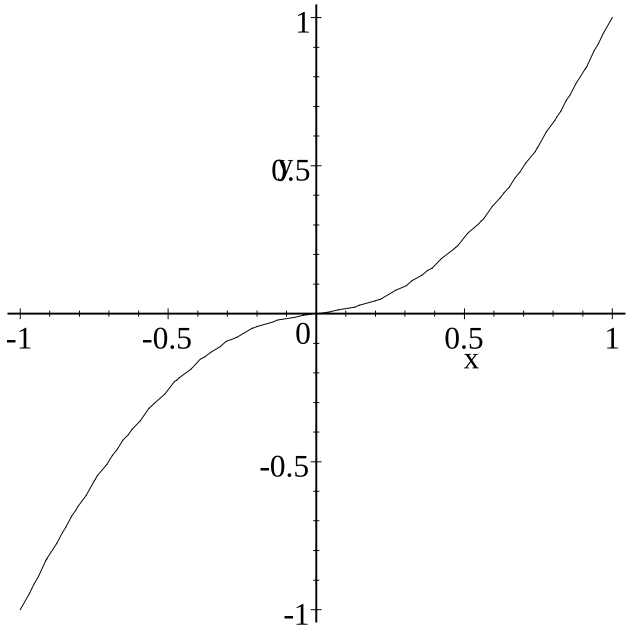
$$\begin{pmatrix} \mathbf{D}_1 \\ \hline \mathbf{D}_2 \end{pmatrix} = \left(\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{1}{15} & 0 & 0 & \frac{1}{3} & \frac{1}{15} & 1 \\
\frac{1}{20} & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{20} & 0 & \frac{1}{2} & \frac{1}{5} & 1 \\
\frac{1}{5} & \frac{1}{15} & 0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{1}{15} & \frac{2}{3} & \frac{2}{5} & 1 \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} & \frac{5}{6} & \frac{2}{3} & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
-1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-\frac{1}{2} & -\frac{1}{3} & -\frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} & -\frac{5}{6} & -\frac{2}{3} & 1 \\
-\frac{1}{5} & -\frac{1}{15} & 0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{1}{15} & -\frac{2}{3} & -\frac{2}{5} & 1 \\
-\frac{1}{20} & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{20} & 0 & -\frac{1}{2} & -\frac{1}{5} & 1 \\
0 & 0 & 0 & 0 & \frac{1}{15} & 0 & 0 & -\frac{1}{3} & -\frac{1}{15} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

Contribution from $\mathbf{D}_2$

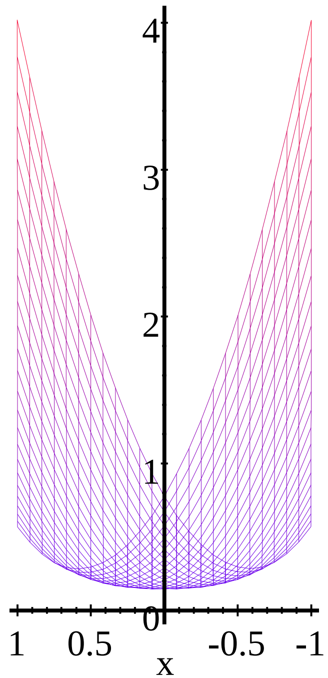
Implicit from combined D

- Eigenvalues 4.25, 3.91, 1.98, 1.31, 0.38, 0.37, 0.11, 0.05, 0.03 and 0.007937. Combining eigenvector with basis functions and plot implicit

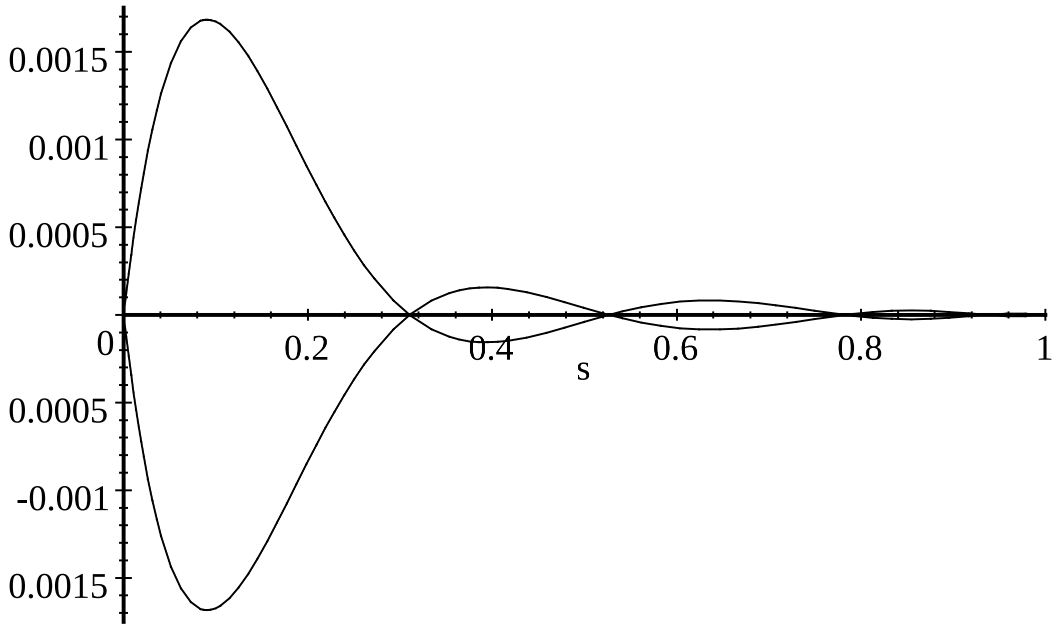
$$\begin{pmatrix} x^3 \\ x^2 y \\ xy^2 \\ y^3 \\ x^2 \\ xy \\ y^2 \\ x \\ y \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -.4602815334 \\ .6983314313 \\ -.5087222523 \\ .1433800204 \\ 0 \\ 0 \\ 0 \\ -.0170228764 \\ .1443197267 \\ 0 \end{pmatrix} = 0$$



Studying the quality of approximation



Length of gradient of q



Error of $\mathbf{p}_1(s)$ and $\mathbf{p}_2(s)$

Example parabolas in power basis

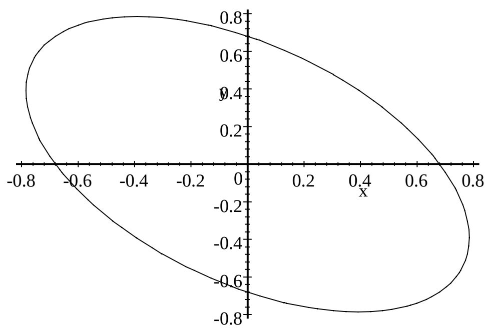
Contribution from $\mathbf{D}_1$

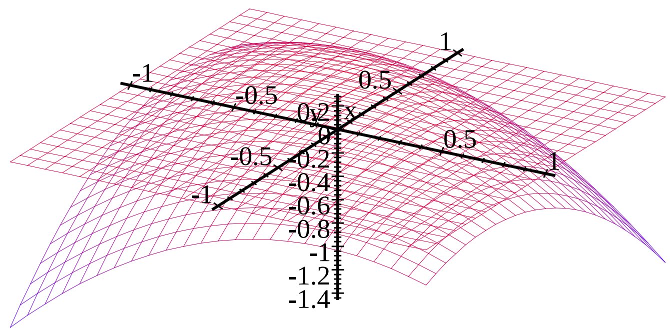
$$\left(\frac{\mathbf{D}_1}{\mathbf{D}_2}\right) = \left(\begin{array}{cccccccccc}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

Contribution from $\mathbf{D}_2$

Implicit combined D power basis

- All eigenvalues larger than one: $\{\sqrt{2}, \sqrt{6}+\sqrt{2}, \sqrt{6}-\sqrt{2}\}$
- No good approximation when power basis used for the basis functions of the parametric curves

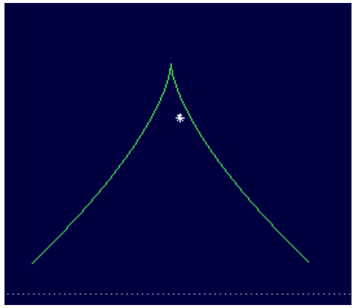
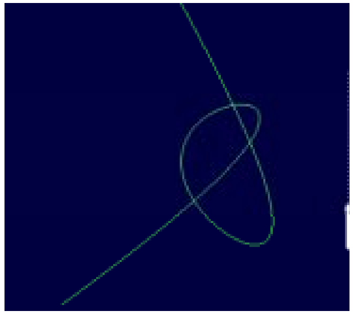
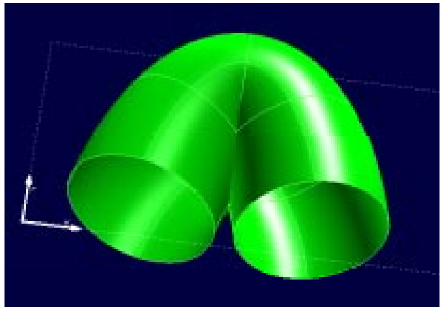
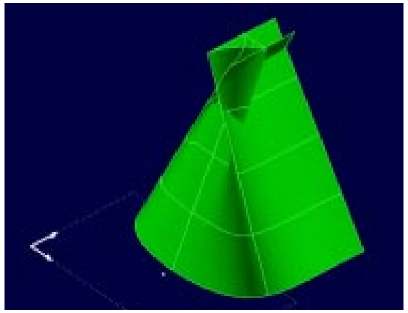




q is very flat around the origin

Idea in the EU GAIA project

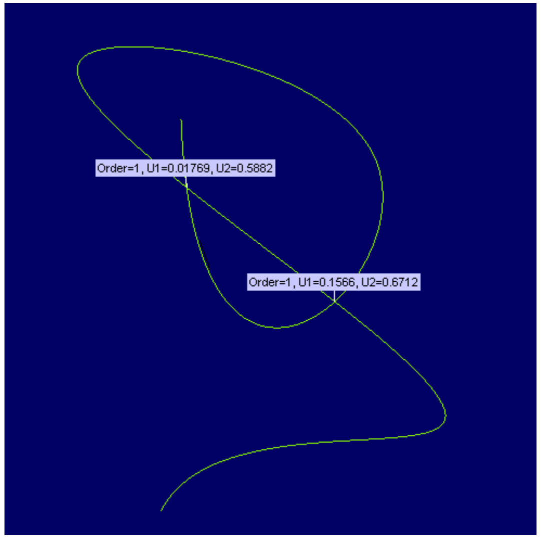
- Use approximate implicitization to detect singularities is NURBS curves and surfaces.
  - Make approximations that aim at enhancing singular behavior.

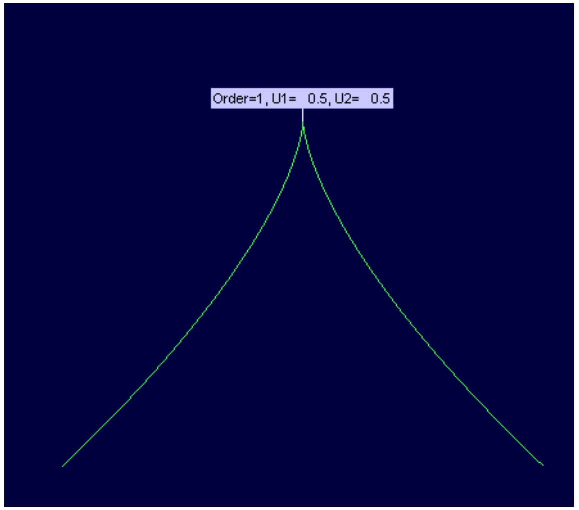
   

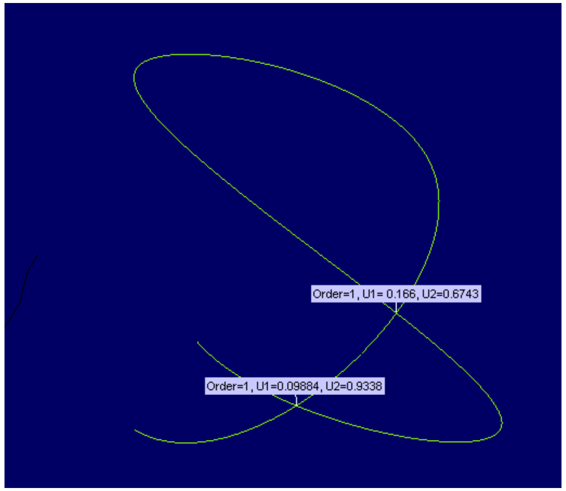
Self-intersection and approximate implicitization

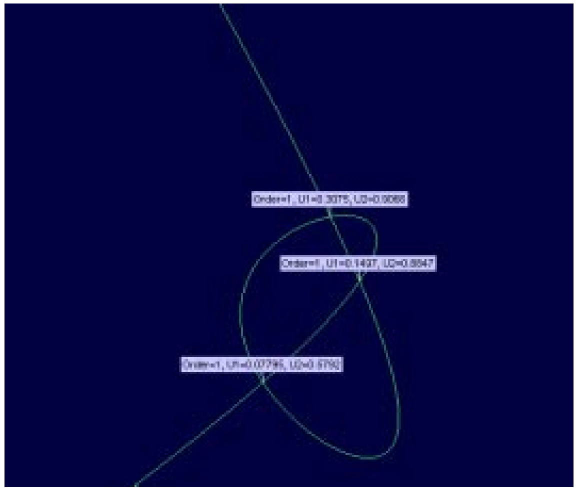
- **Let** $q(x,y)=0$ **be an approximation to an interval of** $\mathbf{p}(s)$**.**
  - If $\nabla q(\mathbf{p}(s))\cdot\, \mathbf{n}(s)$ change sign then the curve has a possible self-intersection close to the sign change
  - If $\nabla q(\mathbf{p}(s))\cdot\, \mathbf{n}(s)=0$ is “small” then the curve has a possible cusp close to the sign change
- **For curves we have used the exact algebraic degree**
- **For surfaces we have used degree 4**

Example curve self-intersection

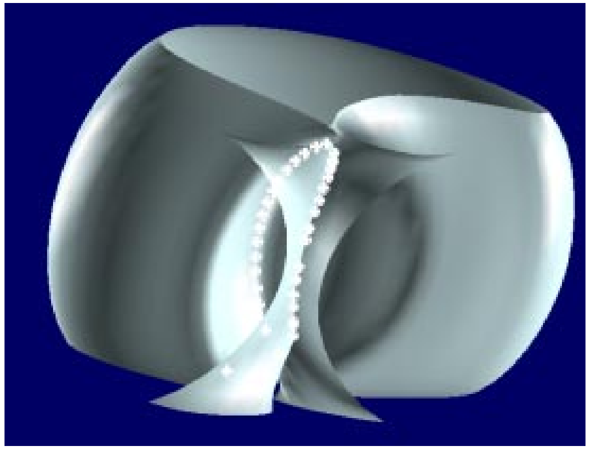


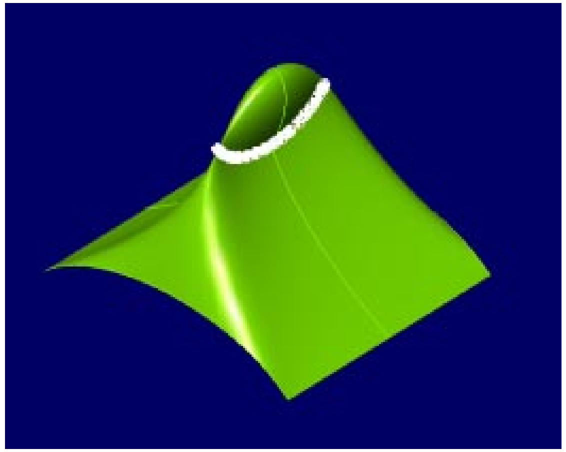


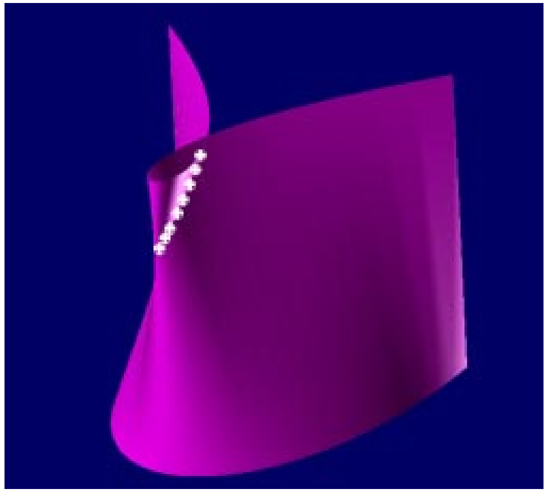


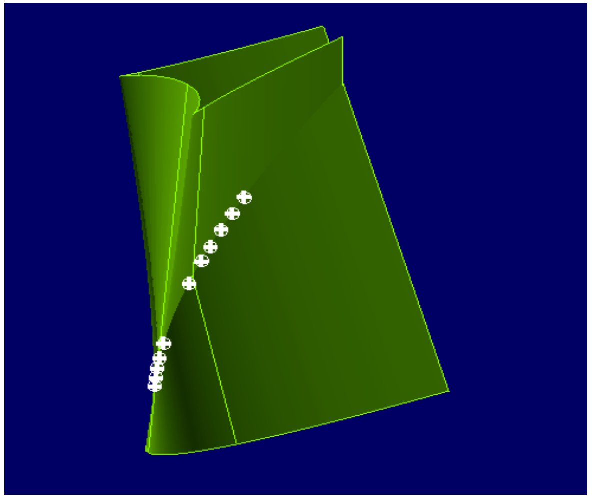


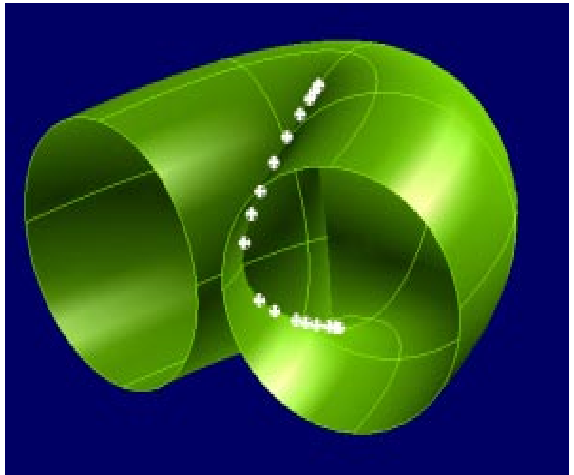
Example surface self-intersection



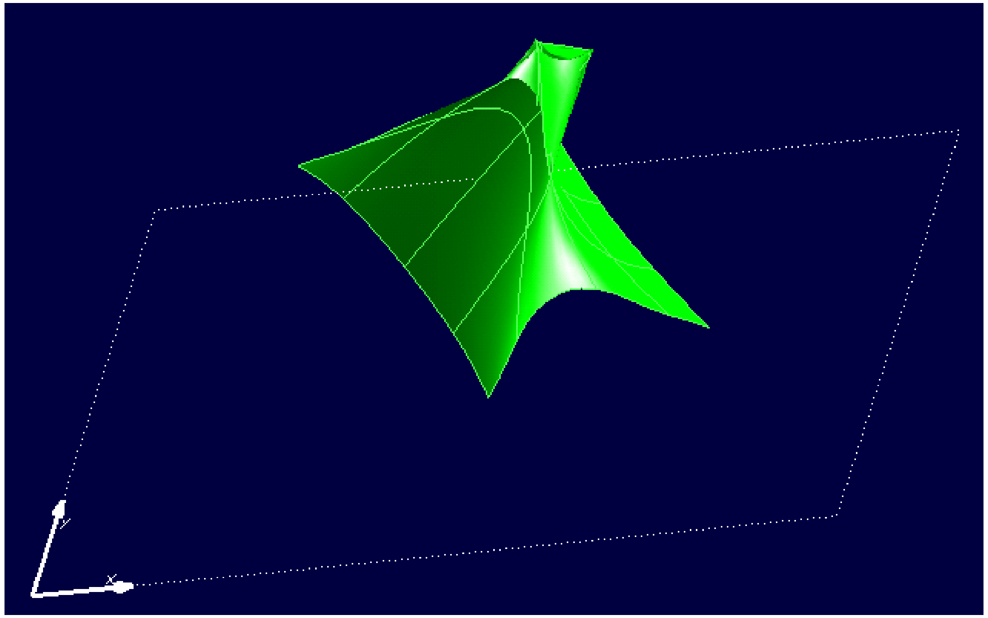


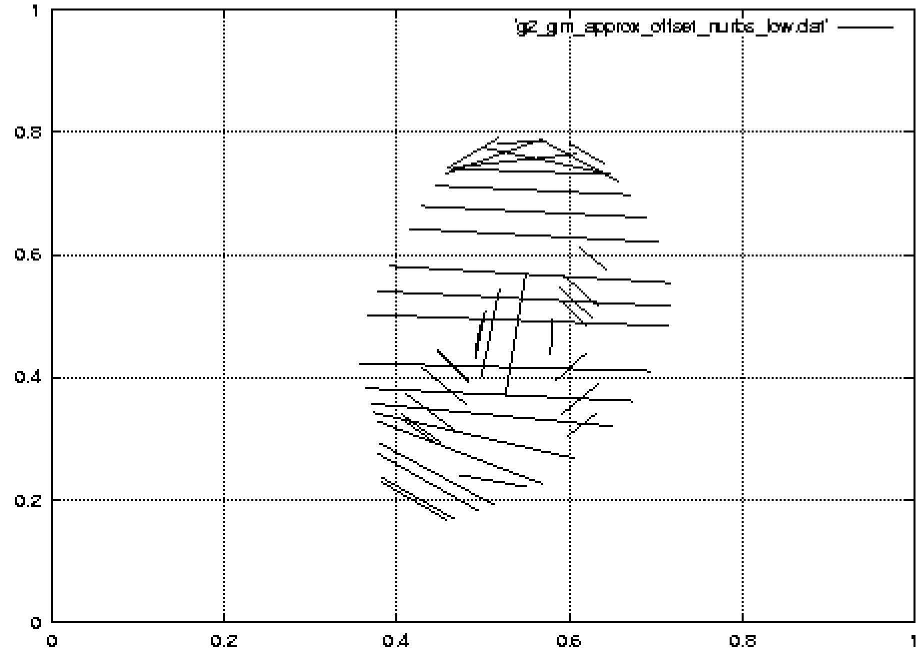






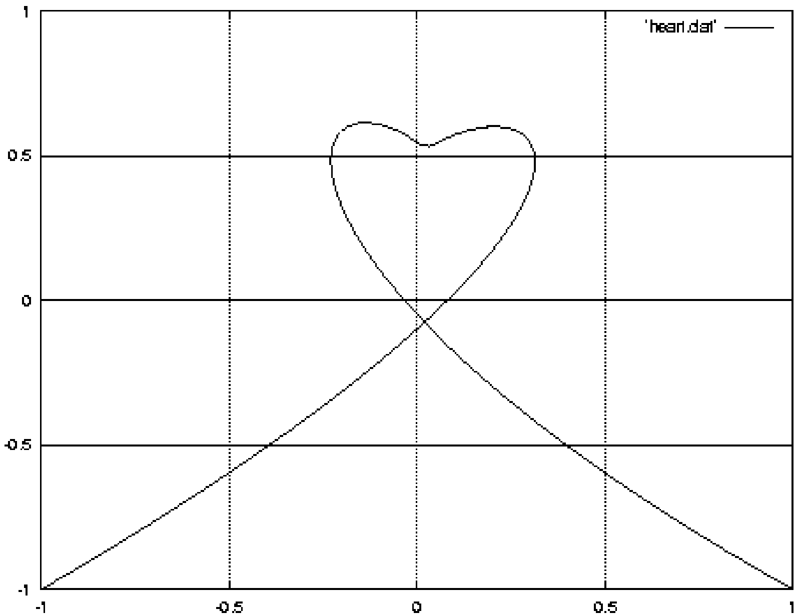
Example self-intersection

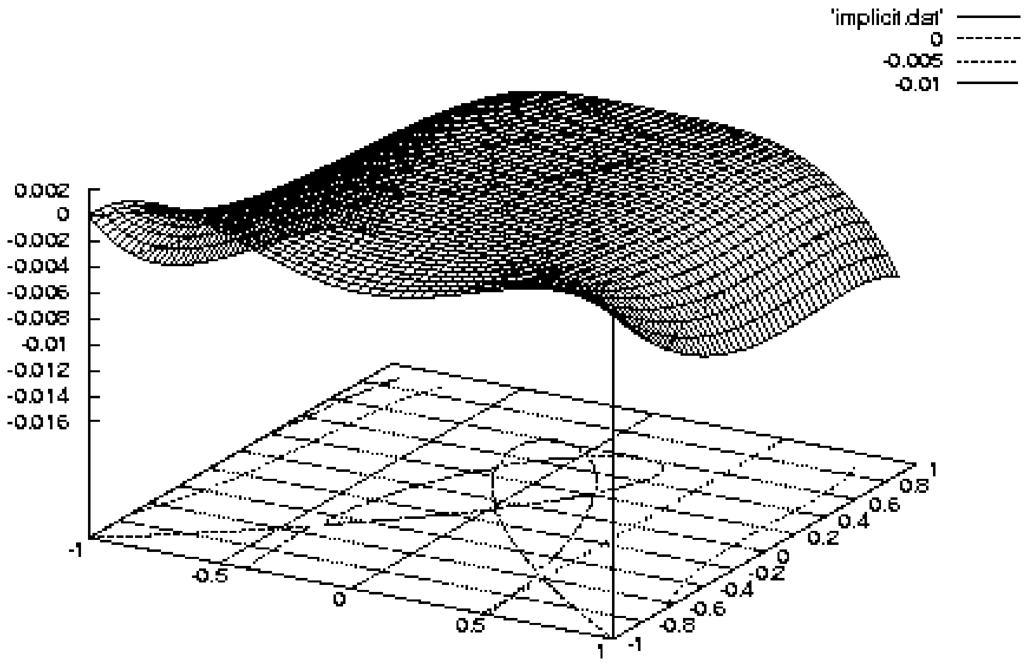




– A selfintersection with many branches. The lines connects points that have the same 3D coordinates.

Two third degree curve segments as degree 6 implicit





- Approximation of two Bezier curves joined with C^0 continuity

Other activities

- An 1-year EU-supported preproject within Future and Emerging Technologies (FET) will started in October 2000 assessing approximate implicitization used on intersection for CAD
  - France: University of Nice, Think3
  - Norway: University of Oslo and SINTEF
- Based on the FET preproject we aim at establishing an EU RTD-project with broad European participation

Algebraic geometry within CAGD

- Knowledge from algebraic geometry has already contributed significantly to CAGD.
- To accelerate the use of algebraic geometry in CAGD a concentrated research effort is necessary.
- Intels IA-64 with multiple parallel instructions (up to 4 floating point operations per clock cycle) and large cache (4Mbyte L3 cache) will be introduce later in 2001. These are well suited for executing approximate algebraic algorithms used within CAGD.