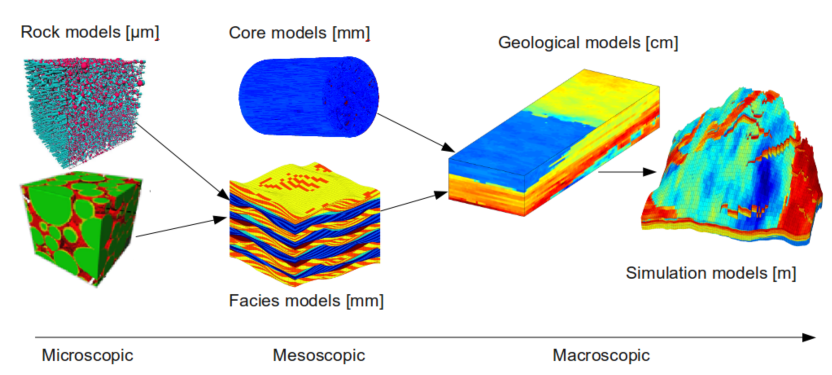
Multiscale Mixed Finite-Element Methods

SINTEF
Knut–Andreas Lie
Department of Applied Mathematics, SINTEF ICT, Oslo, Norway

Motivation

Flow and transport in subsurface rocks are multiscale phenomena that involve a large range of physical scales. Upscaling is therefore inevitable in reservoir modeling. However, upscaling is either a manual and *time-consuming* process, or when automated, not sufficiently robust.



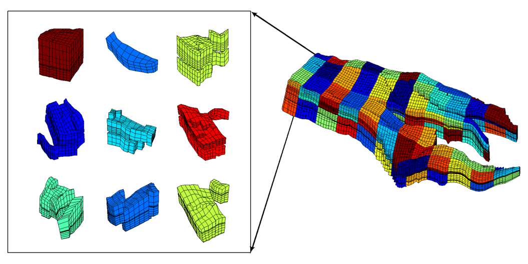
Multiscale Methods

- Methods that incorporate fine-scale information into a set of coarse-scale equations in a way which is consistent with the local property of the differential operator.
- Multiscale pressure solvers perform up/down-scaling in a single step and provide both coarse-scale and fine-scale resolution.
- By clever reuse of computations, these methods promise a significant speedup and can enable simulation on grids with seismic/geological resolution. Advantages:
  - faster model building and history matching,
  - better estimation of uncertainty by running alternative models,
  - makes inversion a better instrument to find remaining oil,

Visual Review of Key Ideas

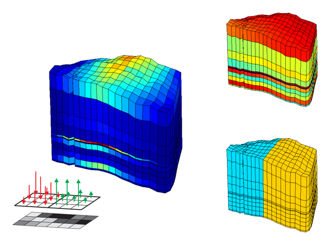
1. Introduce a coarse grid in which each block consists of a connected collection of cells from the underlying fine grid.

For corner-point grids, the simplest and yet most effective approach is to partition the input grid in IJK space. This will often give blocks with highly irregular geometries, but will preserve geological structures, simplify the coupling in the linear system, and enable automatic upgridding.



2. Postprocess to ensure all blocks are connected and detect all connections between adjacent gridblocks.

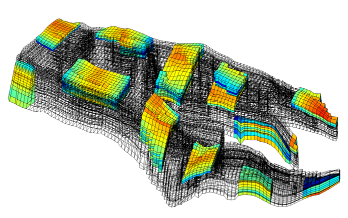
3. Compute local flow problems:
   - One for each pair of gridblocks using no-flow boundary conditions, or
   - one for each coarse-block interface with prescribed interface flow.



Almost the same as for flow-based upscaling.

4. Collect local solutions as basis functions

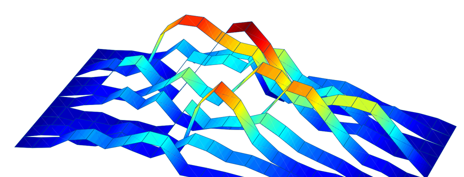
Use these to construct a global solution from a mixed finite-element formulation. The figure shows a few basis functions for a Norwegian Sea field.



Mathematical Background

Consider (for simplicity) incompressible single-phase flow

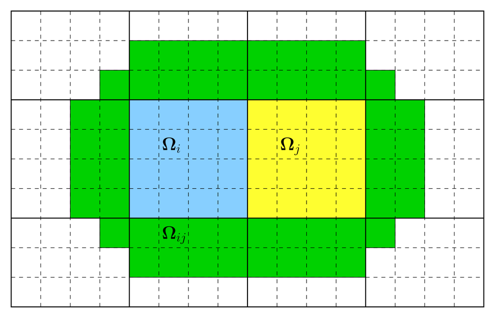
$$\nabla \cdot \vec{u} = 0, \qquad \vec{u} = -\mathbf{K}\nabla p$$



Describe the solution as a sum of multiscale basis functions $\vec{u} = \sum u_{ij}\vec{\psi}_{ij}$. Each basis function is determined by solving a localized flow problem, e.g.,

$$\vec{\psi}_{ij} = -\mathbf{K}\nabla\phi_{ij}, \quad \nabla\cdot\vec{\psi}_{ij} = \begin{cases} w_i(x), & x \in \Omega_i \\ -w_j(x), & x \in \Omega_j \\ 0, & \text{otherwise} \end{cases}$$

with no-flow boundary conditions. The source term $w(x) > 0$ is normalized over Ω_i and drives a unit flow from Ω_i to Ω_j. (For homogeneous, isotropic $\mathbf{K}$ and rectangular blocks with no overlap, this construction reproduces the standard RT0 basis.)



Ω_i and Ω_j: two gridblocks that are connected via a single interface in the coarse grid,
Ω_{ij} a single-connected domain that contains the two

Linear Algebra

Fine-scale system (mixed/mimetic):

$$\begin{bmatrix} \mathbf{B} & \mathbf{C} & \mathbf{D} \\ \mathbf{C}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{D}^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ -\mathbf{p} \\ \pi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{B}$: inner product of velocity basis functions
$\mathbf{C}$: integral of the divergence of velociy b.f.
$\mathbf{D}$: map from local to global face numbering

Coarse-scale system (algebraic reduction):

$$\begin{bmatrix} \mathbf{\Psi}^T\mathbf{B}\mathbf{\Psi} & \mathbf{\Psi}^T\mathbf{C}\mathcal{I} & \mathbf{\Psi}^T\mathbf{D}\mathcal{J} \\ \mathcal{I}^T\mathbf{C}^T\mathbf{\Psi} & 0 & 0 \\ \mathcal{J}^T\mathbf{D}^T\mathbf{\Psi} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^c \\ -\mathbf{p}^c \\ \pi^c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{\Psi}$: matrix containing multiscale basis functions
$\mathcal{I}$: prolongation from coarse blocks to cells in fine grid
$\mathcal{J}$: prolongation from block faces to cell faces

The indefinite systems can be made symmetric positive-definite by using a standard Schur complement technique ($\mathbf{u}$ = flux, $\mathbf{p}$=cell pressures, π=face pressures).

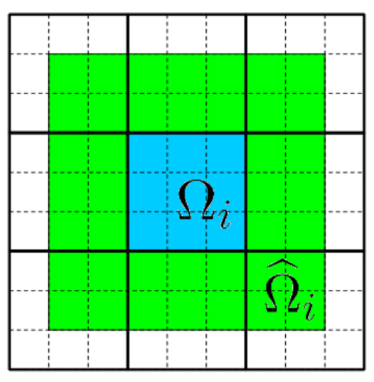
Parabolic Problems (Compressibility)

In the multiscale finite-volume framework, compressibility is handled by introducing a set of correction functions. For MsMFEM, we propose a residual formulation, in which the elliptic multiscale basis functions act as a predictor and a parabolic correction is computed using a standard domain-decomposition method.

- Construct elliptic basis functions initially
- Residual formulation of linearized flow equations:

$$\begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{P} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\rm ms} + \hat{\mathbf{u}}^{\nu+1} \\ \mathbf{p}_{\rm ms} + \hat{\mathbf{p}}^{\nu+1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{P}\mathbf{p}^n + \mathbf{V}\mathbf{u}^\nu \end{bmatrix}$$

- (Non)overlapping Scharz method with localization:
  - zero right-hand-side in $\widehat{\Omega}_i \setminus \Omega_i$
  - zero flux BCs on $\partial\widehat{\Omega}_i$
- Iterate on multiscale and residuals until convergence

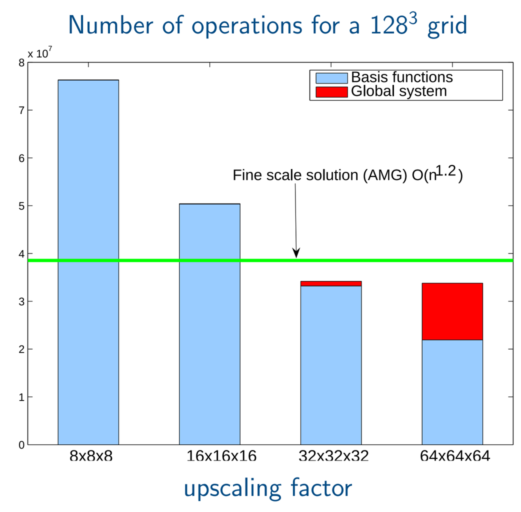


Ω_i: coarse grid block
$\widehat{\Omega}_i$: overlapping subdomains in DD formulation

Examples: Efficiency

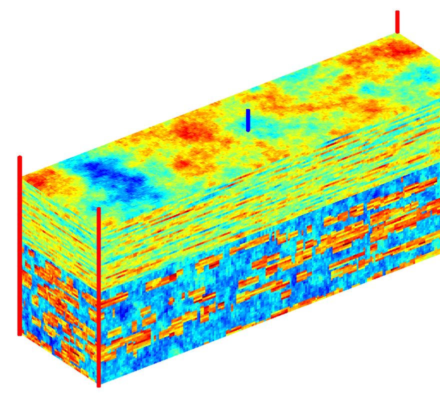
- Reuse of basis functions for time-dependent problems: compute them initially and update infrequently, e.g., only for large changes in total mobility in a block.
- Parallelization: all basis functions can be computed independently

Theoretical Number of Operations



The plot shows that once the basis functions have been computed, the cost of each new pressure solution is relatively low.

Model 2 from SPE10

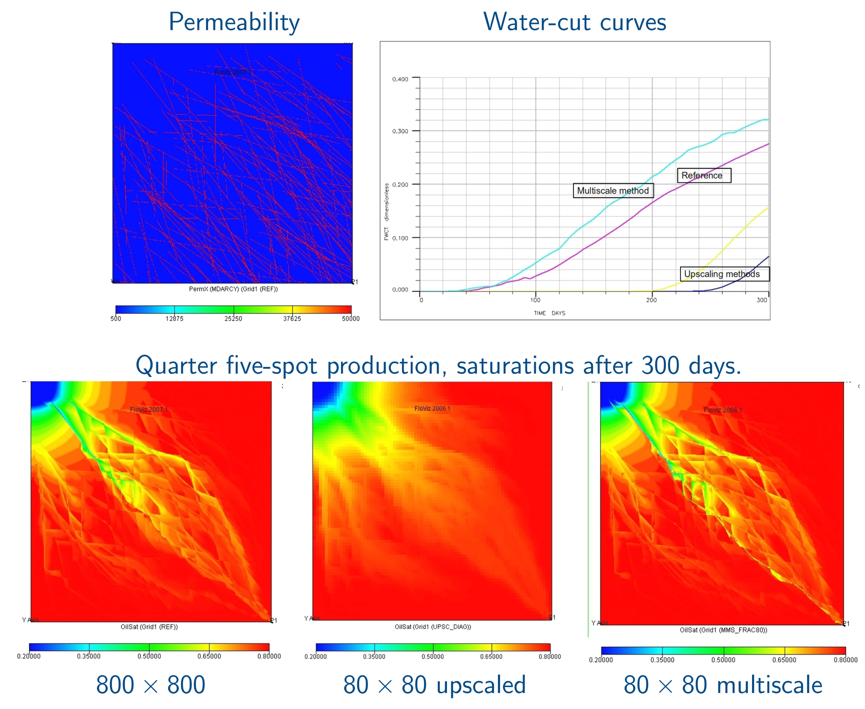


Inhouse code (multiscale + streamlines) from 2005:
Multiscale: 2 min and 20 sec
Multigrid: 8 min and 36 sec

Simplification: no gravity, no compressibility ⟶ a few percent away from reference solution

Example: Fracture Corridors

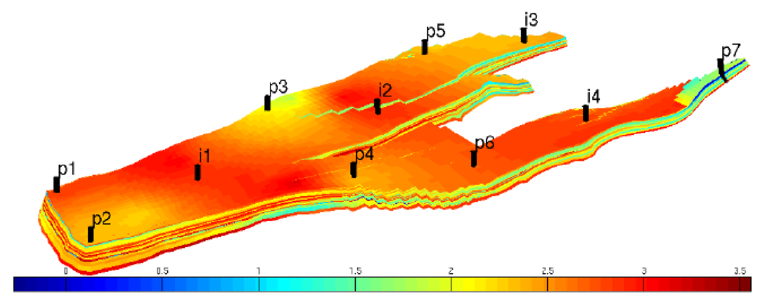
- 200 linear fractures randomly distributed
- Fracture permeability, 50 darcies
- 800 × 800 Cartesian grid
- Homogeneous background permeability, 500 mD.
- Flow-based diagonal upscaling to 80 × 80 grid
- Multiscale method on 80 × 80 grid with extra coarse blocks added to represent the fractures

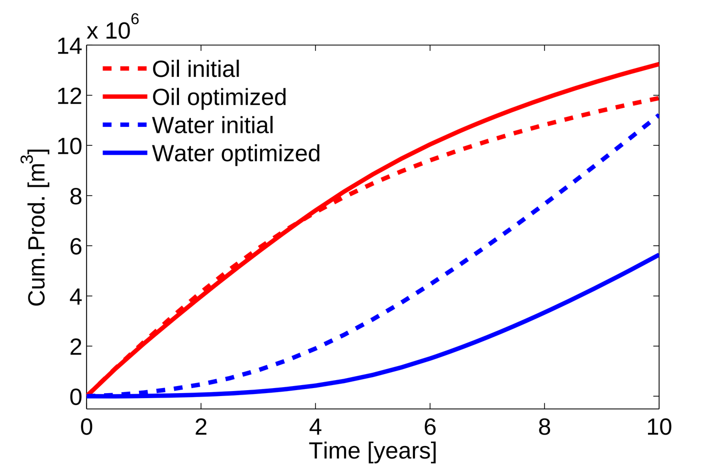


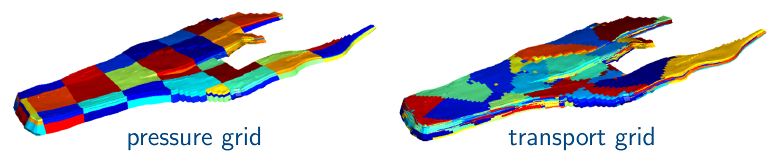
More details: J. R. Natvig et al. Multiscale mimetic solvers for efficient streamline simulation of fractured reservoirs. SPE 119132, 2009 RSS.

Example: Optimizing Net-Present Value

- Layered model from the Norwegian Sea
- Initially filled with oil
- Seven producers, four injectors
- Simulation model: $F(\mathbf{x}^n, \mathbf{x}^{n-1}, \mathbf{c}^n)$
  $\mathbf{x}$=state variables, $\mathbf{c}$=controls.
- Objective $J(\mathbf{x}, \mathbf{c})$, net-present value
- Optimization: adjoint formulation
  $$\frac{\partial F^T}{\partial \mathbf{x}}\lambda = -\frac{\partial J^T}{\partial \mathbf{x}}.$$
- Solver: multiscale pressure, flow-based coarsening for transport







Key step: efficient communication between the two grids using precomputed mappings.

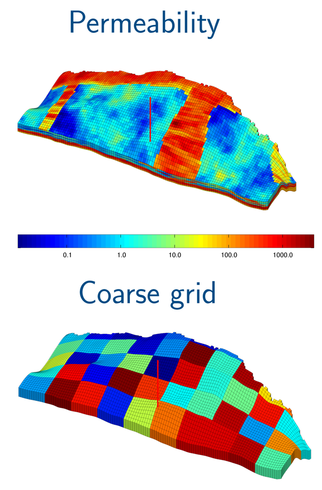
Forward simulations:
44 927 cells, 20 time steps, runtime less than 5 secs in Matlab: ∼ 100× speedup

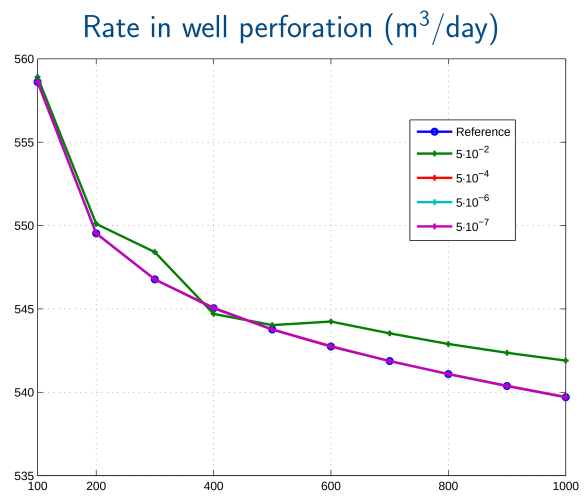
More details: S. Krogstad et al. Adjoint multiscale mixed finite elements. SPE 119112. 2009 Reservoir Simulation Symposium.

Examples: Compressibility / Black-Oil

Primary Production from a Gas Reservoir

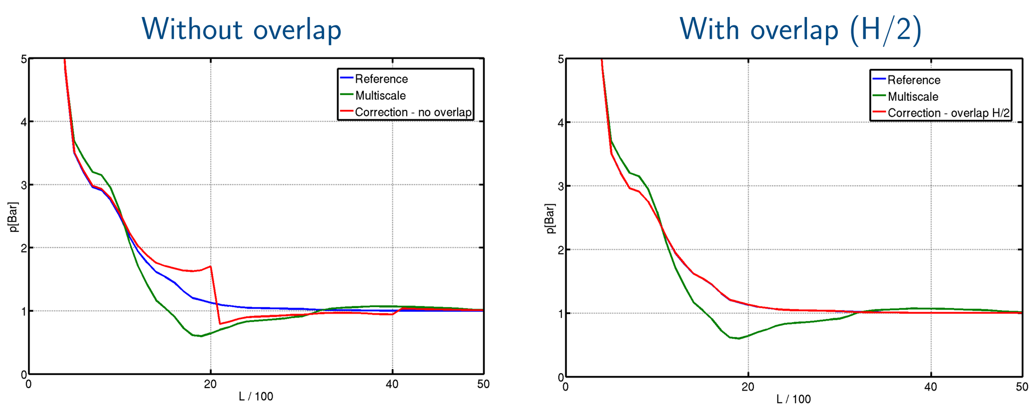
- Shallow-marine reservoir (realization from SAIGUP)
- Model size: 40 × 120 × 20
- Initially filled with gas, 200 bar
- Single producer, bhp=150 bar
- Multiscale solution for different tolerences compared with fine-scale reference solution.





Strong Compressibility

- 1-D heterogeneous
- Filled with air (1 bar)
- Left: inject air (10 bar)
- Right: produce air (1 bar)
- Fine grid: 100 cells
- Coarse grid: 5 blocks



The figures show the multiscale solution in the left half of the reservoir before and after the correction step compared with the fine-grid solution. Using overlap in the domain decomposition method is essential to correct the errors at coarse-grid interfaces.

References

- S. Krogstad et al. A multiscale mixed finite-element solver for three-phase black-oil flow. SPE 118993, 2009 SPE Reservoir Simulation Symposium. DOI: 10.2118/118993-MS.
- J. E. Aarnes, S. Krogstad, and K.-A. Lie. Multiscale mixed/mimetic methods on corner-point grids. Comp. Geosci. 12(3):297–315, 2008. DOI: 10.1007/s10596-007-9072-8.
- V. Kippe, J. E. Aarnes, and K.-A. Lie. A comparison of multiscale methods for elliptic problems in porous media flow. Comp. Geosci. 12(3):377–398, 2008. DOI: 10.1007/s10596-007-9074-6.
- Matlab Reservoir Simulation Toolbox. http://www.sintef.no/Projectweb/MRST/