Ensemble Kalman Filter

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The Ensemble Kalman Filter (EnKF)

- Represents error statistics using an ensemble of model states.
- Evolves error statistics by ensemble integrations.
- "Variance minimizing" analysis scheme operating on the ensemble.

$$\Downarrow$$

- Monte Carlo, low rank, error subspace method.
- Converges to the Kalman Filter with increasing ensemble size.
- Fully nonlinear error evolution, contrary to EKF.
- Assumption of Gaussian statistics in analysis scheme.

The error covariance matrix

Define ensemble covariances around the ensemble mean

$$\boldsymbol{C}^{\mathrm{f}}_{\psi\psi} \simeq (\boldsymbol{C}^{\mathrm{f}}_{\psi\psi})^{\mathrm{e}} = \overline{(\boldsymbol{\psi}^{\mathrm{f}} - \overline{\boldsymbol{\psi}^{\mathrm{f}}})(\boldsymbol{\psi}^{\mathrm{f}} - \overline{\boldsymbol{\psi}^{\mathrm{f}}})^{\mathrm{T}}}$$

$$\boldsymbol{C}^{\mathrm{a}}_{\psi\psi} \simeq (\boldsymbol{C}^{\mathrm{a}}_{\psi\psi})^{\mathrm{e}} = \overline{(\boldsymbol{\psi}^{\mathrm{a}} - \overline{\boldsymbol{\psi}^{\mathrm{a}}})(\boldsymbol{\psi}^{\mathrm{a}} - \overline{\boldsymbol{\psi}^{\mathrm{a}}})^{\mathrm{T}}}$$

- The ensemble mean $\overline{\psi}$ is the best-guess.
- The ensemble spread defines the error variance.
- The covariance is determined by the smoothness of the ensemble members.
- A covariance matrix can be represented by an ensemble of model states (not unique).

Dynamical evolution of error statistics

- Each ensemble member evolves according to the model dynamics which is expressed by a stochastic differential equation

$$d\boldsymbol{\psi} = \boldsymbol{g}(\boldsymbol{\psi})dt + \boldsymbol{h}(\boldsymbol{\psi})d\boldsymbol{q}.$$

- The probability density f then evolve according to Kolmogorov's equation

$$\frac{\partial f}{\partial t} + \sum_i \frac{\partial (g_i f)}{\partial \psi_i} = \frac{1}{2} \sum_{i,j} \frac{\partial^2 f (\boldsymbol{h}\boldsymbol{C}_{qq}\boldsymbol{h}^T)_{ij}}{\partial \psi_i \partial \psi_j}.$$

- This is the fundamental equation for evolution of error statistics and can be solved using Monte Carlo methods.

Analysis scheme (1)

- Given an ensemble of model forcasts, ψ_j^{f}, defining forecast error covariance

$$\boldsymbol{C}_{\psi\psi}^{\mathrm{f}} \simeq (\boldsymbol{C}_{\psi\psi}^{\mathrm{f}})^{\mathrm{e}} = \overline{(\boldsymbol{\psi}^{\mathrm{f}} - \overline{\boldsymbol{\psi}^{\mathrm{f}}})(\boldsymbol{\psi}^{\mathrm{f}} - \overline{\boldsymbol{\psi}^{\mathrm{f}}})^{\mathrm{T}}}.$$

- Create an ensemble of observations

$$\boldsymbol{d}_j = \boldsymbol{d} + \boldsymbol{\epsilon}_j,$$

with

  - $\boldsymbol{d}$, the real observations,
  - $\boldsymbol{\epsilon}_j$, a vector of observation noise,
  - $\overline{\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathrm{T}}} = (\boldsymbol{C}_{\epsilon\epsilon})^{\mathrm{e}} \simeq \boldsymbol{C}_{\epsilon\epsilon}$.

Analysis scheme (2)

- Update each ensemble member according to

$$\psi_j^{\mathrm{a}} = \psi_j^{\mathrm{f}} + (\boldsymbol{C}_{\psi\psi}^{\mathrm{f}})^{\mathrm{e}} \boldsymbol{\mathcal{M}}^{\mathrm{T}} \left(\boldsymbol{\mathcal{M}} (\boldsymbol{C}_{\psi\psi}^{\mathrm{f}})^{\mathrm{e}} \boldsymbol{\mathcal{M}}^{\mathrm{T}} + \boldsymbol{C}_{\epsilon\epsilon} \right)^{-1} \left(\boldsymbol{d}_j - \boldsymbol{\mathcal{M}} \psi_j^{\mathrm{f}} \right)$$

- Thus, the update of the mean becomes

$$\overline{\psi^{\mathrm{a}}} = \overline{\psi^{\mathrm{f}}} + (\boldsymbol{C}_{\psi\psi}^{\mathrm{f}})^{\mathrm{e}} \boldsymbol{\mathcal{M}}^{\mathrm{T}} \left(\boldsymbol{\mathcal{M}} (\boldsymbol{C}_{\psi\psi}^{\mathrm{f}})^{\mathrm{e}} \boldsymbol{\mathcal{M}}^{\mathrm{T}} + \boldsymbol{C}_{\epsilon\epsilon} \right)^{-1} \left(\boldsymbol{d} - \boldsymbol{\mathcal{M}} \overline{\psi^{\mathrm{f}}} \right).$$

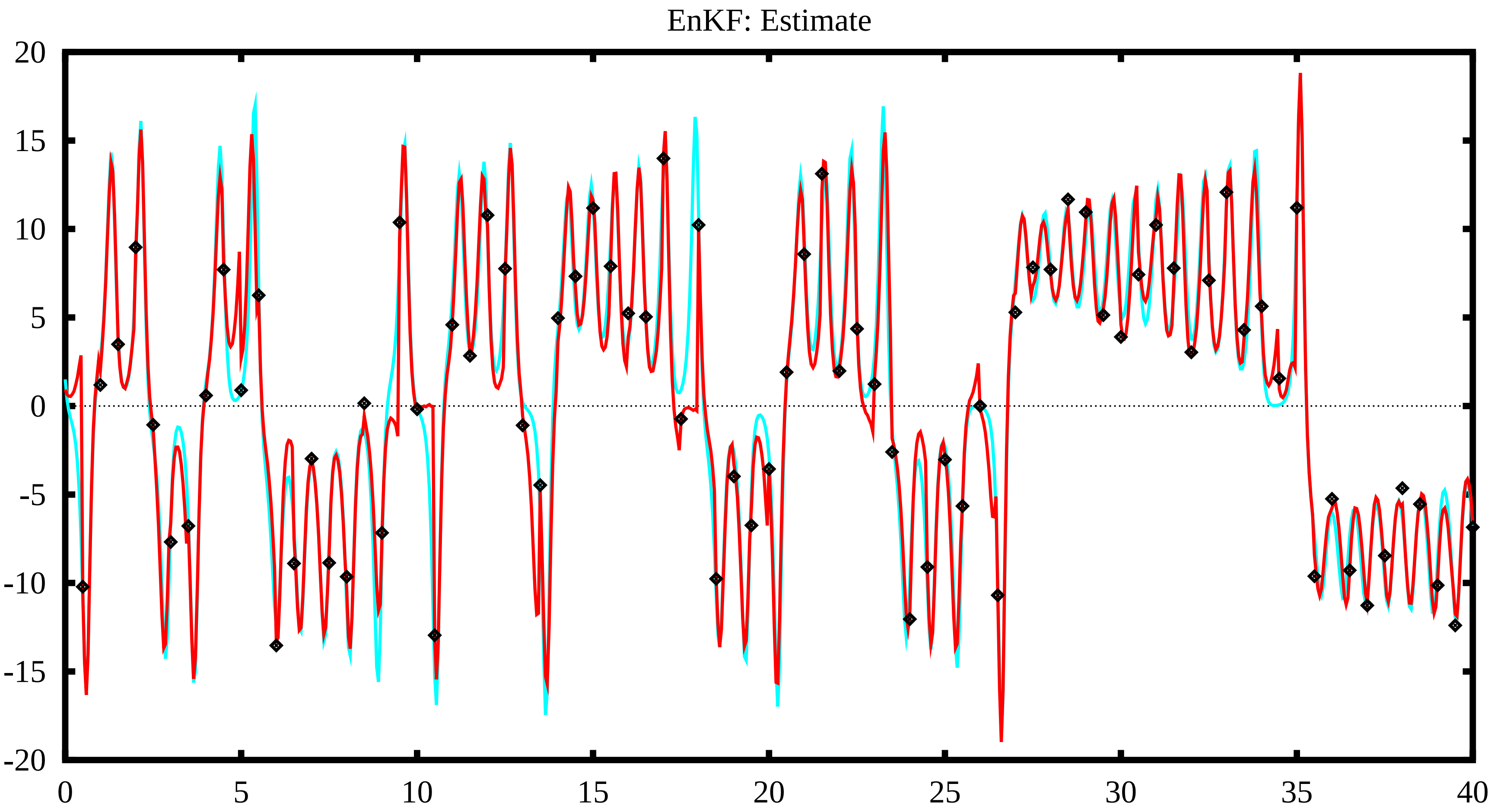
- The posterior error covariance becomes

$$\begin{aligned}(\boldsymbol{C}_{\psi\psi}^{\mathrm{a}})^{\mathrm{e}} &= (\boldsymbol{C}_{\psi\psi}^{\mathrm{f}})^{\mathrm{e}} \\ &\quad - (\boldsymbol{C}_{\psi\psi}^{\mathrm{f}})^{\mathrm{e}} \boldsymbol{\mathcal{M}}^{\mathrm{T}} \left(\boldsymbol{\mathcal{M}} (\boldsymbol{C}_{\psi\psi}^{\mathrm{f}})^{\mathrm{e}} \boldsymbol{\mathcal{M}}^{\mathrm{T}} + \boldsymbol{C}_{\epsilon\epsilon} \right)^{-1} \boldsymbol{\mathcal{M}} (\boldsymbol{C}_{\psi\psi}^{\mathrm{f}})^{\mathrm{e}}.\end{aligned}$$

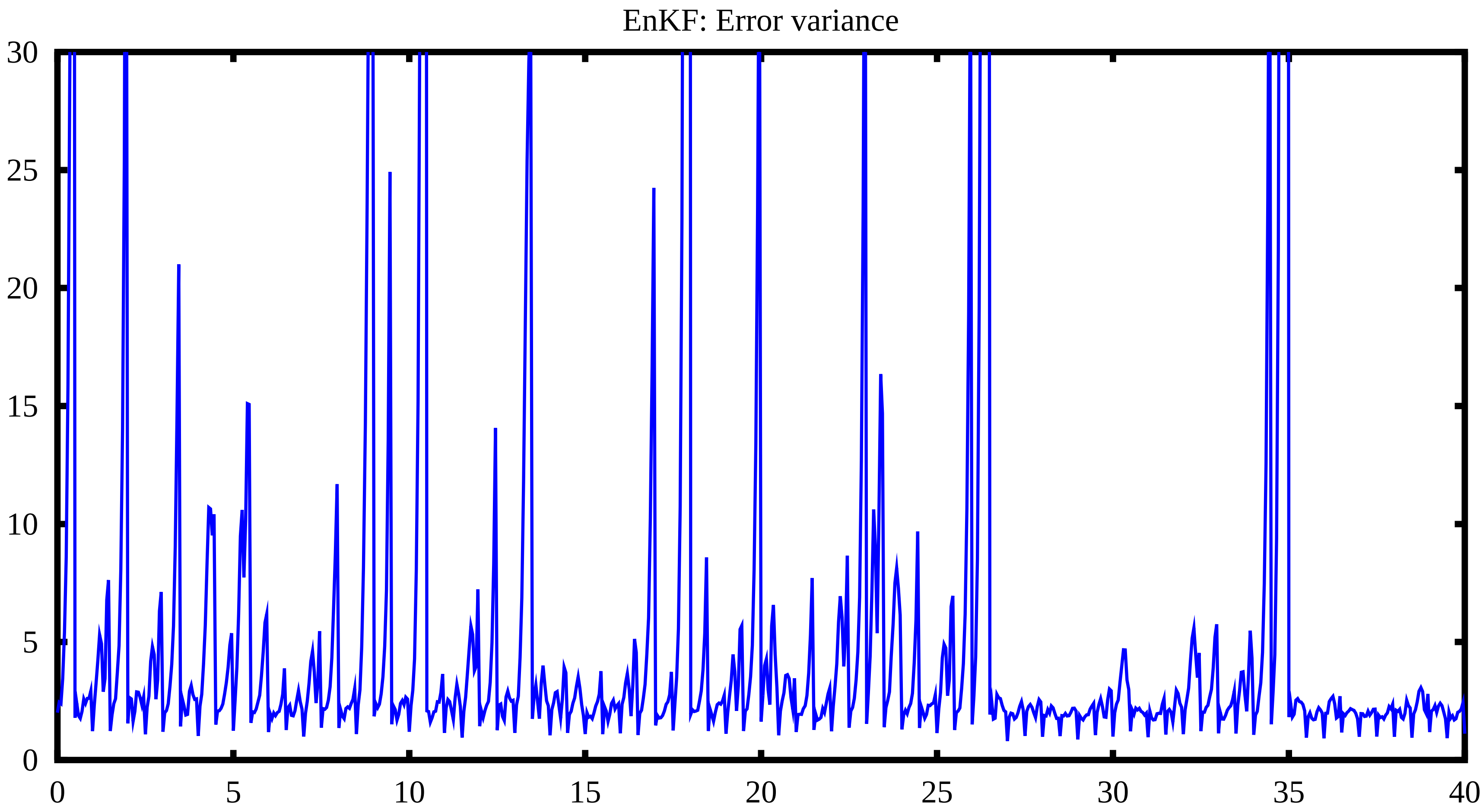
Example: Lorenz model

- Application with the chaotic Lorenz model
- Illustrates properties with higly nonlinear dynamical models.
- From Evensen (1997), MWR.

EnKF solution



EnKF error variance



Summary: Lorenz model

- The EnKF works well with highly nonlinear dynamical models.
- There is no linearization in the evolution of error statistics.
- Methods using tangent linear or adjoint operators have problems with the Lorenz equations:
  - limited by the predictability time,
  - limited by the validity time of tangent linear operator.
- Can we expect the same to be true for more complex models with large state spaces?

Analysis equation (1)

- Define the ensemble matrix

$$\boldsymbol{A} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_N) \in \Re^{n \times N}.$$

- The ensemble mean is (defining $\mathbf{1}_N \in \Re^{N \times N} \equiv 1/N$)

$$\overline{\boldsymbol{A}} = \boldsymbol{A}\mathbf{1}_N.$$

- The ensemble perturbations become

$$\boldsymbol{A}' = \boldsymbol{A} - \overline{\boldsymbol{A}} = \boldsymbol{A}(\boldsymbol{I} - \mathbf{1}_N).$$

- The ensemble covariance matrix $(\boldsymbol{C}_{\psi\psi})^{\mathrm{e}} \in \Re^{n \times n}$ becomes

$$(\boldsymbol{C}_{\psi\psi})^{\mathrm{e}} = \frac{\boldsymbol{A}'(\boldsymbol{A}')^{\mathrm{T}}}{N-1}.$$

Analysis equation (2)

- Given a vector of measurements $\boldsymbol{d} \in \Re^m$, define

$$\boldsymbol{d}_j = \boldsymbol{d} + \boldsymbol{\epsilon}_j, \quad j = 1, \dots, N,$$

stored in

$$\boldsymbol{D} = (\boldsymbol{d}_1, \boldsymbol{d}_2, \dots, \boldsymbol{d}_N) \in \Re^{m \times N}.$$

- The ensemble perturbations are stored in

$$\boldsymbol{E} = (\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \dots, \boldsymbol{\epsilon}_N) \in \Re^{m \times N},$$

thus, the measurement error covariance matrix becomes

$$\boldsymbol{C}^{\mathrm{e}}_{\epsilon\epsilon} = \frac{\boldsymbol{E}\boldsymbol{E}^{\mathrm{T}}}{N-1}.$$

Analysis equation (3)

- The analysis equation can now be written

$$\boldsymbol{A}^{\mathrm{a}} = \boldsymbol{A} + (\boldsymbol{C}_{\psi\psi})^{\mathrm{e}} \boldsymbol{\mathcal{M}}^{\mathrm{T}} \left(\boldsymbol{\mathcal{M}} (\boldsymbol{C}_{\psi\psi})^{\mathrm{e}} \boldsymbol{\mathcal{M}}^{\mathrm{T}} + \boldsymbol{C}_{\epsilon\epsilon} \right)^{-1} \left(\boldsymbol{D} - \boldsymbol{\mathcal{M}} \boldsymbol{A} \right).$$

- Defining the innovations $\boldsymbol{D}' = \boldsymbol{D} - \boldsymbol{\mathcal{M}} \boldsymbol{A}$ and using previous definitions:

$$\boldsymbol{A}^{\mathrm{a}} = \boldsymbol{A} + \boldsymbol{A}' (\boldsymbol{\mathcal{M}} \boldsymbol{A}')^{\mathrm{T}} \left((\boldsymbol{\mathcal{M}} \boldsymbol{A}')(\boldsymbol{\mathcal{M}} \boldsymbol{A}')^{\mathrm{T}} + \boldsymbol{C}_{\epsilon\epsilon} \right)^{-1} \boldsymbol{D}'.$$

  i.e., analysis expressed entirely in terms of the ensemble

Analysis equation (4)

- Define $\boldsymbol{S} = \boldsymbol{\mathcal{M}}\boldsymbol{A}'$ and $\boldsymbol{C} = \boldsymbol{S}\boldsymbol{S}^{\mathrm{T}} + \boldsymbol{C}_{\epsilon\epsilon}$.
- Use $\boldsymbol{A}' = \boldsymbol{A}(\boldsymbol{I} - \mathbf{1}_N)$.
- Use $\mathbf{1}_N\boldsymbol{S}^{\mathrm{T}} \equiv \mathbf{0}$.

$$
\begin{aligned}
\boldsymbol{A}^{\mathrm{a}} &= \boldsymbol{A} + \boldsymbol{A}'\boldsymbol{S}^{\mathrm{T}}\left(\boldsymbol{S}\boldsymbol{S}^{\mathrm{T}} + \boldsymbol{C}_{\epsilon\epsilon}\right)^{-1}\boldsymbol{D}' \\
&= \boldsymbol{A} + \boldsymbol{A}(\boldsymbol{I} - \mathbf{1}_N)\boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{D}' \\
&= \boldsymbol{A}\left(\boldsymbol{I} + \boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{D}'\right) \\
&= \boldsymbol{A}\boldsymbol{X}
\end{aligned}
$$

Remarks

- $(\boldsymbol{C}_{\psi\psi})^{\mathrm{e}}$ never computed but indirectly used to determine $\boldsymbol{\mathcal{M}}(\boldsymbol{C}_{\psi\psi})^{\mathrm{e}}\boldsymbol{\mathcal{M}}^{\mathrm{T}} = \boldsymbol{S}\boldsymbol{S}^{\mathrm{T}}$.
- Covariances only needed between observed variables at measurement locations.
- Analysis may be interpreted as:
  - combination of forecast ensemble members, or,
  - forecast plus combination of covariance functions.
- Accuracy of analysis is determined by:
  - the accuracy of $\boldsymbol{X}$,
  - the properties of the ensemble error space.

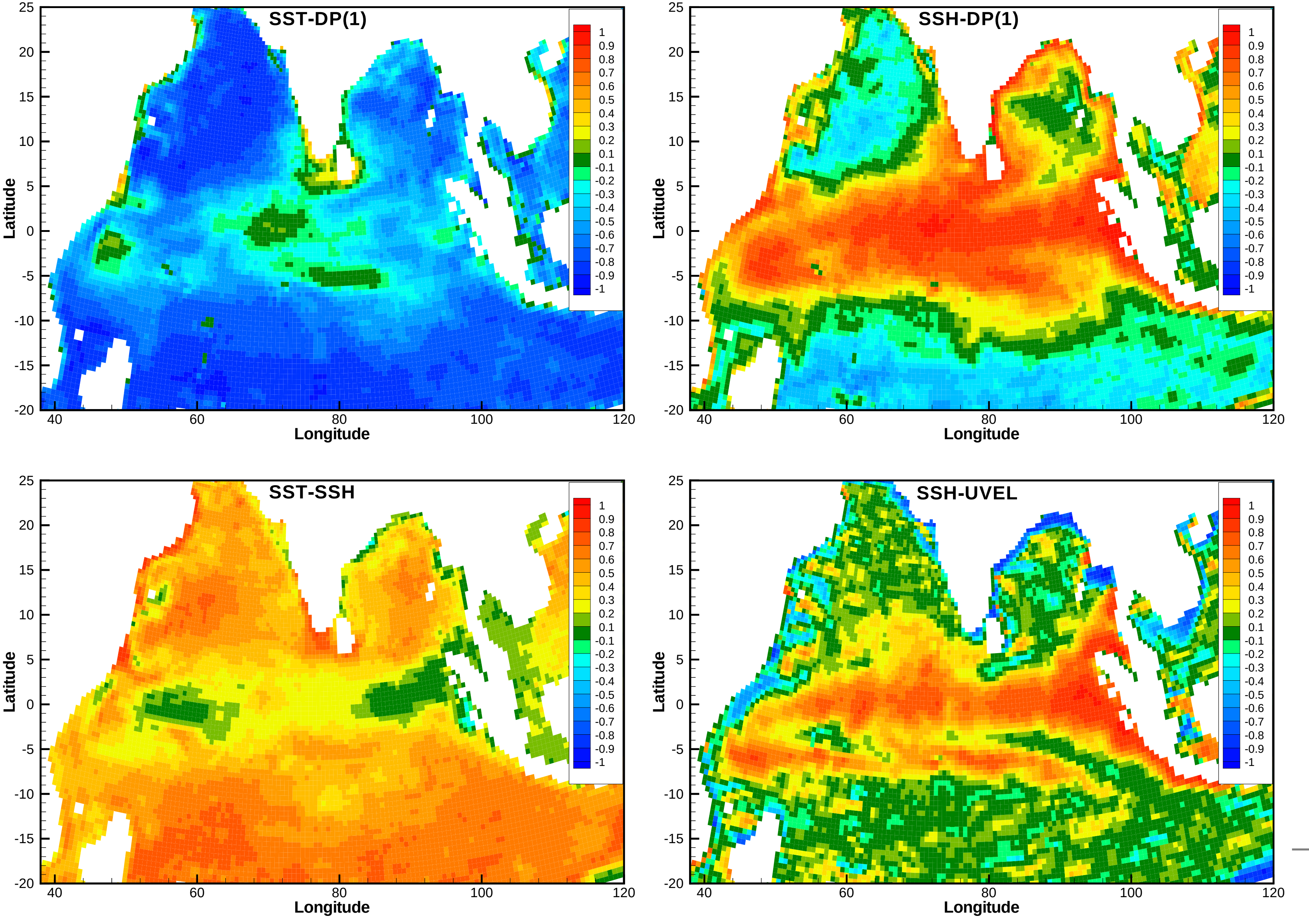
Remarks

- For a linear model, any choice of X will result in an analysis which is also a solution of the model.

Examples of ensemble statistics

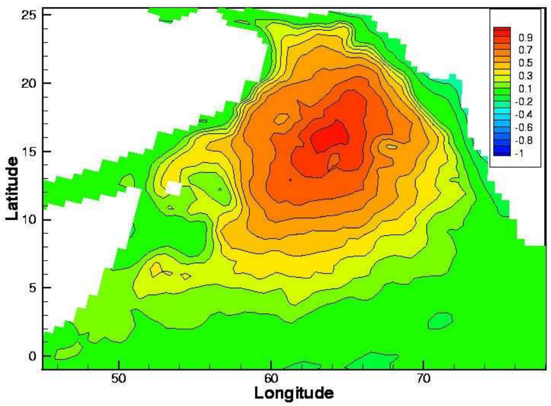
- Taken from Haugen and Evensen (2002), Ocean Dynamics.
- OGCM (MICOM) for the Indian Ocean.
- Assimilation of SST and SLA data.

Spatial correlations

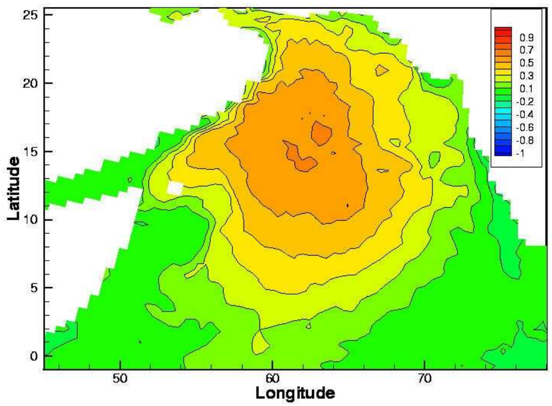


Correlation functions

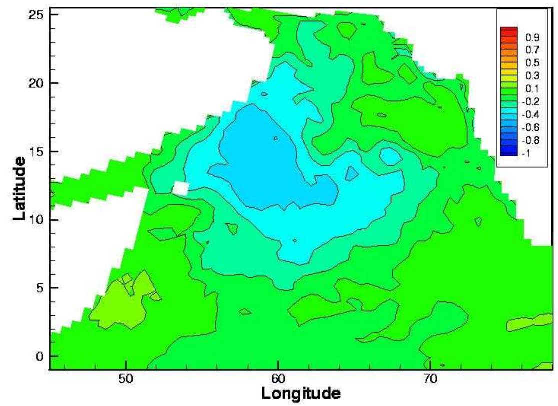
SSH–SSH



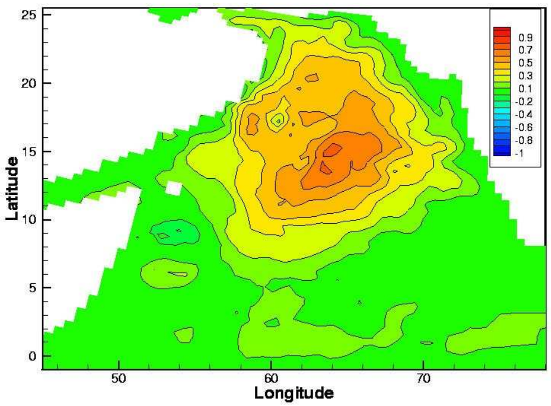
SSH–SST



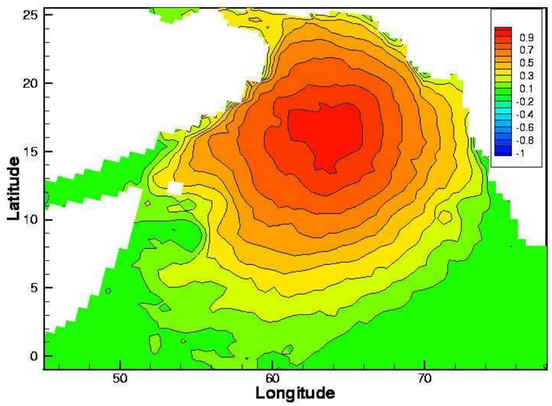
SSH–DP



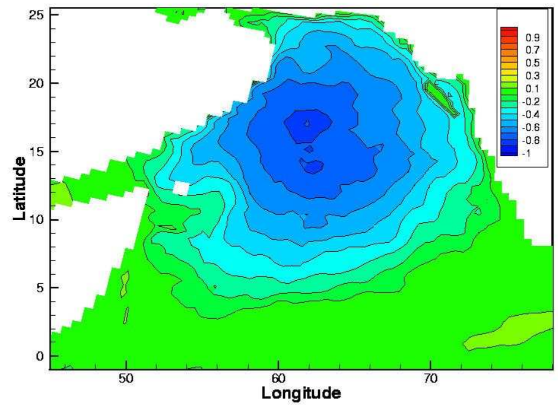
SST–SSH



SST–SST

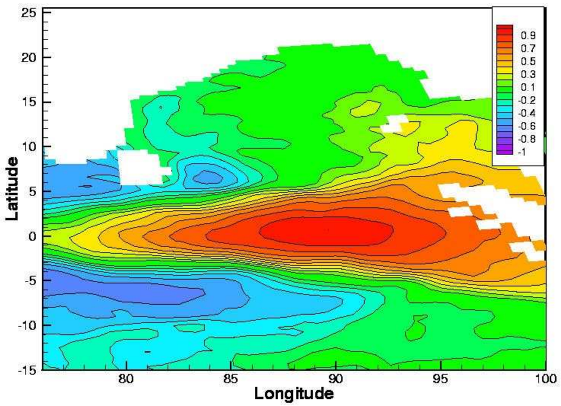


SST–DP

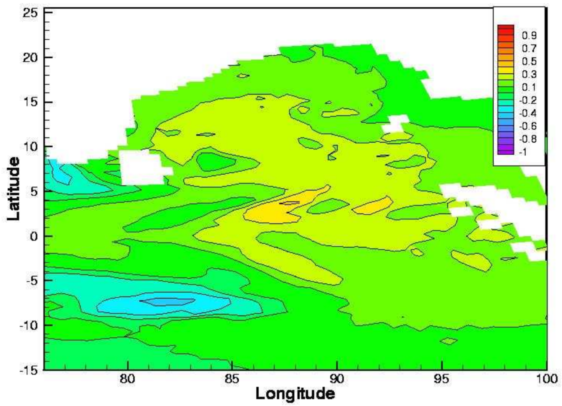


Correlation functions

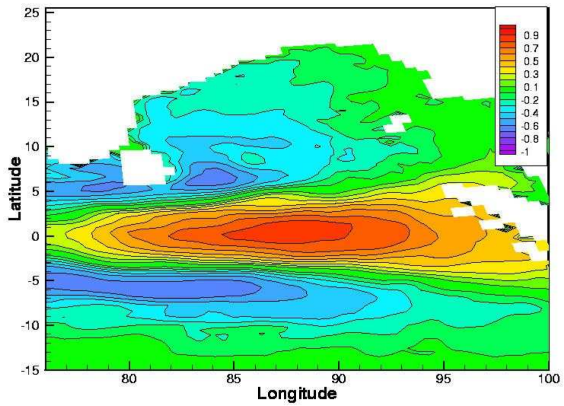
SSH–SSH



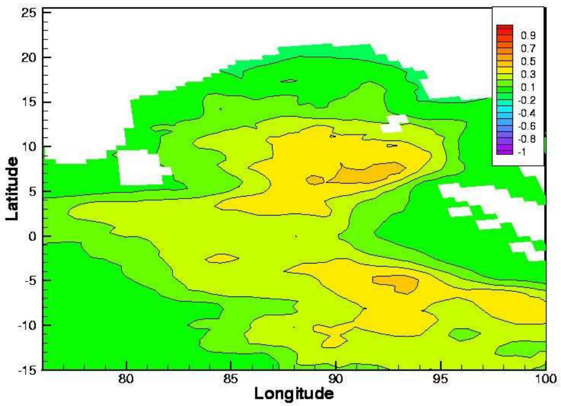
SSH–SST



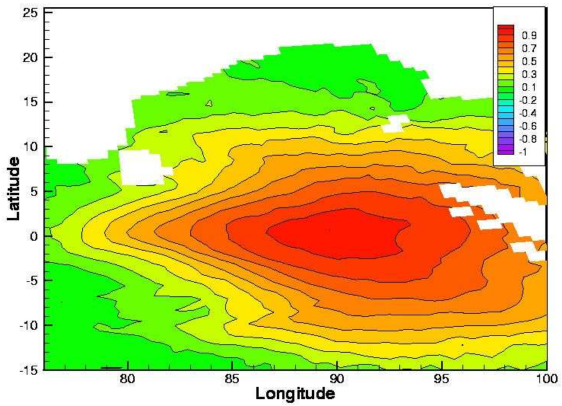
SSH–DP



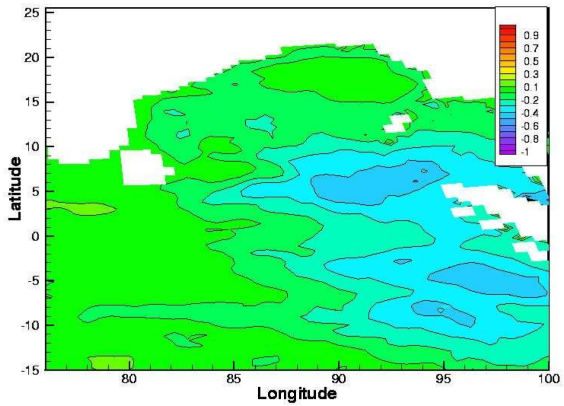
SST–SSH



SST–SST



SST–DP



Time–Depth: Temperature