Kalman Filtering

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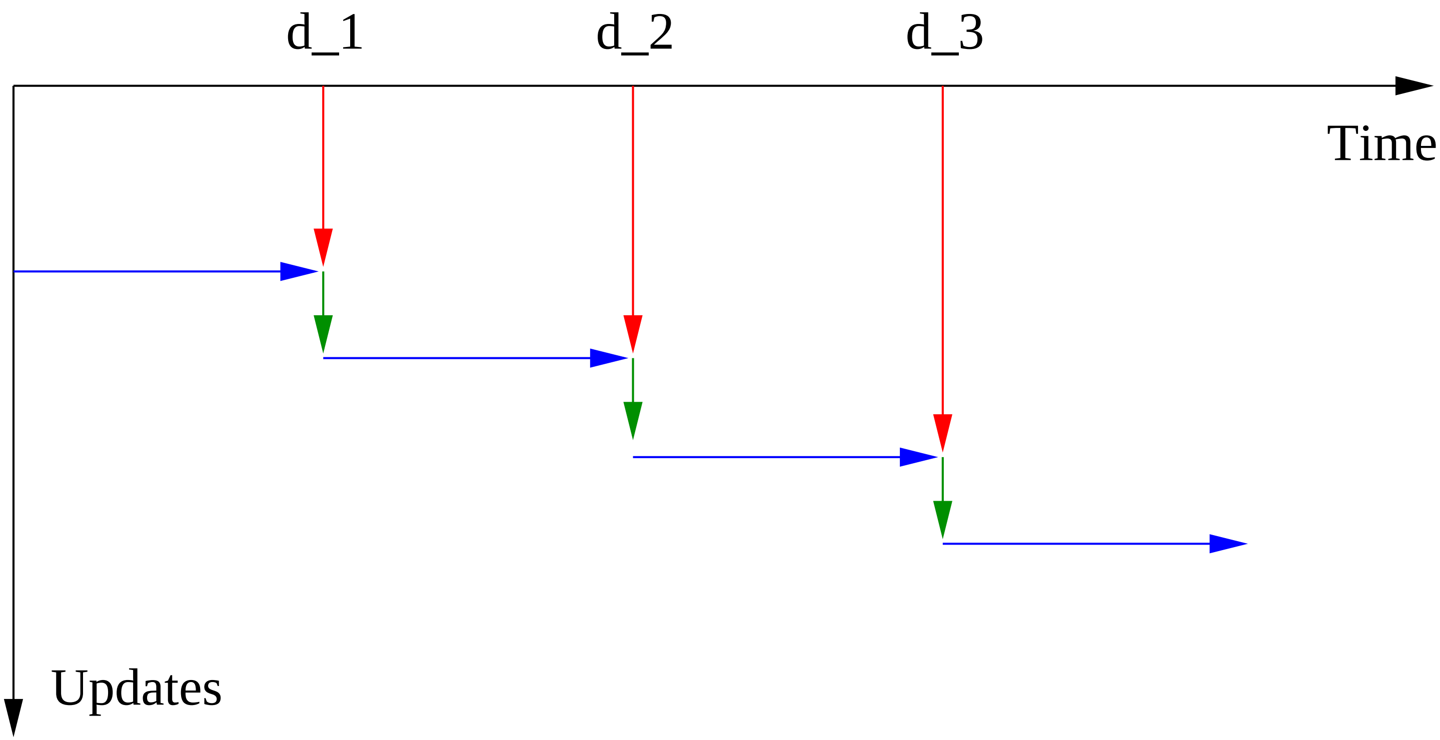
Kalman Filter

- “Approximate” solution of inverse problem
- Sequentially updates model state and uncertainty
- Variance minimizing update step
- Estimate improves and uncertainty reduces at each update



Analysis scheme

Given at time t:

- First guess estimate, $\boldsymbol{\psi}^{\mathrm{f}}(\boldsymbol{x})$.
- Error covariance of first guess, $\boldsymbol{C}^{\mathrm{f}}_{\boldsymbol{\psi\psi}}(\boldsymbol{x}_1, \boldsymbol{x}_2)$.
- Vector of measurements, $\boldsymbol{\mathcal{M}}\psi^{\mathrm{t}} = \boldsymbol{d} + \boldsymbol{\epsilon}$.
- Error covariance for measurements, $\boldsymbol{C}_{\epsilon\epsilon}$.

and assume Gaussian statistics!

$$\mathcal{J}[\psi] = \iint_{\mathcal{D}} \big(\boldsymbol{\psi} - \boldsymbol{\psi}^{\mathrm{f}}\big)\big(\boldsymbol{C}^{\mathrm{f}}_{\psi\psi}\big)^{-1}\big(\boldsymbol{\psi} - \boldsymbol{\psi}^{\mathrm{f}}\big)^{\mathrm{T}} d\boldsymbol{x}_1 d\boldsymbol{x}_2$$
$$+ \big(\boldsymbol{\mathcal{M}}\psi - \boldsymbol{d}\big)\boldsymbol{C}^{-1}_{\epsilon\epsilon}\big(\boldsymbol{\mathcal{M}}\psi - \boldsymbol{d}\big)^{\mathrm{T}}$$

Analysis scheme

An optimal estimator is then

- Analysis update

$$\boldsymbol{\psi}^{\mathrm{a}} = \boldsymbol{\psi}^{\mathrm{f}} + \boldsymbol{C}^{\mathrm{f}}_{\psi\psi}\boldsymbol{\mathcal{M}}^{\mathrm{T}}\left(\boldsymbol{\mathcal{M}}\boldsymbol{C}^{\mathrm{f}}_{\psi\psi}\boldsymbol{\mathcal{M}}^{\mathrm{T}} + \boldsymbol{C}_{\epsilon\epsilon}\right)^{-1}\left(\boldsymbol{d} - \boldsymbol{\mathcal{M}}\boldsymbol{\psi}^{\mathrm{f}}\right)$$

- Error covariance update

$$\boldsymbol{C}^{\mathrm{a}}_{\psi\psi} = \boldsymbol{C}^{\mathrm{f}}_{\psi\psi} - \boldsymbol{C}^{\mathrm{f}}_{\psi\psi}\boldsymbol{\mathcal{M}}^{\mathrm{T}}\left(\boldsymbol{\mathcal{M}}\boldsymbol{C}^{\mathrm{f}}_{\psi\psi}\boldsymbol{\mathcal{M}}^{\mathrm{T}} + \boldsymbol{C}_{\epsilon\epsilon}\right)^{-1}\boldsymbol{\mathcal{M}}\boldsymbol{C}^{\mathrm{f}}_{\psi\psi}.$$

Analysis scheme

Properties of the estimator

- Linear, unbiased and variance minimizing.
- MLH estimate for Gaussian statistics.
- Dimension of problem equals number of measurements.
- Can be derived from statistical or variational formulation.
- Consistent with Bayes.

Kalman Filter (KF)

- Includes time dimension.
- Idea is to:
  1. predict $\psi^{\mathrm{f}}(\boldsymbol{x})$ and $C^{\mathrm{f}}_{\psi\psi}(\boldsymbol{x}_1, \boldsymbol{x}_2)$,
  2. use analysis scheme to update these whenever measurements are available.
- Sequential filtering method!
  - Information from measurments carried forward in time.

KF: Error evolution

Derivation for linear scalar model

- Evolution of true state

$$\psi_k^{\mathrm{t}} = F\psi_{k-1}^{\mathrm{t}} + q_{k-1}$$

- Our model is

$$\psi_k^{\mathrm{f}} = F\psi_{k-1}^{\mathrm{a}},$$

  with forecast f and analysis a.

- Difference is

$$\psi_k^{\mathrm{t}} - \psi_k^{\mathrm{f}} = F(\psi_{k-1}^{\mathrm{t}} - \psi_{k-1}^{\mathrm{a}}) + q_{k-1}$$

KF: Error covariance evolution

Error covariance equation

$$
\begin{aligned}
C^{\mathrm{f}}_{\psi\psi}(t_k) &= \overline{(\psi^{\mathrm{t}}_k - \psi^{\mathrm{f}}_k)^2} \\
&= F^2\overline{(\psi^{\mathrm{t}}_{k-1} - \psi^{\mathrm{a}}_{k-1})^2} + \overline{q^2_{k-1}} + 2F\overline{(\psi^{\mathrm{t}}_{k-1} - \psi^{\mathrm{a}}_{k-1})q_{k-1}} \\
&= F^2 C^{\mathrm{a}}_{\psi\psi}(t_{k-1}) + C_{qq}(t_{k-1}).
\end{aligned}
$$

with

- $C^{\mathrm{a}}_{\psi\psi}(t_{k-1}) = \overline{(\psi^{\mathrm{t}}_{k-1} - \psi^{\mathrm{a}}_{k-1})^2}$,
- $C_{qq}(t_{k-1}) = \overline{(q_{k-1})^2}$,
- model errors uncorrelated with state error.

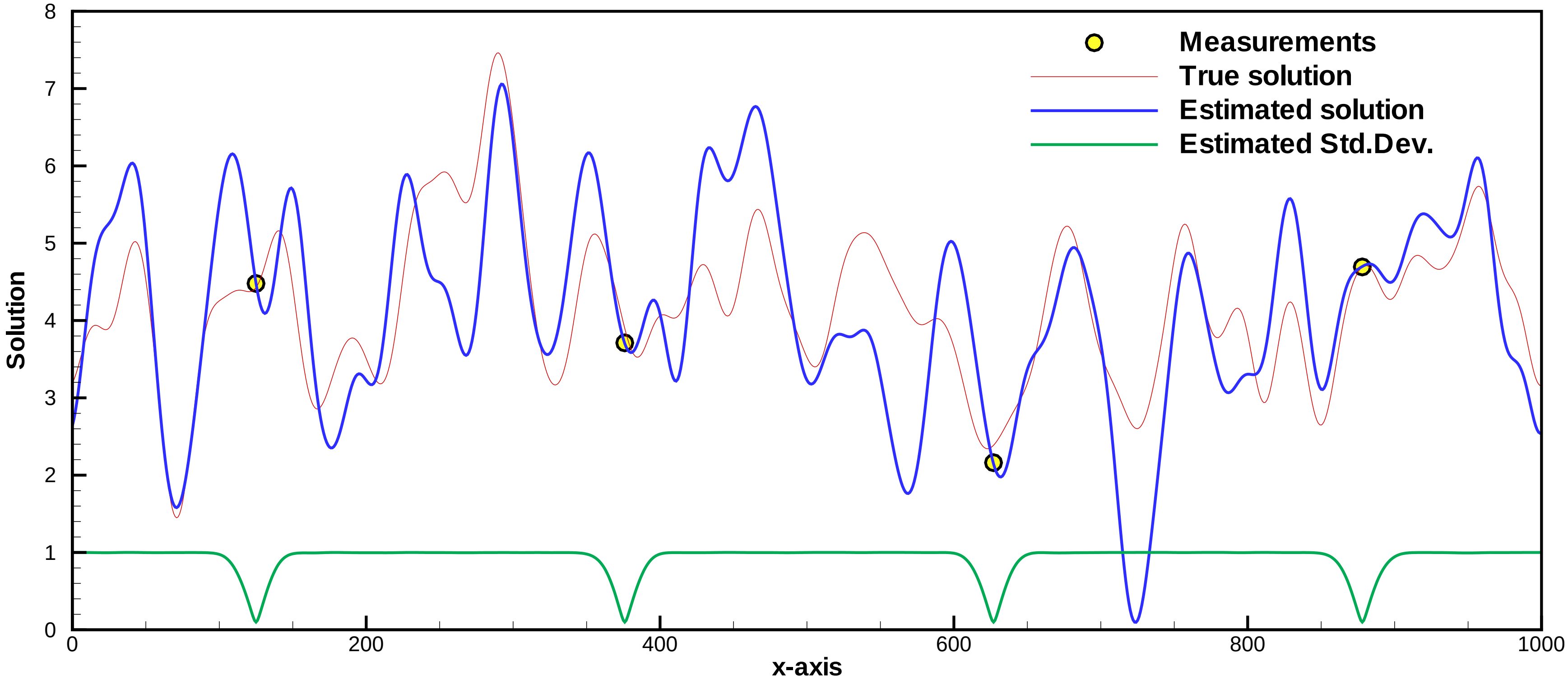
Kalman Filter Example

Model

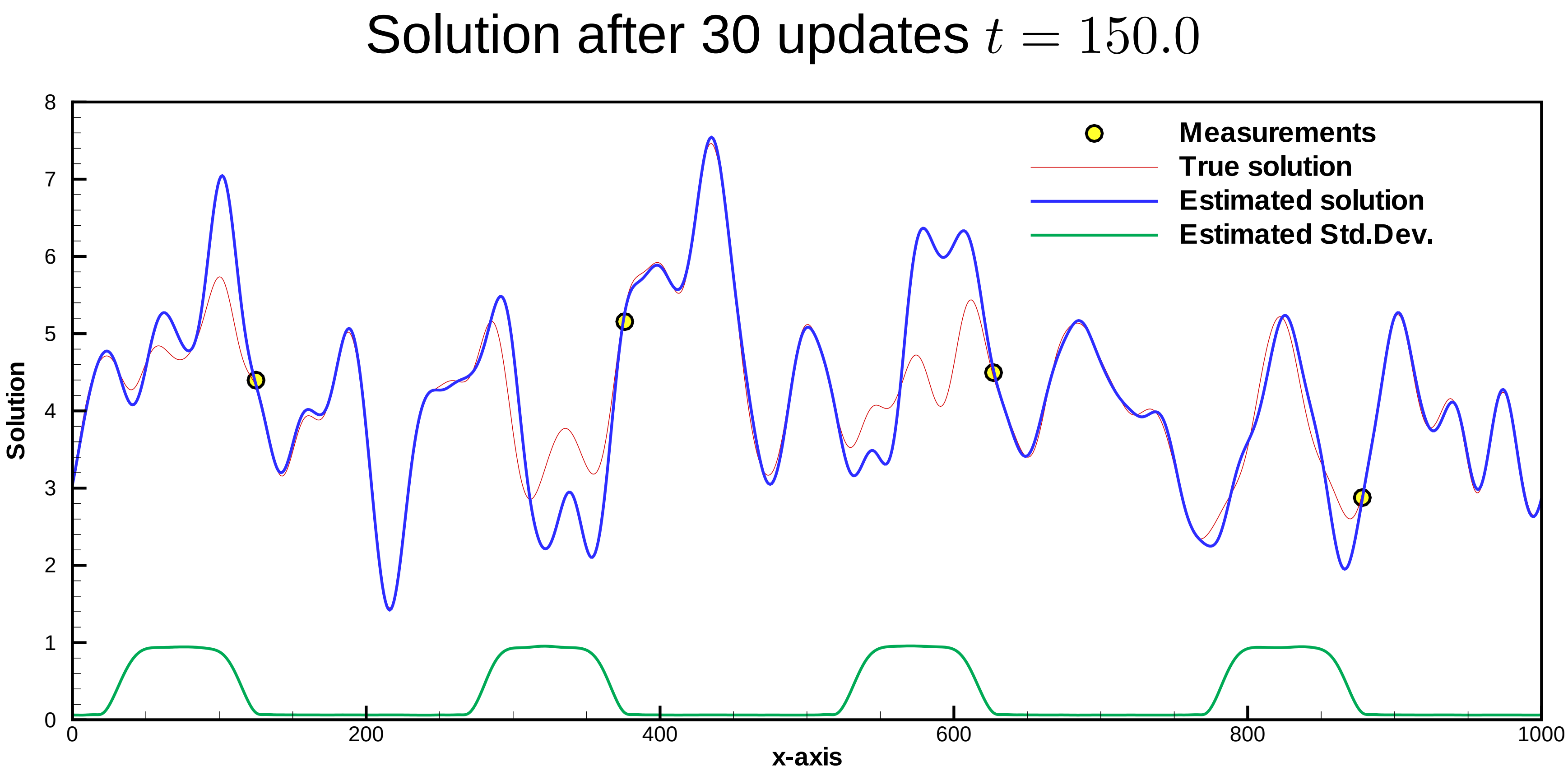
- Linear advection equation
- Periodic domain
- $u = \Delta t = \Delta x = 1.0$
- Random reference solution.
- First guess is reference plus random perturbation.
- Initial variance is 1.0
- Four measurements every 5 time units.
- Measurement variance is 0.01.
- Cases without and including system noise = 0.0004.

Kalman Filter Example: Case A

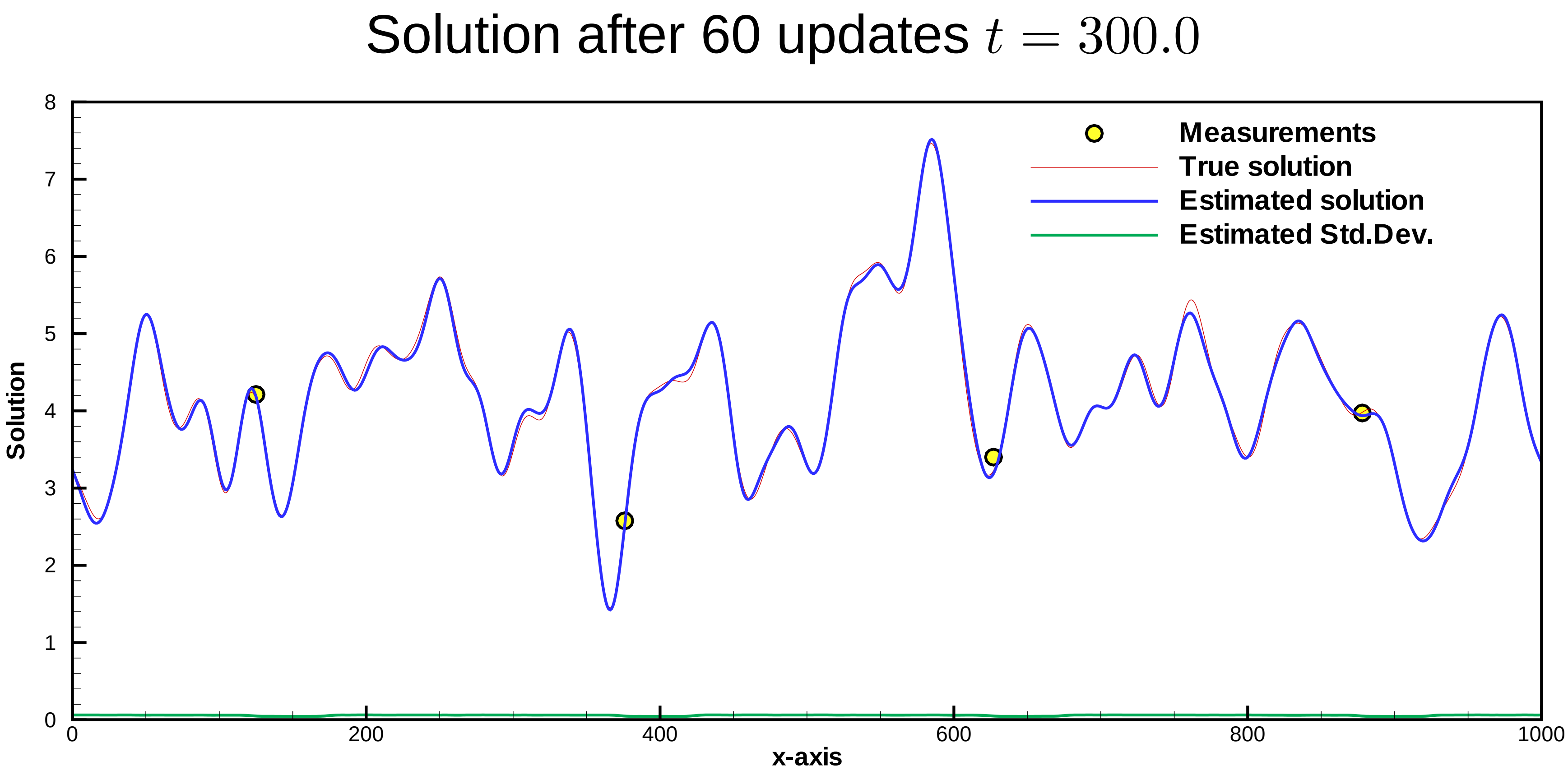
Solution after first update $t = 5.0$



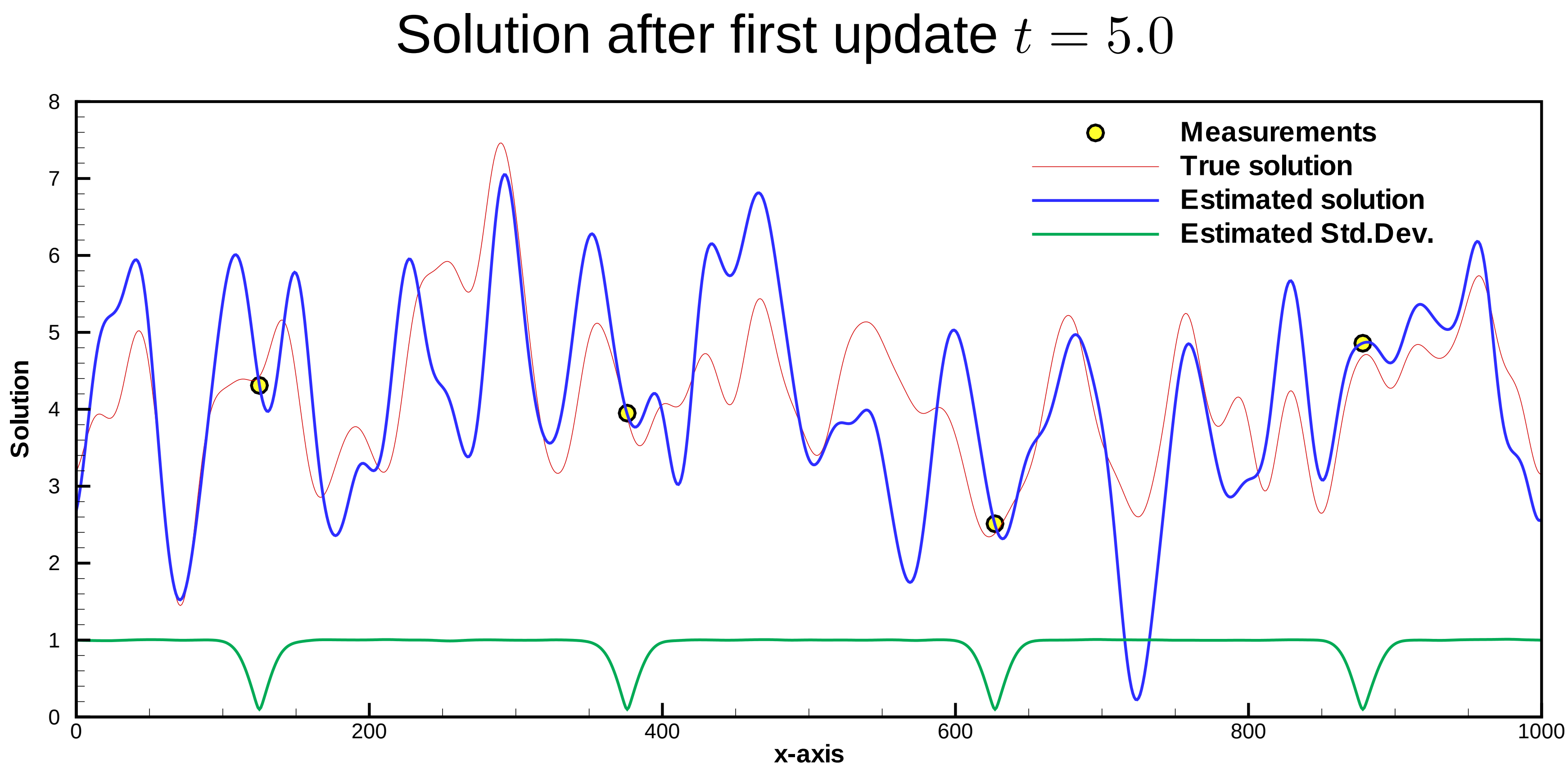
Kalman Filter Example: Case A



Kalman Filter Example: Case A

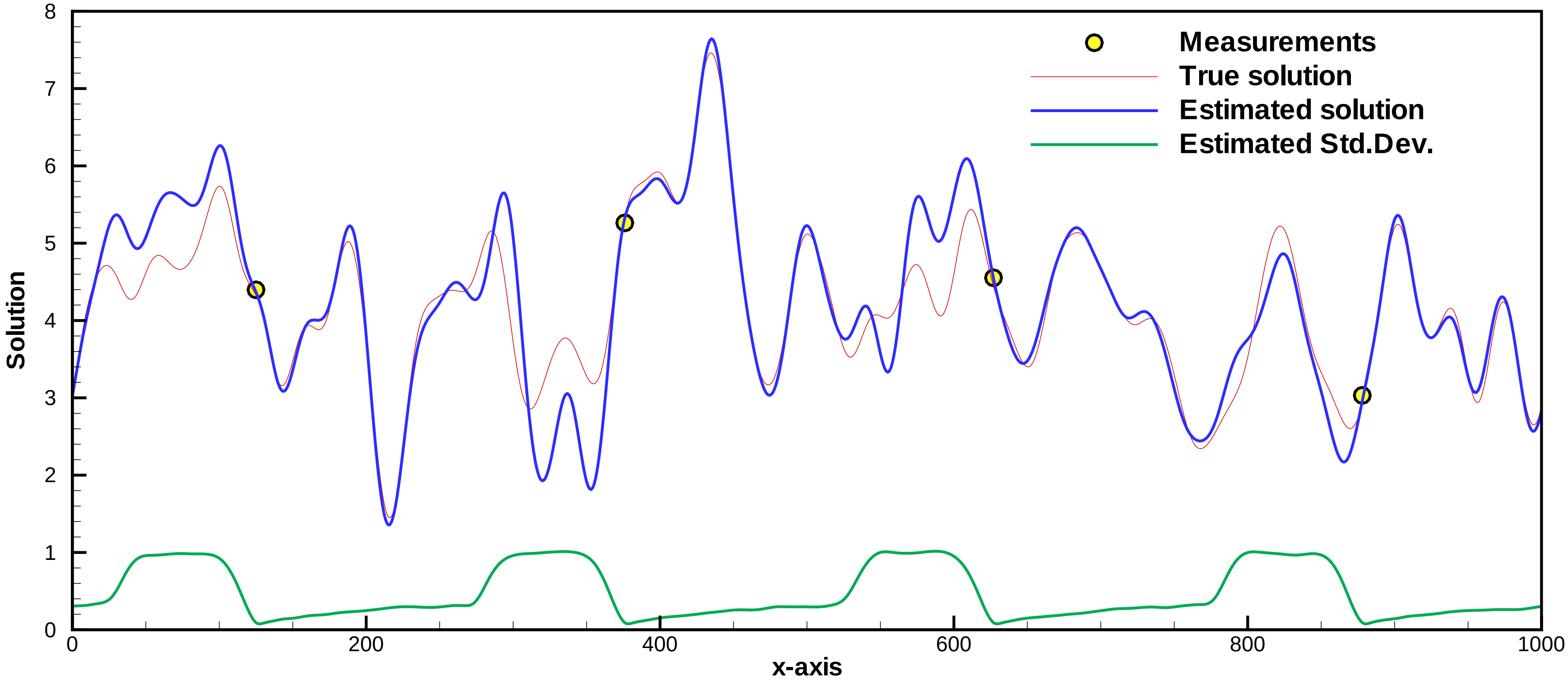


Kalman Filter Example: Case B

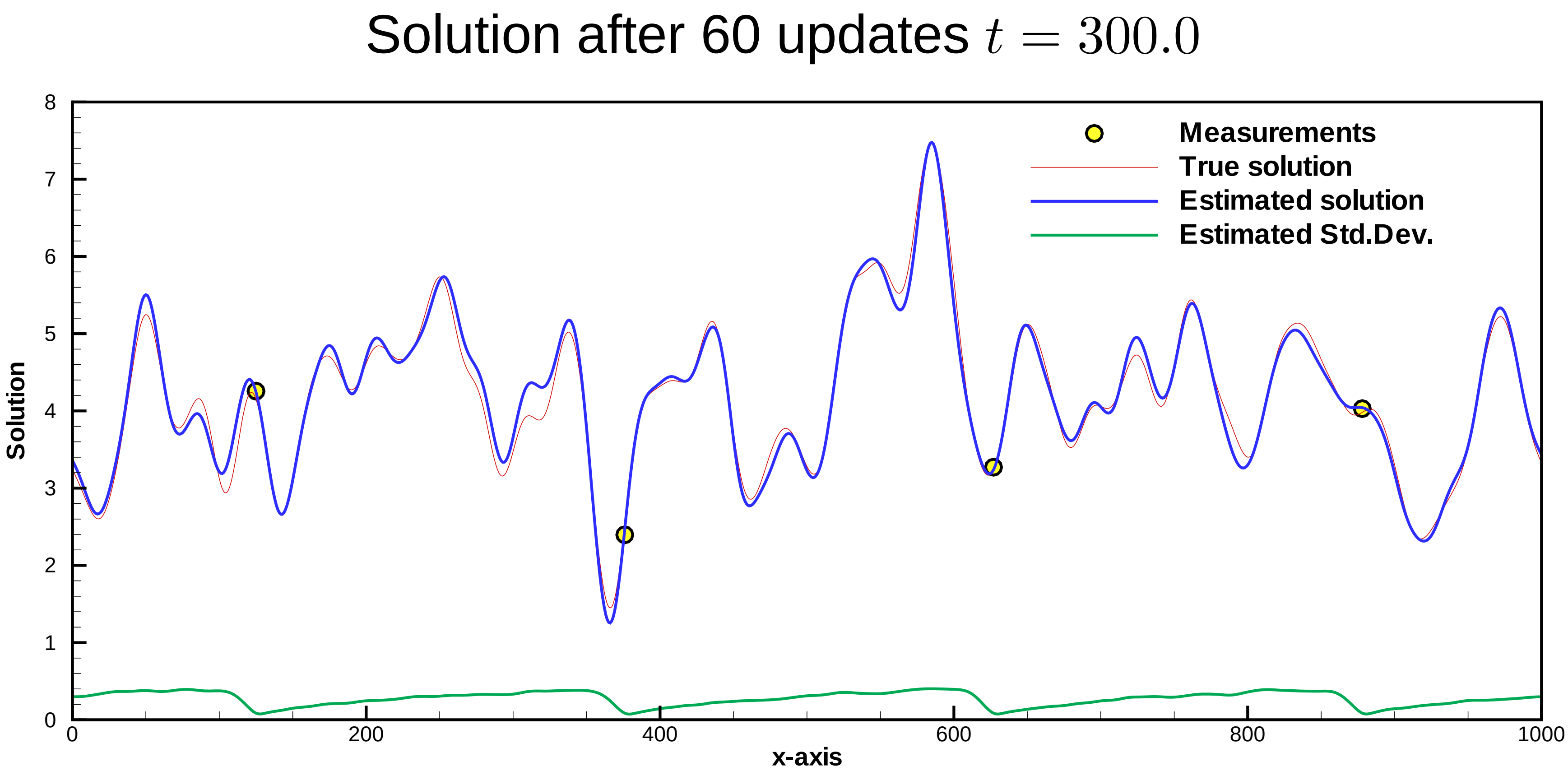


Kalman Filter Example: Case B

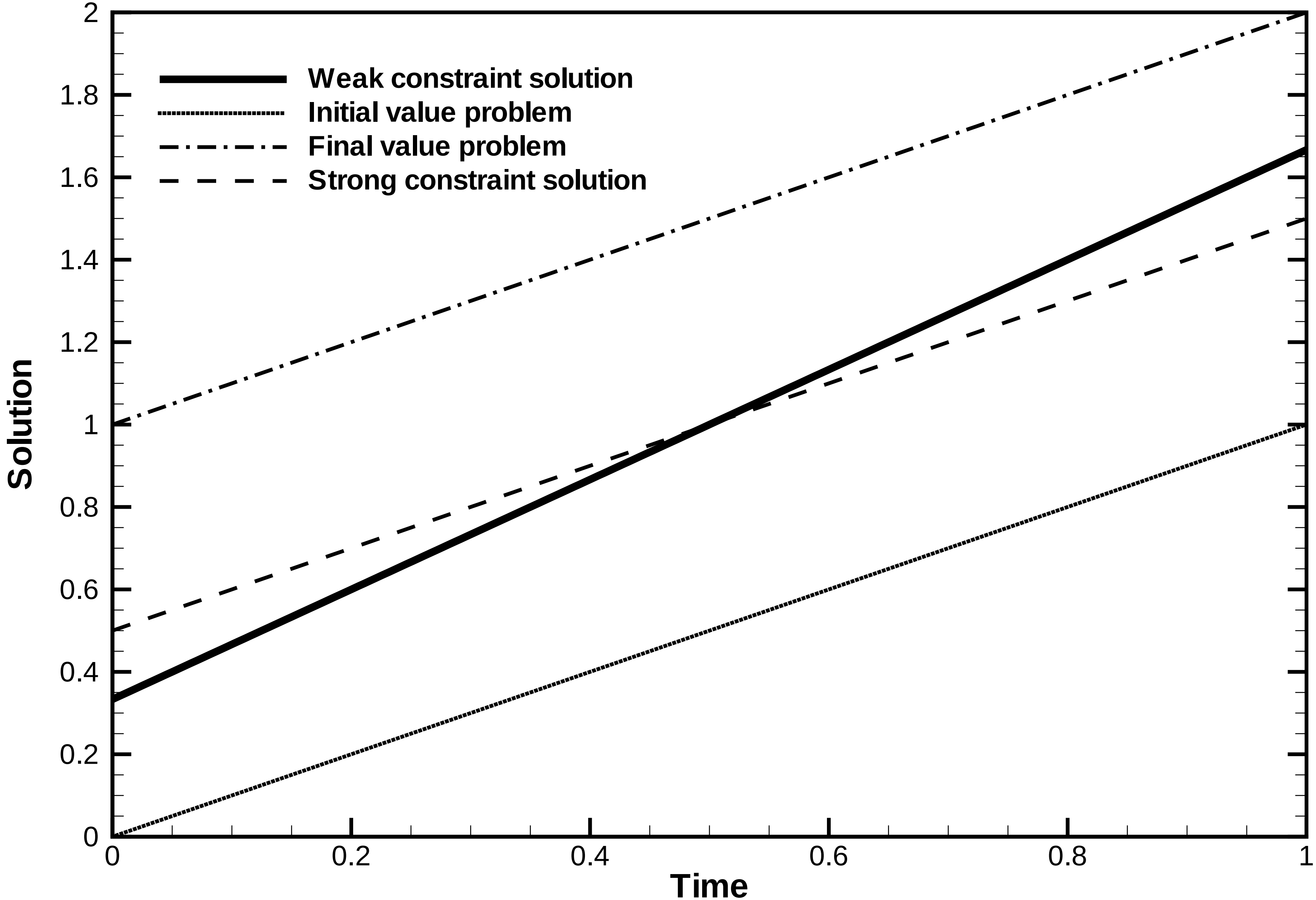
Solution after 30 updates $t = 150.0$



Kalman Filter Example: Case B



Inverse problem revisited



KF solution

What is the KF solution for the linear inverse problem?

- Solve initial value problem until $t = 1$.
- The predicted error variance equals $2C$ at $t = 1$.
- Update at $t = 1$ is

$$
\begin{aligned}
\psi^{\mathrm{a}} &= \psi^{\mathrm{f}} + \frac{C^{\mathrm{f}}_{\psi\psi}}{C_{\epsilon\epsilon} + C^{\mathrm{f}}_{\psi\psi}}(d - \psi^{\mathrm{f}}) \\
&= 1 + \frac{2C}{C + 2C}(2 - 1) \\
&= 5/3
\end{aligned}
$$

- KF solution at final time equals weak constraint variational solution.

Nonlinear dynamics

- Derivation of Extended Kalman Filter (EKF)

Nonlinear scalar model

$$\psi_k^{\mathrm{t}} = F(\psi_{k-1}^{\mathrm{t}}) + q_{k-1},$$
$$\psi_k^{\mathrm{f}} = F(\psi_{k-1}^{\mathrm{a}}),$$
$$\psi_k^{\mathrm{t}} - \psi_k^{\mathrm{f}} = F(\psi_{k-1}^{\mathrm{t}}) - F(\psi_{k-1}^{\mathrm{a}}) + q_{k-1}.$$

Use Taylor expansion

$$\begin{aligned} F(\psi_{k-1}^{\mathrm{t}}) &= F(\psi_{k-1}^{\mathrm{a}}) + F'(\psi_{k-1}^{\mathrm{a}})(\psi_{k-1}^{\mathrm{t}} - \psi_{k-1}^{\mathrm{a}}) \\ &\quad + \frac{1}{2}F''(\psi_{k-1}^{\mathrm{a}})(\psi_{k-1}^{\mathrm{t}} - \psi_{k-1}^{\mathrm{a}})^2 + \cdots . \end{aligned}$$

EKF: Derivation

Difference becomes

$$
\begin{aligned}
\psi_k^{\mathrm{t}} - \psi_k^{\mathrm{f}} &= F'(\psi_{k-1}^{\mathrm{a}})(\psi_{k-1}^{\mathrm{t}} - \psi_{k-1}^{\mathrm{a}}) \\
&\qquad + \frac{1}{2} F''(\psi_{k-1}^{\mathrm{a}})(\psi_{k-1}^{\mathrm{t}} - \psi_{k-1}^{\mathrm{a}})^2 + \cdots + q_{k-1}.
\end{aligned}
$$

By squaring and taking the expectation we get

$$
\begin{aligned}
C_{\psi\psi}^{\mathrm{f}}(t_k) &= \overline{(\psi_k^{\mathrm{t}} - \psi_k^{\mathrm{f}})^2} \\
&= \overline{(\psi_{k-1}^{\mathrm{t}} - \psi_{k-1}^{\mathrm{a}})^2} (F'(\psi_{k-1}^{\mathrm{a}}))^2 + \frac{1}{2} \overline{(\psi_{k-1}^{\mathrm{t}} - \psi_{k-1}^{\mathrm{a}})^3} F'(\psi_{k-1}^{\mathrm{a}}) F''(\psi \\
&+ \frac{1}{4} \overline{(\psi_{k-1}^{\mathrm{t}} - \psi_{k-1}^{\mathrm{a}})^4} (F''(\psi_{k-1}^{\mathrm{a}}))^2 + \cdots + C_{qq}(t_{k-1}).
\end{aligned}
$$

EKF: Error evolution

Close by discarding high order moments to get

$$\psi_k^{\mathrm{f}} = F(\psi_{k-1}^{\mathrm{a}}),$$

$$C_{\psi\psi}^{\mathrm{f}}(t_k) \simeq C_{\psi\psi}^{\mathrm{a}}(t_{k-1})\left(F'(\psi_{k-1}^{a})\right)^2 + C_{qq}(t_{k-1}),$$

together with standard analysis equations.

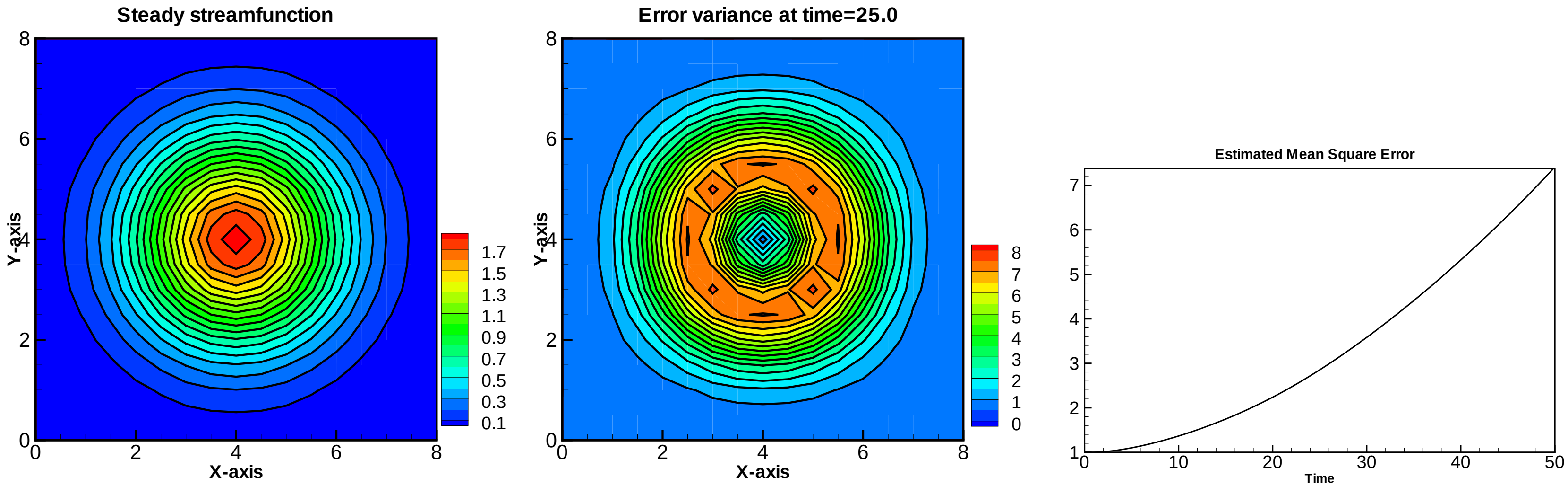
Example of EKF

Nonlinear quasi-geostrophic model

- Steady stream function solution.
- Curved and sheared flow.
- Supports instability.
- Initial variance is 1.0.

Example of EKF

Results (from Evensen 1992):



- Linear closure approximation not valid!
- Leads to linear instability and exponential error growth.

EKF: Summary

- KF is optimal linear filter method!
- EKF applies closure approximation in error covariance equation.
- Does not work for strongly nonlinear models.
  - May lead to linear instabilities in error covariance equation.
  - To simple closure.
- Furthermore, the model equation is not the correct one!

EKF: Equation for the mean

- EKF integrates equation for “central forecast”.
- For nonlinear dynamics the central forecast and mean differs.
- An equation for the mean is easily derived as

$$\overline{\boldsymbol{\psi}_k} = \boldsymbol{F}(\overline{\boldsymbol{\psi}_{k-1}}) + \frac{1}{2}\mathcal{H}_{k-1}\boldsymbol{C}_{\psi\psi}(t_{k-1}) + \cdots .$$

with $\mathcal{H}$ the Hessian of F.

Kalman filtering summary

- Cost of KF/EKF is
  1. storage of $\mathcal{O}(n^2)$ elements,
  2. integration of $2n$ models,
  3. implementation of tangent linear model in EKF,
  4. implementation of Hessian operator and evaluation of $\mathcal{H}_{k-1}\boldsymbol{C}_{\psi\psi}(t_{k-1})$ if equation for the mean is used.
- Consistency limited by
  1. linear equation for error covariance evolution,
  2. normally the use of the central forecast equation.