The inverse problem

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Model equations

$$
\begin{aligned}
\frac{\partial \boldsymbol{\psi}}{\partial t} &= \boldsymbol{g}(\boldsymbol{\psi}), \\
\boldsymbol{\psi}|_{t_0} &= \boldsymbol{\Psi}_0,
\end{aligned}
\tag{1}
$$

$\Longrightarrow$ Well-posed problem with unique deterministic solution.

- $\boldsymbol{\psi}(\boldsymbol{x}, t)$ is model state vector.

Model equations and measurements

$$\frac{\partial \boldsymbol{\psi}}{\partial t} = \boldsymbol{g}(\boldsymbol{\psi}),$$
$$\boldsymbol{\psi}\big|_{t_0} = \boldsymbol{\Psi}_0,$$
$$\mathcal{M}\boldsymbol{\psi} = \boldsymbol{d}.$$

$\Longrightarrow$ Over-determined problem with no solution.

Example of direct measurement of $\psi(t)$:

$$\psi(t_i) = \mathcal{M}_i\psi = \int \delta(t - t_i)\psi(t)dt.$$

Allow for errors

Assume stochastic errors $\boldsymbol{q}(\boldsymbol{x},t)$, $\boldsymbol{a}(\boldsymbol{x})$ and $\boldsymbol{\epsilon}$:

$$\frac{\partial \boldsymbol{\psi}}{\partial t} = \boldsymbol{g}(\boldsymbol{\psi}) + \boldsymbol{q},$$

$$\boldsymbol{\psi}|_{t_0} = \boldsymbol{\Psi}_0 + \boldsymbol{a},$$

$$\mathcal{M}\boldsymbol{\psi}^{\mathrm{t}} = \boldsymbol{d} + \boldsymbol{\epsilon}.$$

$\Longrightarrow$ Infinitively many solutions.

- Must specify statistics for error terms!
- Least squares problem.
- Find estimate for ψ which “minimizes” errors.

State estimation

"Find an estimate of the state given a dynamical model and measurements."

- Standard data assimilation problem.
- Minimize errors in model and measurements.
- Solved by e.g. adjoint, representer or Kalman filter methods.

Simple example

Given the model

$$
\begin{aligned}
\frac{d\psi}{dt} &= 1, \\
\psi(0) &= 0, \\
\psi(1) &= 2,
\end{aligned}
$$

- Overdetermined.
- No solution.

Allowing for errors

Relax model and conditions

$$\frac{d\psi}{dt} = 1 + q,$$
$$\psi(0) = 0 + a,$$
$$\psi(1) = 2 + b.$$

- Underdetermined.
- Infinitively many solutions.

Statistical assumption

Statistical null hypothesis, $\mathcal{H}_0$:

$$
\begin{aligned}
\overline{q(t)} &= 0, & \overline{q(t_1)q(t_2)} &= C_0\delta(t_1 - t_2), & \overline{q(t)a} &= 0,\\
\overline{a} &= 0, & \overline{a^2} &= C_0, & \overline{ab} &= 0,\\
\overline{b} &= 0, & \overline{b^2} &= C_0, & \overline{q(t)b} &= 0.
\end{aligned}
$$

Makes it possible to seek a solution which:

- is close to the conditions,
- almost satisfies the model,

by minimizing error terms.

Penalty function

- Define penalty function

$$\mathcal{J}[\psi] = W_0 \int_0^1 \left(\frac{d\psi}{dt} - 1 \right)^2 dt + W_0 \big(\psi(0) - 0\big)^2 + W_0 \big(\psi(1) - 2\big)^2$$

  with $W_0 = C_0^{-1}$.

- Then ψ is an extremum if

$$\delta \mathcal{J}[\psi] = \mathcal{J}[\psi + \delta\psi] - \mathcal{J}[\psi] = \mathcal{O}\big(\delta\psi^2\big)$$

  when $\delta\psi \to 0$.

Variation of penalty function

We have

$$\mathcal{J}[\psi + \delta\psi] = W_0 \int_0^1 \left(\frac{d\psi}{dt} - 1 + \frac{d\delta\psi}{dt} \right)^2 dt$$
$$+ W_0 \big(\psi(0) - 0 + \delta\psi(0)\big)^2 + W_0 \big(\psi(1) - 2 + \delta\psi(1)\big)^2$$

and we must have

$$\int_0^1 \frac{d\delta\psi}{dt} \left(\frac{d\psi}{dt} - 1 \right) dt + \delta\psi(0)\big(\psi(0) - 0\big) + \delta\psi(1)\big(\psi(1) - 2\big) = 0,$$

From integration by part we get

$$\delta\psi \left(\frac{d\psi}{dt} - 1 \right)\Big|_0^1 - \int_0^1 \delta\psi \frac{d^2\psi}{dt^2} dt + \delta\psi(0)\big(\psi(0) - 0\big) + \delta\psi(1)\big(\psi(1) - 2\big) = 0.$$

Minimium of penalty function

This gives the following system of equations

$$\delta\psi(0)\left(-\frac{d\psi}{dt}+1+\psi\right)\bigg|_{t=0}=0,$$

$$\delta\psi(1)\left(\frac{d\psi}{dt}-1+\psi-2\right)\bigg|_{t=1}=0,$$

$$\int_0^1 \delta\psi\left(\frac{d^2\psi}{dt^2}\right)dt=0,$$

or since $\delta\psi$ is arbitrary....

Euler-Lagrange equation

The Euler–Lagrange equation

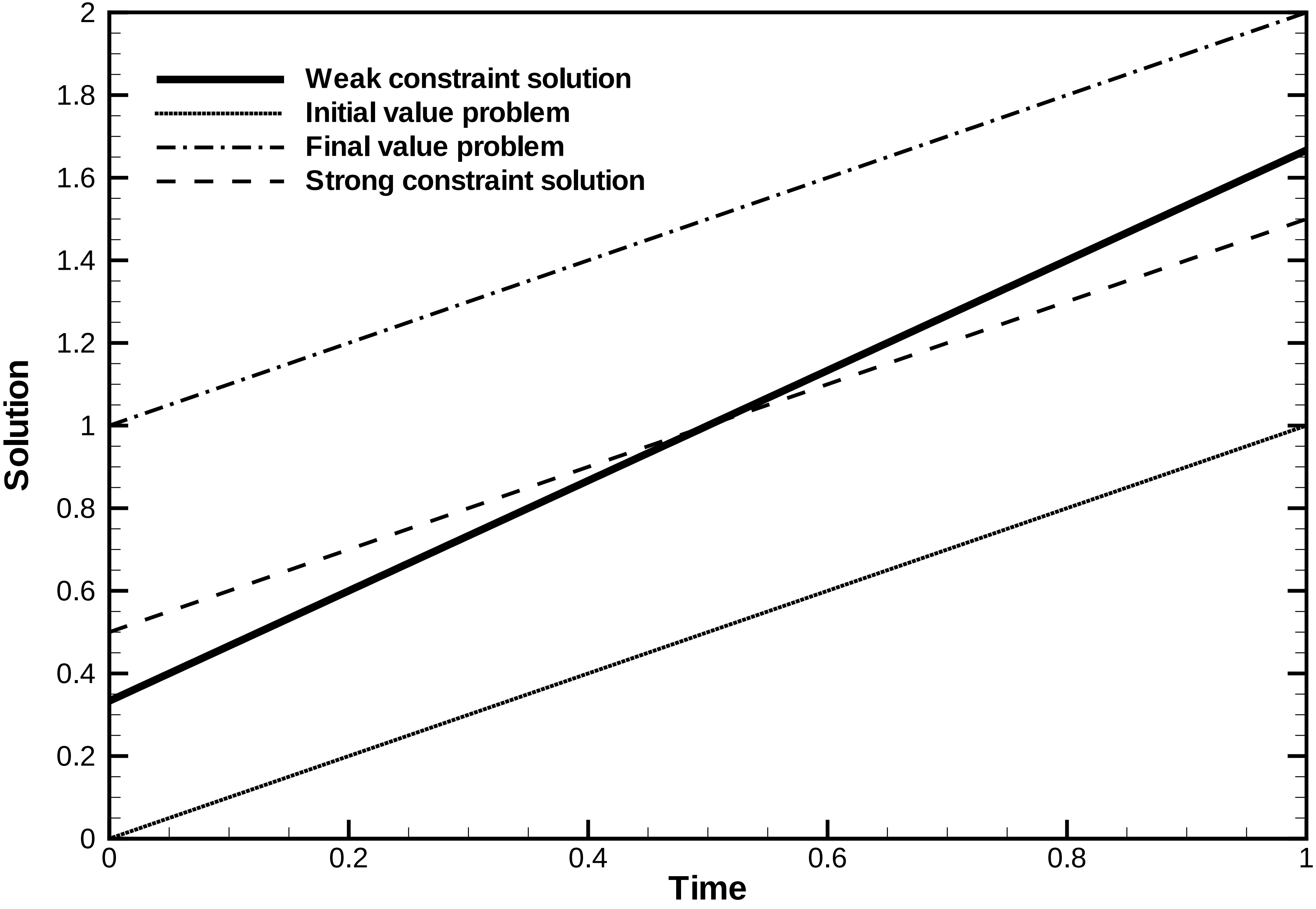
$$\frac{d\psi}{dt} - \psi = 1 \qquad \text{for } t = 0,$$
$$\frac{d\psi}{dt} + \psi = 3 \qquad \text{for } t = 1,$$
$$\frac{d^2\psi}{dt^2} = 0.$$

- Elliptic boundary value problem in time.
- It has a unique solution.

$$\psi = c_1 t + c_2,$$

with $c_1 = 4/3$ and $c_2 = 1/3$.

Results



Weak constraint solution
Initial value problem
Final value problem
Strong constraint solution
Solution
Time
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Summary

- Additional data makes problem over determined
- Allowing for errors gives variational inverse problem
- Weigthed least squares solution
- Solution almost satisfy dynamics and data