Pipe networks: coupling constants in a junction for the isentropic Euler equations

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Introduction

• Heat exchangers with parallel channels
  - Pressure drop dependent on phase composition
  - May result in wrong flow distribution
  - May cause instabilities
• Need to model junctions dynamically
  - F.ex. main inlet pipe to tubes
  - Repartition of mass flow in each pipe
Overview

- Numerical modelling of flow in pipes
- The model: Isentropic Euler equations
- The Riemann problem
  - Mathematical notions
  - Coupling of the pipes through the generalised Riemann problem
  - The right coupling condition
- Physical interpretation
- Numerical examples
  - Entropy condition at the junction
  - Conservation of energy in junctions
Modelling of flow in pipes

- One-dimensional models

![Diagram of a pipe with flow direction](image-url)
Modelling of flow in pipes

- One-dimensional models
- Finite-volume method

![Diagram showing a cell and its average]
Modelling of flow in pipes

- One-dimensional models
- Finite-volume method
- Boundary conditions with ghost cells

Ghost cell  Cell  Cell average
Junctions

- Several pipe connected together
Junctions

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- How to represent a junction?
  - Pipes: one-dimensional models
  - Junction: multi-dimensional flow
Junctions

- Several pipe connected together
- How to represent a junction?
  - Pipes: one-dimensional models
  - Junction: multi-dimensional flow
- Describe the junction with ghost cells
  - Solve the pipes as independent domains
  - Set the ghost cells
- The junction has no volume
The model

Isentropic Euler equations

- Conservation of mass:
  \[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0 \]

- Conservation of momentum:
  \[ \frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x} \left( \rho v^2 + p \right) = 0 \]

Equation of state for isentropic flow

\[ p = k \rho^\gamma \quad \text{(then, } s(x,t) = \text{const)} \]

In quasilinear form

\[ \frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0 \]

\[ U = \begin{pmatrix} \rho \\ \rho v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ a^2 - v^2 & 2v \end{pmatrix} \]

where \( a^2 = \left( \frac{\partial p}{\partial \rho} \right)_s = \frac{\gamma p}{\rho} \)
The Riemann problem

• Eigenvalues of the Jacobian
  \( v - a \), \( v + a \)
  with eigenvectors
  \[
  \begin{pmatrix} 1 \\ v - a \end{pmatrix}, \begin{pmatrix} 1 \\ v + a \end{pmatrix}
  \]

• Shock or rarefaction wave
The Riemann problem

- Eigenvalues of the Jacobian $\nu - a$, $\nu + a$
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- Shock or rarefaction wave

- The star-state \( U^* \)
  - Related to \( U_L \) and \( U_R \)
    through the wave of family 1 and 2, respectively
The Riemann problem

Equations for the waves of the second family

\[ v^* (\rho^*; \rho_R, v_R)_{R2} = v_R \]
\[ + \frac{2\sqrt{yk}}{y-1} \left( \rho^* \frac{y-1}{2} - \rho_R \frac{y-1}{2} \right), \quad 0 < \rho^* \leq \rho_R \]
\[ v^* (\rho^*; \rho_R, v_R)_{S2} = v_R \]
\[ + \sqrt{\frac{k(\rho^* - \rho_R)(\rho^* y - \rho_R y)}{\rho^* \rho_R}}, \quad \rho^* > \rho_R \]

Hugoniot Locus: points connected by a curve separated by one wave
The Riemann problem

Pipe initialised with a Riemann problem
The Riemann problem

After evolution. At the initial discontinuity, the $U^*$-state
The Riemann problem

• Pipe cut in two at the initial discontinuity, 
  - $U^*$ as initial value in the boundary cells.
• The same waves propagate to the left and to the right as in the whole pipe.
• The two half-pipes are coupled using the $U^*$-state.
The generalised Riemann problem

• We can couple 2 pipes and get the same behaviour as if we had a single pipe.
  - $U^*$-state is the only information needed to couple them
• Can we find a $U^*$-state for more than 2 pipes?
The generalised Riemann problem

• We can couple 2 pipes and get the same behaviour as if we had a single pipe.
  – $U^*$-state is the only information needed to couple them
• Can we find a $U^*$-state for more than 2 pipes?
  – Yes, but slightly more complicated
The generalised Riemann problem

- Each pipe section has its own $U^*_k$-state,
- following the conditions:

\[ U^*_1, \ldots, U^*_N \text{ are related together} \Rightarrow \text{Junction condition} \]
The generalised Riemann problem

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  - Each $U_k^*$ is related to the initial $U_k$ in the $k^{th}$ section
    ⇒ Wave equation (positive speed or stationary)
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    ⇒ Wave equation (positive speed or stationary)
  - $U_1^*, \ldots, U_N^*$ are related together
    ⇒ Junction condition
- Reminder: the junction has no volume
The coupling conditions

Two conditions are needed to close the system

\[
\sum_{k=1}^{N} (A_k \rho^*_k v^*_k) = 0
\]

What about momentum?

– It is a vector quantity
– Conserved as a scalar in 1D-models
– Conserved as a vector in 3D

Junctions are 3D objects
The coupling conditions

Two conditions are needed to close the system

• Conservation of mass is an obvious condition

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\sum_{k=1}^{N} \left( A_k \rho_k^* \nu_k^* \right) = 0
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Momentum condition expressed as a coupling constant

For all $k$, $\mathcal{H} \left( \rho_k^*, \nu_k^* \right) = \tilde{\mathcal{H}}$

- The quantity $\mathcal{H} \left( \rho_k^*, \nu_k^* \right)$, function of the $U_k^*$-state,
- is equal to a unique $\tilde{\mathcal{H}}$ for all the pipe sections.
- It is called the coupling constant.
The coupling conditions

Momentum condition expressed as a coupling constant

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What should be the coupled quantity $H\left(\rho, v\right)$?

- The pressure?
  $$H_p(\rho, v) = p = k\rho^\gamma$$
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  ✗

- The momentum flux? (Conservation of momentum)
  
  \[ \mathcal{H}_{MF}(\rho, v) = \rho v^2 + p = \rho v^2 + k\rho^y \]
The coupling conditions

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The coupling conditions

Momentum is a vector quantity

- If the flow derives from a scalar potential field, it is possible to return to a scalar coupling condition
- The Bernoulli invariant is the scalar potential
- It is the stagnation enthalpy $h + \frac{1}{2}v^2$
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In terms of coupled quantity $\mathcal{H}(\rho, v)$

$$\mathcal{H}_{BI}(\rho, v) = h + \frac{1}{2}v^2 = \frac{ky}{\gamma - 1} \rho^{\gamma - 1} + \frac{1}{2}v^2$$

- The flow is virtually brought to a rest at the junction.
- It then flows in the pipe sections from the stagnation state, independently of the other sections.
The coupling conditions

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The coupling conditions

To summarise

- Need to find the stagnation enthalpy (identical for all $U_k^*$)
- Such that, in each pipe section $k$, $U_k^*$ and $U_k$ are related by the relevant wave equation
- One stagnation enthalpy, $N$ different $U_k^* = (\rho_k^*, u_k^*)$
Numerical results

• Examples of why the Bernoulli-based coupling is right
  – Entropy condition at an isolated junction
  – Energy in a closed system
• Simulated with a Roe scheme
Entropy condition in a junction

- Three pipe sections
  - Junction at one end
  - Extrapolation at the other end (infinite pipe)
- Initialised with $v_3$ either 0 m/s or 50 m/s.
- The junction reaches steady state

<table>
<thead>
<tr>
<th>Section</th>
<th>Pressure (bar)</th>
<th>Velocity (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 1</td>
<td>1</td>
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</tr>
<tr>
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<td>1.5</td>
<td>0</td>
</tr>
<tr>
<td>Section 3</td>
<td>1.4</td>
<td>$v_3$</td>
</tr>
</tbody>
</table>
Entropy condition in a junction

Entropy condition

\[ \sigma_J = \sum_{k=1}^{N} A_k \rho_k^* \nu_k^* \left( h_k^* + \frac{1}{2} \nu_k^* \right) \leq 0 \]

Value of the entropy condition at steady state

<table>
<thead>
<tr>
<th>Equal pressure</th>
<th>Momentum flux</th>
<th>Stagnation enthalpy</th>
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</thead>
<tbody>
<tr>
<td>( \nu_3 = 0 \text{ m/s} )</td>
<td>( 1.1 \times 10^5 \text{ J/s} )</td>
<td>( -8.2 \times 10^4 \text{ J/s} )</td>
</tr>
<tr>
<td>( \nu_3 = 50 \text{ m/s} )</td>
<td>( -6.6 \times 10^4 \text{ J/s} )</td>
<td>( 9.8 \times 10^4 \text{ J/s} )</td>
</tr>
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</table>
Energy balance in a closed system

- Three pipe sections
  - Junctions at each end
- The system’s energy content is followed
  - Should decrease, because shocks dissipate energy

Initial conditions

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Energy balance in a closed system

$\Rightarrow$ Wrong coupling constants in the junctions cause a non-physical production of energy.
Summary

• Using the wrong coupling quantity breaks the laws of physics
  – In particular, energy conservation
• Rather theoretical derivation, proved for isentropic Euler equations
• Coupling multiphase flow models with real thermodynamics
  – Physical interpretation hints that stagnation enthalpy should play a role
• Energy is a scalar quantity: same conservation principle as mass?

Proofs in:
Acknowledgment

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