An A Posteriori Error Estimate for Discontinuous Galerkin Methods

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Outline

- We present an a posteriori error estimate in a **mesh dependent energy norm** for a general class of dG methods for elliptic problems.
- *Energy norm a posteriori error estimation for discontinuous Galerkin methods* Comput. Methods Appl. Mech. Engrg., Volume 192, Issues 5-6, 2003, Pages 723-733.
- Joint work with Roland Becker and Peter Hansbo.

A model problem

Find $u : \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
-\nabla \cdot \sigma(u) &= f && \text{in } \Omega \\
u &= g_D && \text{on } \Gamma_D \\
\sigma_n(u) := n \cdot \sigma(u) &= g_N && \text{on } \Gamma_N
\end{aligned}
$$

where Ω denotes a bounded polygonal domain in $\mathbb{R}^d$, $d = 2$ or 3, with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, and $\sigma(u)$ is defined by

$$\sigma(u) = \nabla u$$

The dG method

Find $U \in \mathcal{V}$ such that

$$a(U, v) = (f, v) \qquad \text{for all } v \in \mathcal{V}$$

Here the bilinear form is defined by

$$a(v, w) = \sum_K (\nabla v, \nabla w)_K - \sum_E (\langle n \cdot \nabla v \rangle, [w])_E$$
$$+ \alpha \sum_E ([v], \langle n \cdot \nabla w \rangle)_E + \beta \sum_E (h_E^{-1} [v], [w])_E$$

with α and β real parameters.

Conservation property

Introducing the discrete normal flux

$$\Sigma_n(U) := \begin{cases} \langle \sigma_n(U) \rangle - \beta h^{-1}[U] & \text{on } \partial K \setminus \Gamma, \\ \sigma_n(U) - \beta h^{-1}(U - g_D) & \text{on } \partial K \cap \Gamma_D, \\ g_N & \text{on } \partial K \cap \Gamma_N \end{cases}$$

we obtain the **elementwise conservation** law

$$\boxed{\int_K f + \int_{\partial K} \Sigma_n(U) = 0}$$

or

$$\boxed{\int_{\partial K} \Sigma_n(U) = \int_{\partial K} \sigma_n(u)}$$

The energy norm

We introduce the following **mesh dependent** energy norm:

$$|||v|||^2 = |||v|||^2_{\mathcal{K}} + |||v|||^2_{\partial\mathcal{K}}$$

where

$$|||v|||^2_{\mathcal{K}} = \sum_{K\in\mathcal{K}} (\sigma(v), \nabla v)_K$$

$$|||v|||^2_{\partial\mathcal{K}} = \sum_{K\in\mathcal{K}} (h^{-1}[v], [v])_{\partial K\setminus\Gamma}/2 + (h^{-1}v, v)_{\partial K\cap\Gamma_D}$$

The a posteriori error estimate

$$\boxed{|||e|||^2 \leq c \left(\sum_{K \in \mathcal{K}} \rho_K^2 \right)}$$

with c independent of h and element indicator

$$\rho_K^2 = h_K^2 \|f + \nabla \cdot \sigma(U)_K\|_K^2 + h_K \|\Sigma_n(U) - \sigma_n(U)\|_{\partial K \setminus \Gamma_D}^2 + h_K^{-1} \|[U]\|_{\partial K \setminus \Gamma_N}^2$$

with $\Sigma_n(U)$ defined by

$$\Sigma_n(U) = \begin{cases} \langle \sigma_n(U) \rangle - \beta h^{-1}[U] & \text{on } \partial K \setminus \Gamma \\ g_N & \text{on } \partial K \cap \Gamma_N \end{cases}$$

Remarks

- Valid for several different dG methods including the symmetric and nonsymmetric formulations.
- Valid on nonconvex polyhedra in 2D and 3D
- Valid for general boundary conditions and variable coefficients.

Helmholtz decomposition of ∇e

There exists $\phi \in H^1(\Omega)$ and $\chi \in H(\mathsf{curl}\,, \Omega)$ such that

$$\boxed{\nabla e = \nabla \phi + \mathsf{curl}\, \chi}$$

with

$$\phi = 0 \text{ on } \Gamma_D \text{ and } n \cdot \mathsf{curl}\, \chi = 0 \text{ on } \Gamma_N$$

and the stability estimate

$$\boxed{\|\nabla \phi\| + \|\mathsf{curl}\, \chi\| \leq c |||e|||_K}$$

holds.

Remarks: Helmholtz decomp

- In 3D the stronger stability estimate

$$\|\chi\|_{[H^1(\Omega)]^d} \leq |||e|||_{\mathcal{K}}$$

  does not hold when Ω is a nonconvex polyhedron, while it holds in the convex case.
- Helmholtz decomposition used for derivation of a posteriori error estimates for nonconforming methods **Dari-Duran-Padra-Vampa 1996**, convex, **Carstensen-Bartels-Jansche 2002**, nonconvex 3D.

Basic idea of proof

Using the Helmholtz decomposition of ∇e we get:

$$
\begin{aligned}
|||e|||_{\mathcal{K}}^2 &= (\nabla e, \nabla e) \\
&= \sum_{K \in \mathcal{K}} (\nabla e, \nabla \phi)_K + (\nabla e, \mathsf{curl}\,\chi)_K \\
&\leq \text{residual}(U) \times \Big(\|\nabla \phi\| + \|\mathsf{curl}\,\chi\| \Big) \\
&\leq \text{residual}(U) \times |||e|||_{\mathcal{K}}
\end{aligned}
$$

Thus

$$\boxed{|||e|||_{\mathcal{K}} \leq \text{residual}(U)}$$

Error representation: A

$$
\begin{aligned}
\sum_{K\in\mathcal{K}} (\sigma(e), \nabla\phi)_K &= \sum_{K\in\mathcal{K}} (\sigma(e), \nabla(\phi - \pi_0\phi))_K \\
&= \sum_{K\in\mathcal{K}} (f + \nabla\cdot\sigma(U), \phi - \pi_0\phi)_K \\
&\quad + (\sigma_n(u) - \sigma_n(U), \phi - \pi_0\phi)_{\partial K\setminus\Gamma_D} \\
&= \sum_{K\in\mathcal{K}} (f + \nabla\cdot\sigma(U), \phi - \pi_0\phi)_K \\
&\quad + (\Sigma_n(U) - \sigma_n(U), \phi - \pi_0\phi)_{\partial K\setminus\Gamma_D}
\end{aligned}
$$

$\pi_0\phi$ is the piecewise constant L^2-projection of ϕ.

Error representation: A

Note that since $\Sigma_n(U)$ is **conservative** we have

$$(\sigma_n(u), \pi_0\phi)_{\partial K\setminus\Gamma_D} = (\Sigma_n(U), \pi_0\phi)_{\partial K\setminus\Gamma_D}$$

for all K not intersecting Γ_D.

Error representation: B

$$\sum_{K\in\mathcal{K}} (\nabla e, \mathsf{curl}\,\chi)_K = \sum_{K\in\mathcal{K}} (u - U, n \cdot \mathsf{curl}\,\chi)_{\partial K \setminus \Gamma_N}$$

$$= \sum_{K\in\mathcal{K}} (v - U, n \cdot \mathsf{curl}\,\chi)_{\partial K \setminus \Gamma_N}$$

We replaced u by an arbitrary function

$$v \in \mathcal{V}_{g_D} = \{v \in H^1 : v = g_D \text{ on } \Gamma_D\}$$

Error representation: A + B

$$
\begin{aligned}
|||e|||_{\mathcal{K}}^2 = \sum_{K \in \mathcal{K}} (f + \nabla \cdot \sigma(U), \phi - \pi_0 \phi)_K \\
&+ (\Sigma_n(U) - \sigma_n(U), \phi - \pi_0 \phi)_{\partial K \setminus \Gamma_D} \\
&+ (v - U, n \cdot \mathsf{curl}\, \chi)_{\partial K \setminus \Gamma_N} \\
= I + II + III
\end{aligned}
$$

for all $v \in \mathcal{V}_{g_D}$.

- The a posteriori error estimate now follows from estimates of terms I–III.

Estimate of *III*

$$
\begin{aligned}
|(v - U, n \cdot \mathsf{curl}\,\chi)_{\partial K \setminus \Gamma_N}| \\
&\leq \|v - U\|_{H^{1/2}(\partial K \setminus \Gamma_N)} \|n \cdot \mathsf{curl}\,\chi\|_{H^{-1/2}(\partial K \setminus \Gamma_N)} \\
&\leq c\|v - U\|_{H^{1/2}(\partial K \setminus \Gamma_N)} \|\mathsf{curl}\,\chi\|_K \\
&\leq c\|v - U\|_{H^{1/2}(\partial K \setminus \Gamma_N)} |||e|||_{\mathcal{K}}
\end{aligned}
$$

We employed the trace inequality

$$
\|n \cdot \mathsf{curl}\,\chi\|_{H^{-1/2}(\partial K)} \leq c\|\mathsf{curl}\,\chi\|_K
$$

Est. *III*: trace inequality

Trace inequality

$$\|n \cdot \mathsf{curl}\, \chi\|_{H^{-1/2}(\partial K)} \leq c\|\mathsf{curl}\, \chi\|_K$$

follows from

$$\|n \cdot w\|_{H^{-1/2}(\partial K)} \leq c\Big(\|w\|_K + h_K\|\nabla \cdot w\|_K\Big)$$

see Girault-Raviart with $w = \mathsf{curl}\, \chi$. Note that $\nabla \cdot \mathsf{curl}\, \chi = 0$.

Est. *III*: Technical lemma

We finally employ the following technical lemma

$$\boxed{\inf_{v \in \mathcal{V}_{g_D}} \sum_{K \in \mathcal{K}} \|v - U\|^2_{H^{1/2}(\partial K \setminus \Gamma_N)} \leq c \sum_{K \in \mathcal{K}} h_K^{-1} \|[U]\|^2_{\partial K \setminus \Gamma_N}}$$

with constant c independent of h.

- see paper for proof.

Sinusoidal hill

Consider

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \Gamma$$

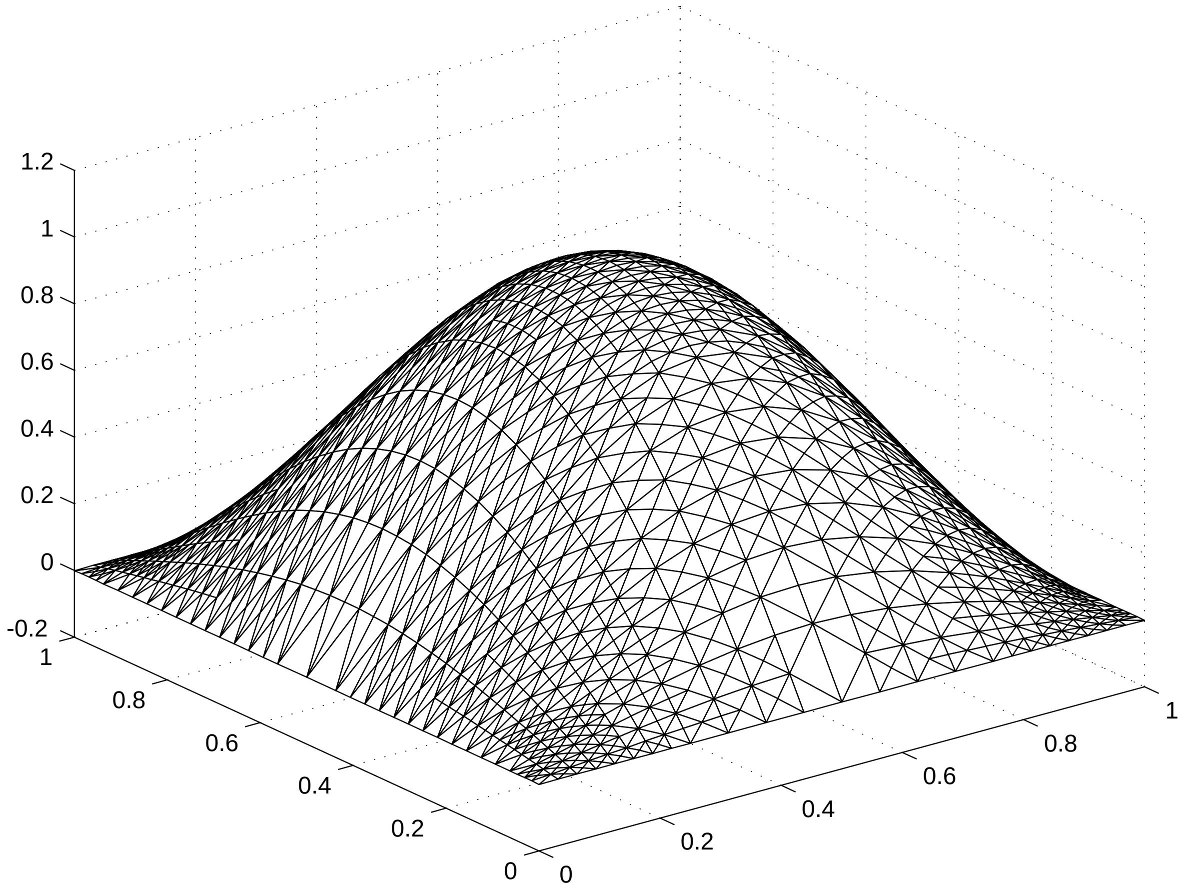
Let

$$\begin{cases} f = 2\,\pi^2 \, \sin(\pi\, x) \, \sin(\pi\, y) \\ \Omega = (0,1) \times (0,1) \end{cases}$$

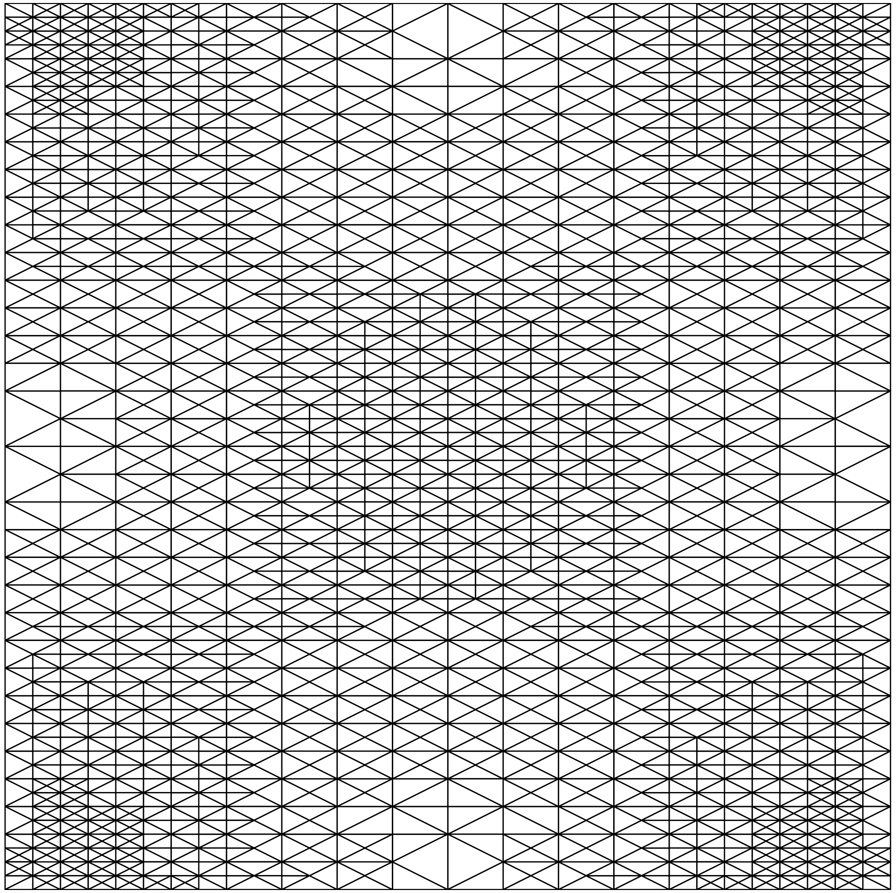
Then

$$u = \sin(\pi\, x) \sin(\pi\, y)$$

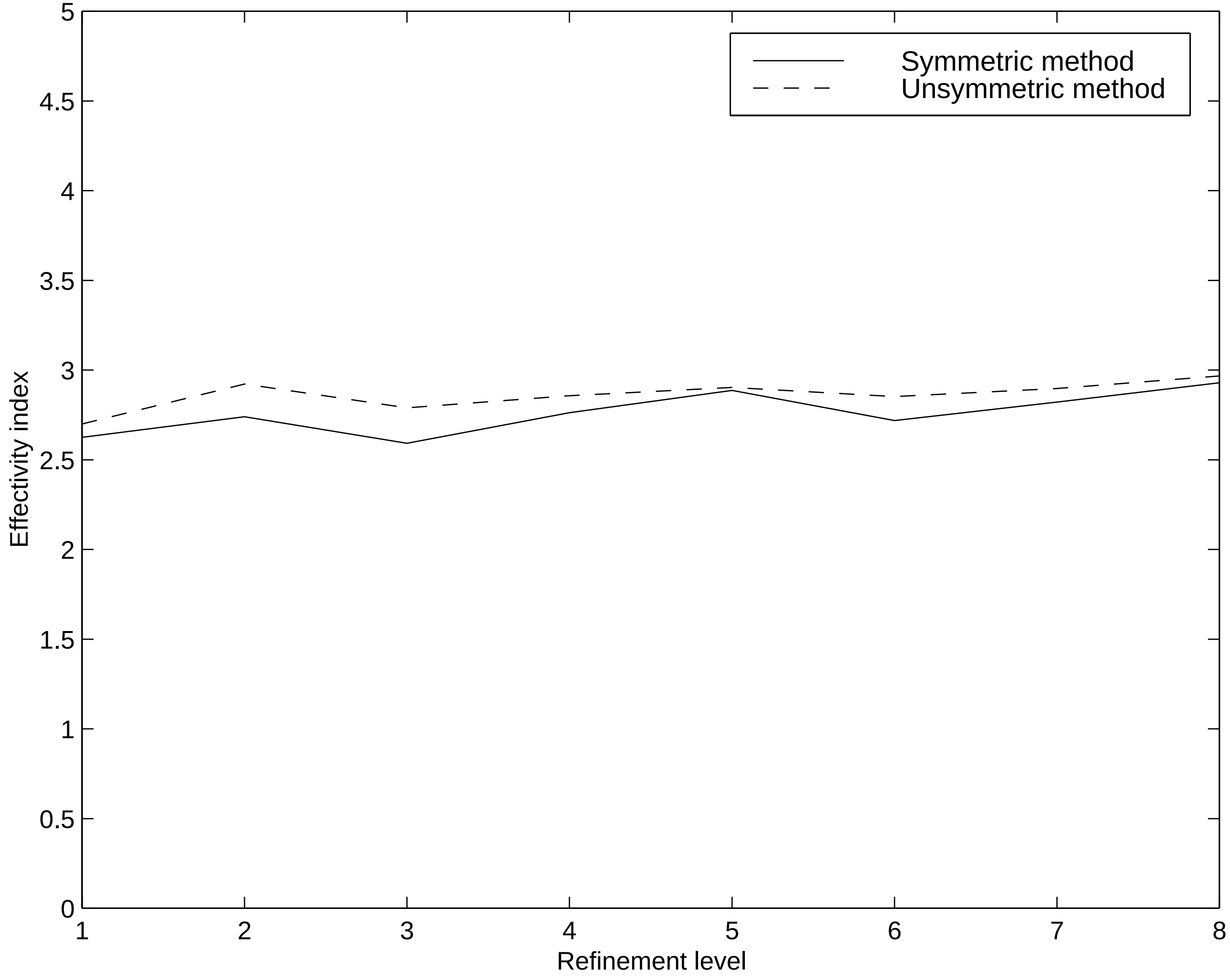
Sinusoidal hill: Solution



Sinusoidal hill: Final mesh



Sinusoidal hill: Effectivity index



Peak function

Consider

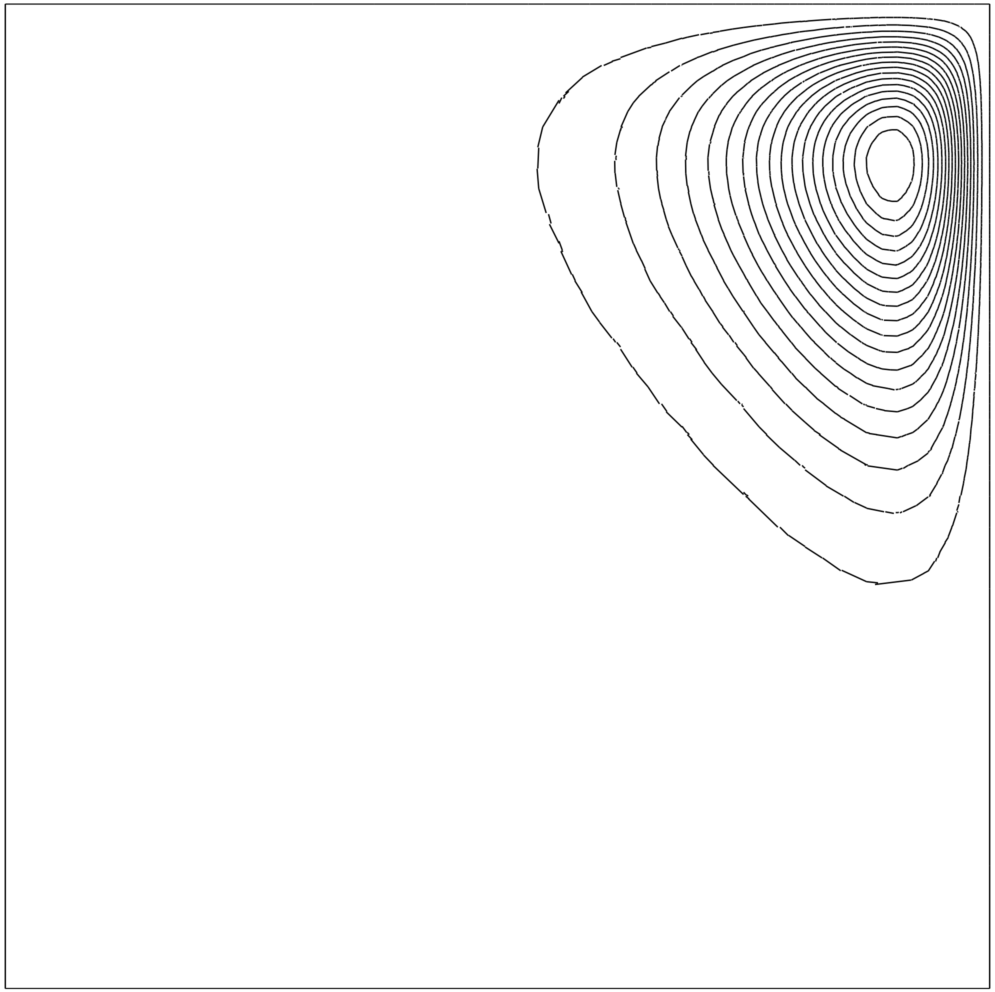
$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \Gamma$$

Chose f such that

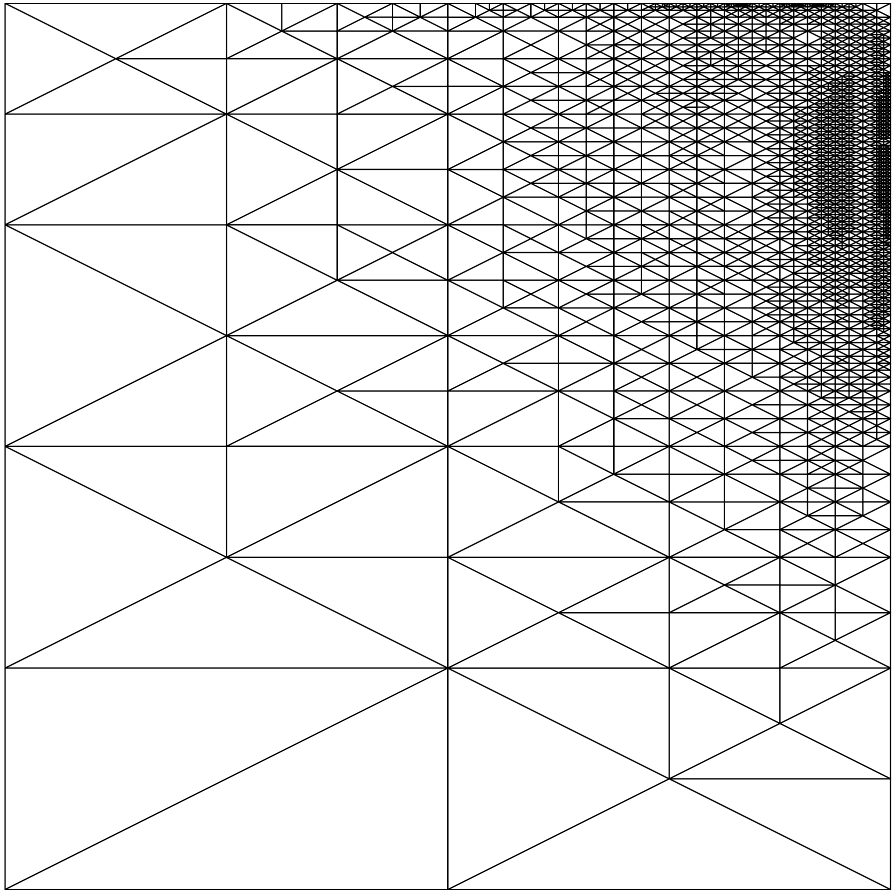
$$u = \frac{e^{10\,x^2+10\,y}\,(1-x)^2\,x^2\,(1-y)^2\,y^2}{2000}$$

on the domain $\Omega = (0,1) \times (0,1)$.

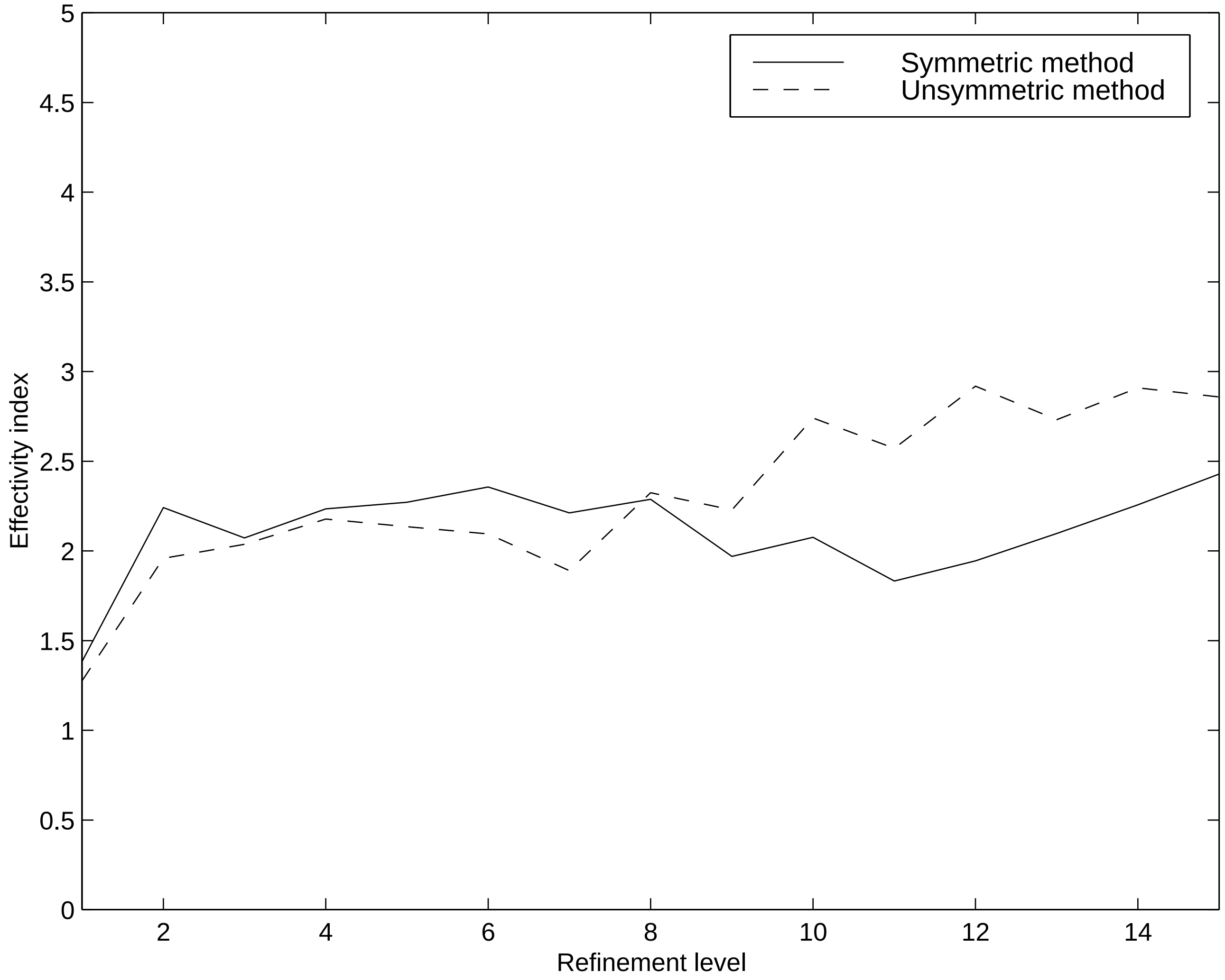
Peak function: Solution



Peak function: Final mesh



Peak function: Effectivity index



Discontinuous Galerkin Methods on Overlapping Grids

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FEM on overlapping grids

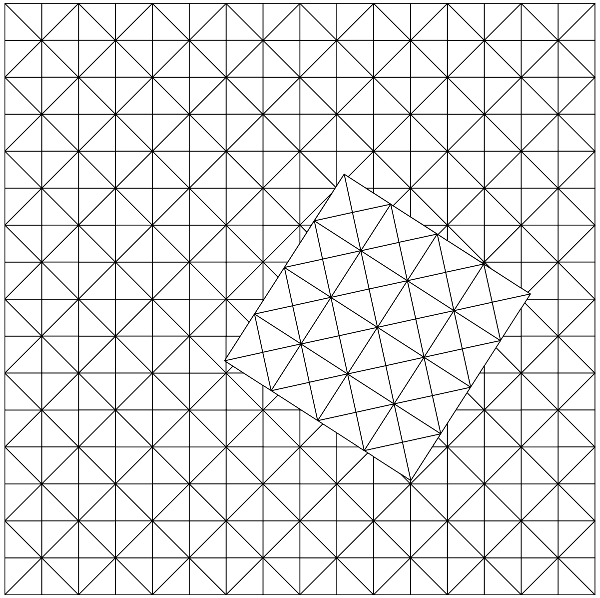
Want to use independent meshes in different regions of the domain for

- Construction of a global mesh for a complex geometry by using overlapping meshes of elementary parts.
- Coupling of unstructured and structured meshes.
- Coupling of boundary fitted meshes to structured or unstructured meshes.

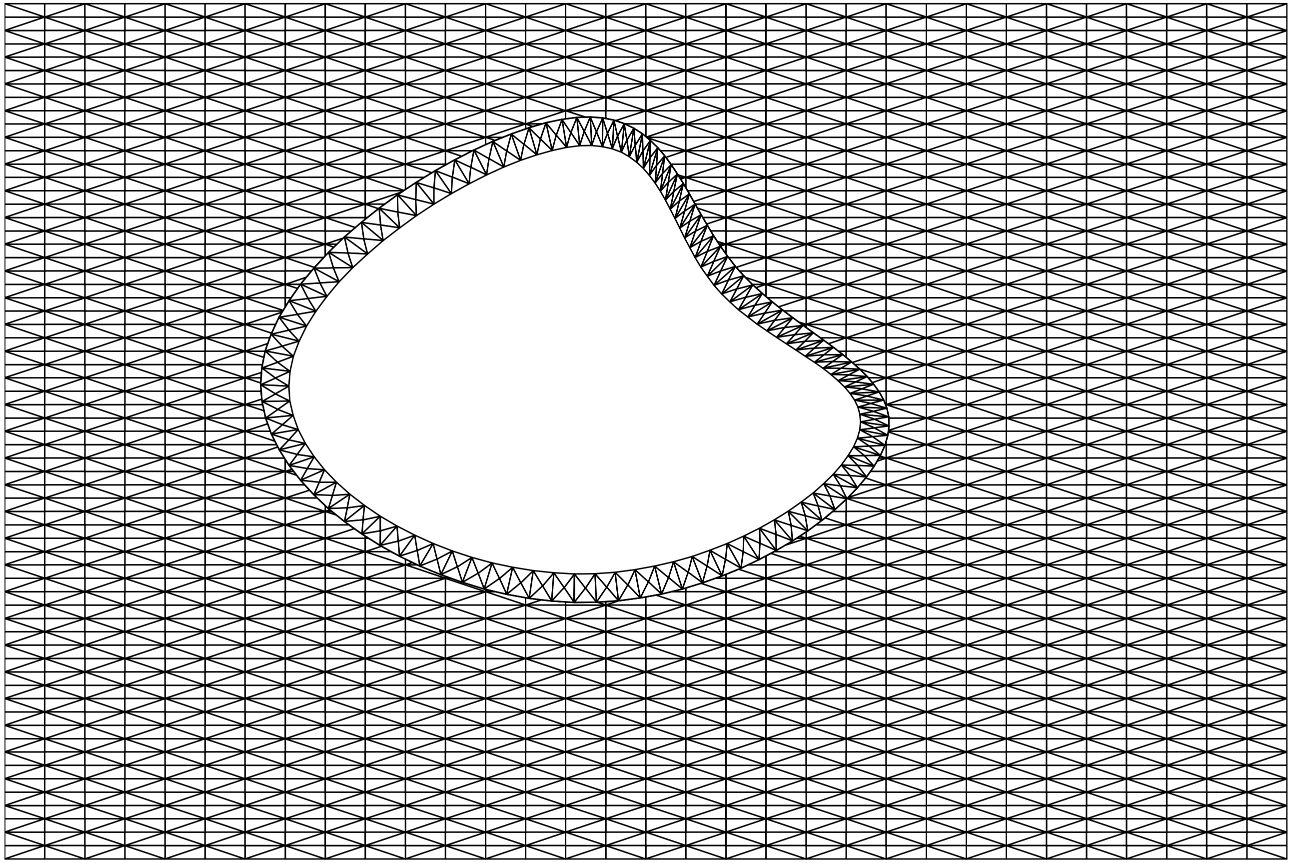
Joint work with A. Hansbo and P. Hansbo.

Overlapping grids

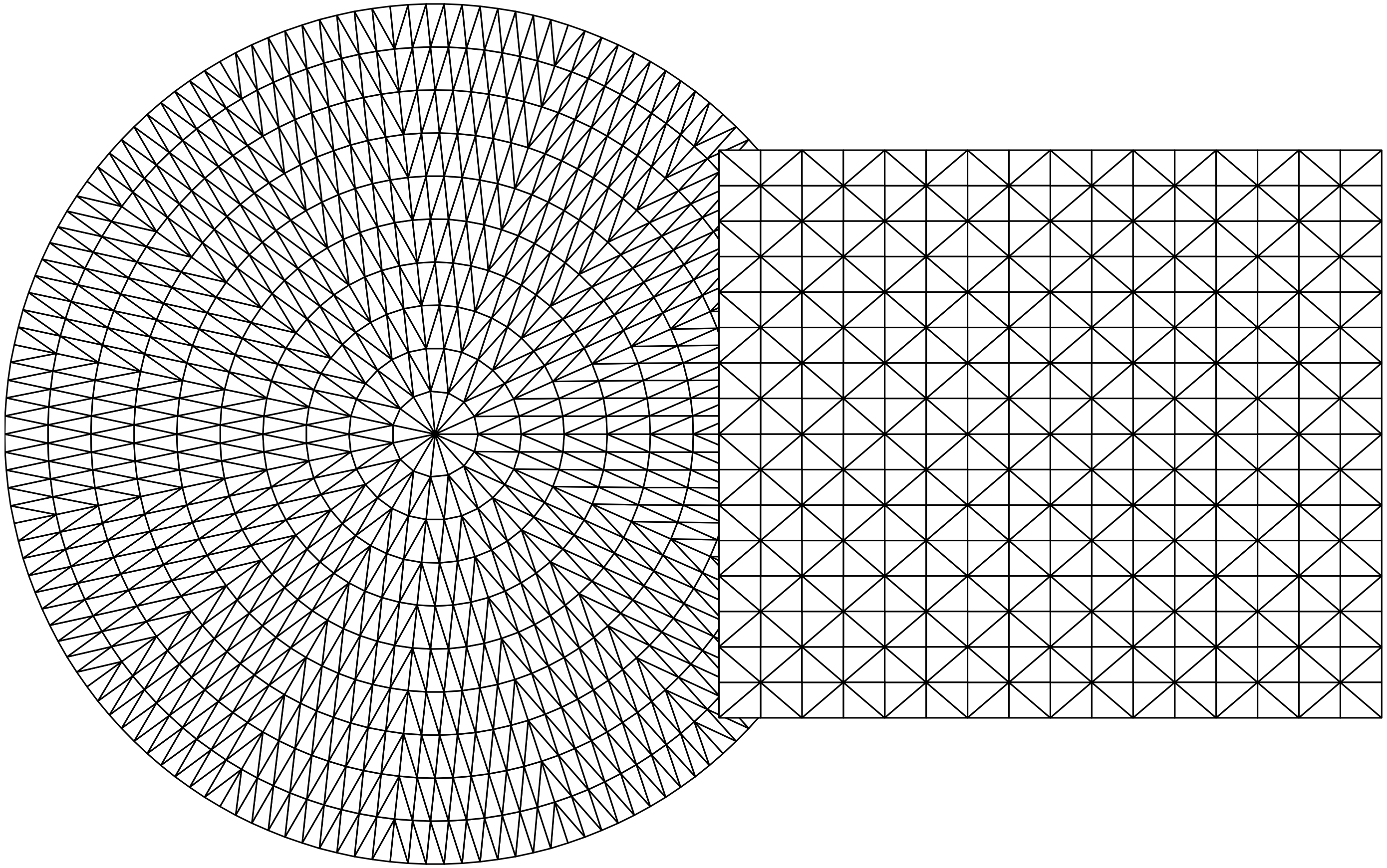
- Ω covered by triangulations $T_{h,1}$ and $T_{h,2}$.
- Γ an **artificial** internal interface **composed of edges from the triangles in one of the meshes** $T_{h,1}$.



Overlapping grids: Example 1



Overlapping grid: Example 2



A simple model problem

Find $u : \Omega \subset \mathbf{R}^d \to \mathbf{R}$ such that

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

This problem has a unique weak solution $u \in H_0^1$ for $f \in H^{-1}$.

A discontinuous space

Let $\mathcal{V}_{c,i}^{*}$ be **continuous** piecewise polynomials on $T_{h,i}$ and

$$\Omega_1 = \cup_{T \in T_{h,1}}, \quad \Omega_2 = \Omega \setminus \Omega_1$$

The discontinuous space is defined by

$$\mathcal{V} = \{ v : v|_{\Omega_1} \in \mathcal{V}_{c,i}, v|_{\Omega_2} = w|_{\Omega_2}, w \in \mathcal{V}_{c,2} \}$$

DG method

Find $U \in \mathcal{V}$ such that

$$\boxed{a(U, v) = (f, v) \quad \text{for all } v \in \mathcal{V}}$$

$$a(U, \phi) = (\nabla U, \nabla \phi)_{\Omega_1 \cup \Omega_2} - (\langle \nabla_{\boldsymbol{n}} U \rangle, [\phi])_\Gamma \\ - ([U], \langle \nabla_{\boldsymbol{n}} \phi \rangle)_\Gamma + (\beta h^{-1} [U], [\phi])_\Gamma,$$

with a **one-sided** approximation of the normal flux:

$$\boxed{\langle \nabla_{\boldsymbol{n}} v \rangle = \nabla_{\boldsymbol{n}} v_1 \qquad \text{on } \Gamma.}$$

Main results

- Stability if the penalty parameter β is large enough.
- Optimal order of convergence in energy and L^2 norm.
- Valid for higher order elements in 2D and 3D and arbitrary intersecions.
- Energy norm a posteriori error estimates.
- Extends to the nonsymmetric case.

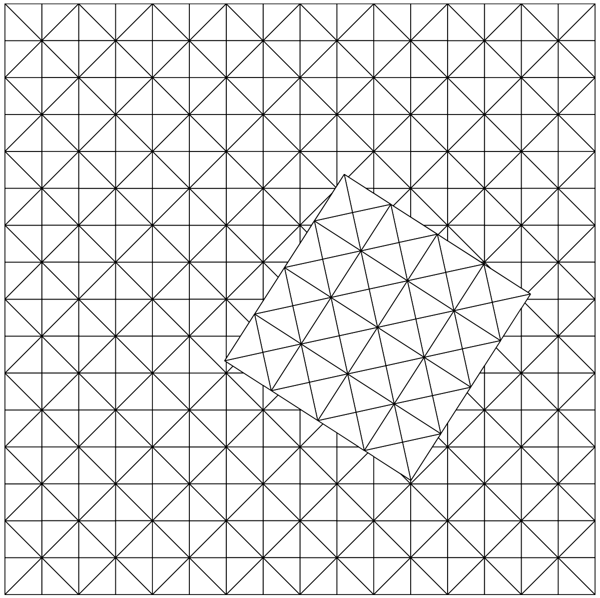
Earlier results

Extends Becker, Stenberg, and Hansbo, where nonmatching meshes with standard mean value flux was used:

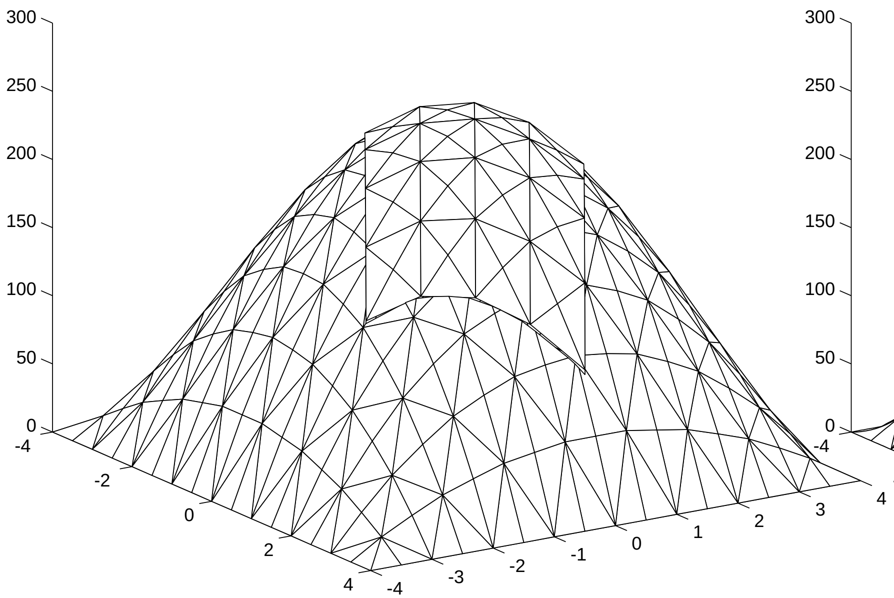
- weaker mesh assumption allowing for composition of arbitrarily overlapping grids (arbitrarily small parts in Ω_2)
- no ad hoc ("saturation") assumptions for the a posteriori result

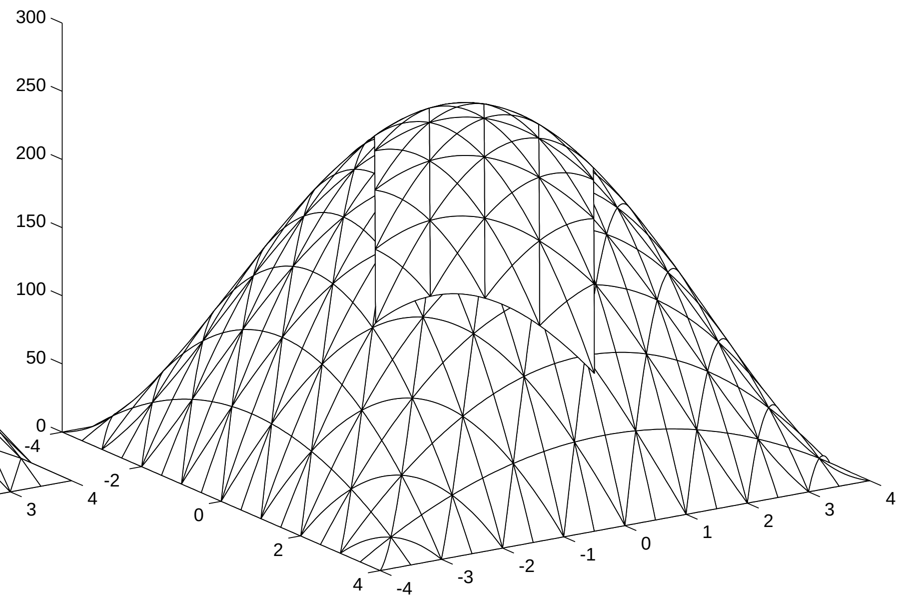
Example

Let $\Omega = (-4, 4) \times (-4, 4)$ and $f = 64 - 2\,x^2 - 2\,y^2$, corresponding to $u = (x - 4)\,(x + 4)\,(y - 4)\,(y + 4)$.



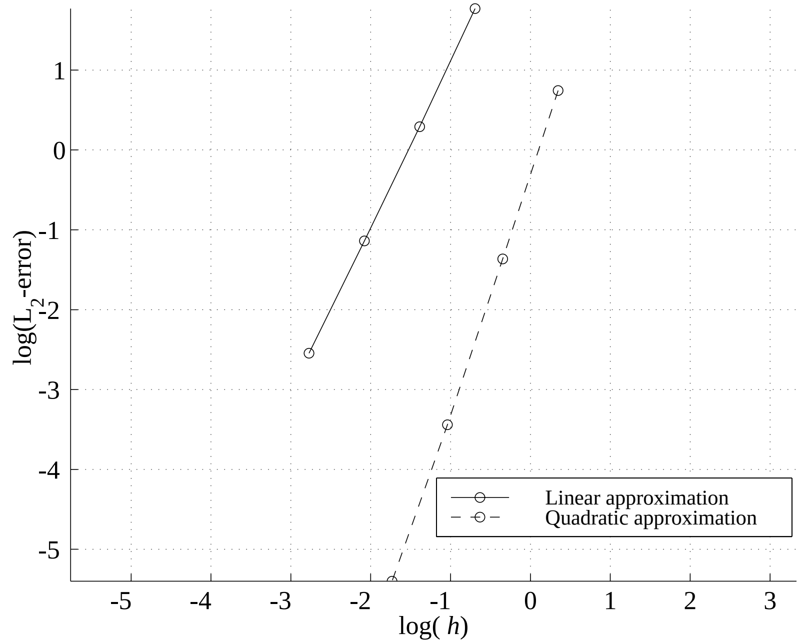
Example: Linears and Quadratics





Example: Convergence in L^2

Convergence in L_2–norm on a sequence of refined meshes. We obtain second and third order convergence for the linear and quadratic approximations, respectively.



Energy norm a posteriori estimate

$$\|\nabla e\|^2_{0,\Omega_1\cup\Omega_2} + \|[e]\|^2_{1/2,h,\Gamma} \leq C \sum_{i=1}^{2} \sum_{K\in T_i^h} \rho^2_{K,i}.$$

The element error indicators $\rho_{K,i}$ are defined by

$$\begin{aligned}\rho^2_{K,i} \;&=\; h_K^2\|f+\Delta U\|^2_{0,P_K} + h_K\|[\boldsymbol{n}_{P_K}\cdot\nabla U]\|^2_{0,\partial P_K} \\ &\quad + h_K^{-1}\|[U]\|^2_{0,\partial P_K\cap\Gamma} + \textstyle\sum_{\Gamma_j\subset\overline{K}} \|[U]\|^2_{1/2,\Gamma_j},\end{aligned}$$

where $P_K = K\cap\Omega_i$ for $K\in T_i^h$.

Computing the $1/2$ norm

- If the interface is one-dimensional:

$$\|[U]\|^2_{1/2,\Gamma_j} := \|[U]\|^2_{0,\Gamma_j} + \int_{\Gamma_j}\int_{\Gamma_j} \frac{|[U](\xi) - [U](\eta)|^2}{|\xi - \eta|^2} d\xi \, d\eta.$$

- $\|[U]\|^2_{1/2,\Gamma_j}$ is difficult to compute in general.

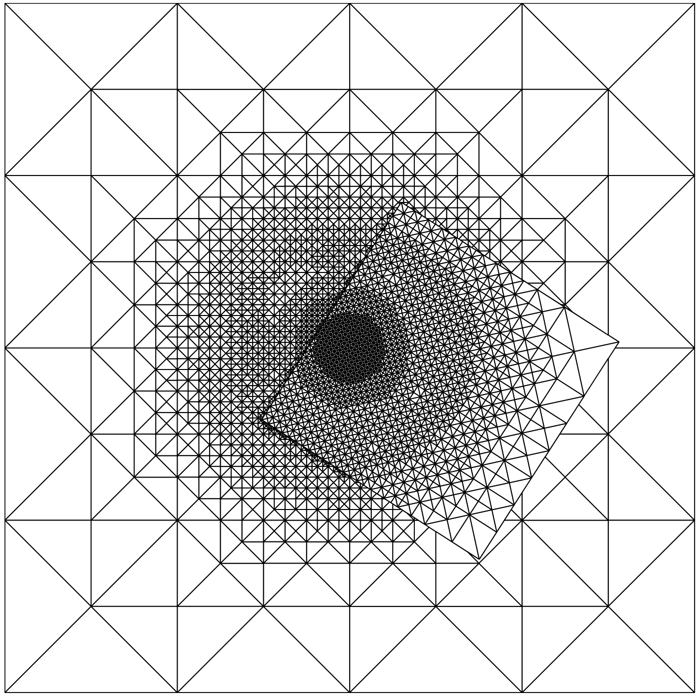
A simplified estimate

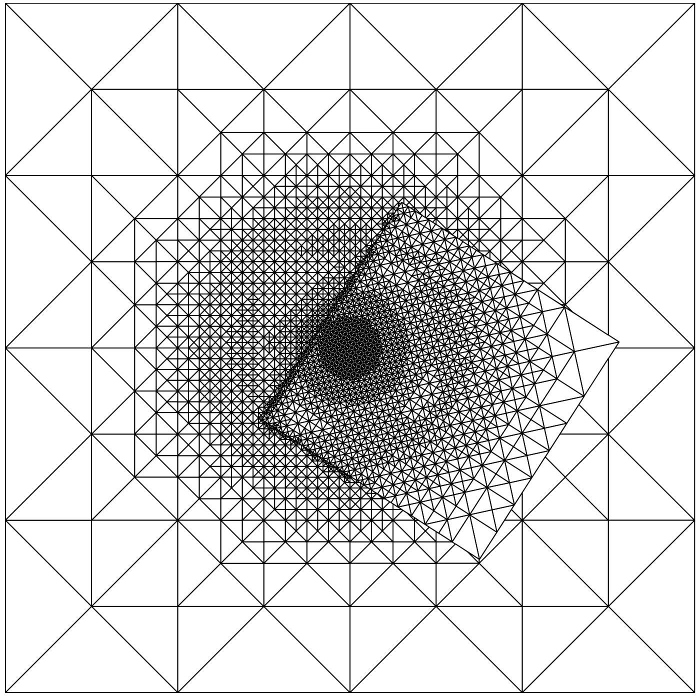
A less sharp but more implementation-friendly variant:

$$
\begin{aligned}
\theta_{K,i}^2 \;&=\; h_K^2 \| f + \Delta U \|_{0,K}^2 + h_K \| [\boldsymbol{n}_K \cdot \nabla U] \|_{0,\partial K \cap \Omega_i^*}^2 \\
&\quad + \sum_{j:\Gamma_j \subset \overline{K}} \Big(h_K \| [\nabla_{\boldsymbol{n}} U]_j \|_{0,\partial K_1^j \cap \Gamma}^2 + h_K^{-1} \| [U]_j \|_{0,\partial K_1^j \cap}^2
\end{aligned}
$$

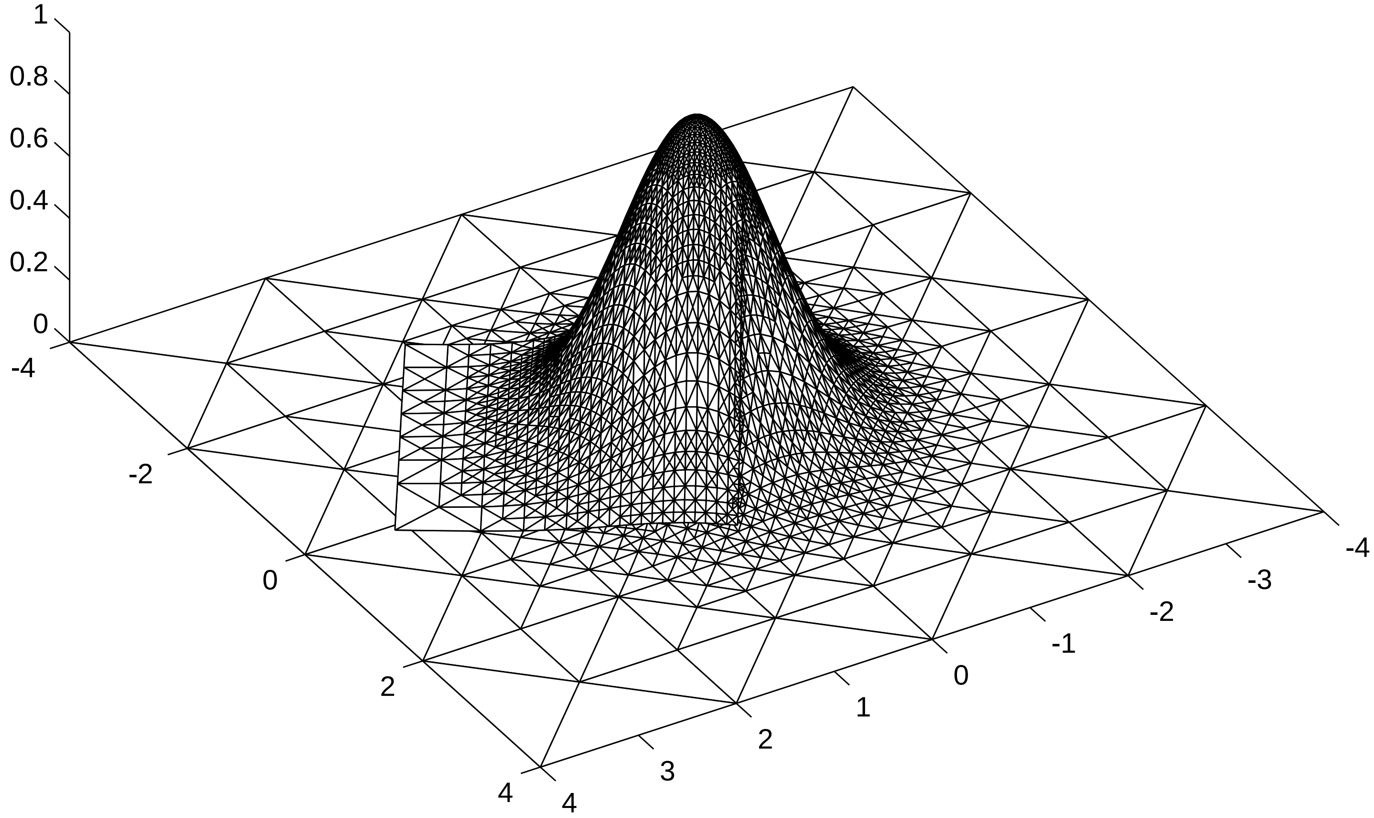
All terms are integrals of single polynomials over the original elements or its edges.

Example: Refined meshes





Example: Refined meshes



The meshes in both cases has a tendency to refine more at the interface. This is because the local error is the largest there. The exact solution is $u = \frac{1}{256} e^{-(x^2+y^2)} \left(4 - x\right) \left(4 + x\right) \left(4 - y\right) \left(4 + y\right)$.